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problem-one

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a) $\begin{pmatrix} 2 & i\sqrt{2} \\ -i\sqrt{2} & 3 \end{pmatrix}\begin{pmatrix} \sqrt{2} \\ -2i \end{pmatrix}$

$= \begin{pmatrix} 2\sqrt{2} + 2\sqrt{2} \\ -i2 - 6i \end{pmatrix}$

$= \begin{pmatrix} 4\sqrt{2} \\ -8i \end{pmatrix}$

b) i) $\begin{pmatrix} 2 & i\sqrt{2} \\ -i\sqrt{2} & 3 \end{pmatrix}\begin{pmatrix} \sqrt{2} \\ -2i \end{pmatrix}$

$= 4 \begin{pmatrix} \sqrt{2} \\ -2i \end{pmatrix}$

$\uparrow$

yes, $\lambda = 4$

ii) $\begin{pmatrix} 2 & i\sqrt{2} \\ -i\sqrt{2} & 3 \end{pmatrix}\begin{pmatrix} \sqrt{2} \\ i \end{pmatrix}$

$$= \begin{pmatrix} 2\sqrt{2} - \sqrt{2} \\ -i2 + 3i \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{2} \\ i \end{pmatrix}$$

$\lambda = 1$ (yes)

c) $\begin{pmatrix} \sqrt{2} & i \end{pmatrix}^* \begin{pmatrix} \sqrt{2} \\ -2i \end{pmatrix}$

$$= 2 - 2 = 0$$

$\Rightarrow$ orthog$^{\underline{n}}$

n.b. H mat: $\lambda \in \mathbb{R}$

$\uparrow$

$H^\dagger = H$ $\quad$ e_λ are orthog.

problem-two

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a) $S_n = \hat{\underline{n}} \cdot \underline{S}$, $\hat{\underline{n}} = \sin\theta\cos\phi\,\hat{\underline{x}} + \sin\theta\sin\phi\,\hat{\underline{y}} + \cos\theta\,\hat{\underline{z}}$

$$\underline{S} = \frac{\hbar}{2} \sum_i \sigma_i x_i$$

$$S_n = \sin\theta\cos\phi\,\sigma_x + \sin\theta\sin\phi\,\sigma_y + \cos\theta\,\sigma_z$$

$$= \left[\begin{pmatrix} 0 & \sin\theta\cos\phi \\ \sin\theta\cos\phi & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i\sin\theta\sin\phi \\ i\sin\theta\sin\phi & 0 \end{pmatrix} + \begin{pmatrix} \cos\theta & 0 \\ 0 & -\cos\theta \end{pmatrix} \right] \frac{\hbar}{2}$$

$$= \frac{\hbar}{2} \begin{pmatrix} \cos\theta & \sin\theta(\cos\phi - i\sin\phi) \\ \sin\theta(\cos\phi + i\sin\phi) & -\cos\theta \end{pmatrix}$$

$$= \frac{\hbar}{2} \begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}$$

$$S_n \chi_n^{\uparrow} = \frac{\hbar}{2} \begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix} \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) e^{i\phi} \end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix} \cos\theta\cos\theta/2 + \sin\theta\sin\theta/2 \\ (\sin\theta\cos\theta/2 - \cos\theta\sin\theta/2)e^{i\phi} \end{pmatrix}$$

$$\sin A\cos B \pm \cos A\sin B = \sin(A \pm B)$$
$$\cos A\cos B \pm \sin A\sin B = \cos(A \mp B)$$

$$= \frac{\hbar}{2}\begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2)e^{i\phi} \end{pmatrix}$$

$$\lambda_{\uparrow} = \frac{\hbar}{2}$$

$$S_n \chi_n^{\downarrow} = \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}\begin{pmatrix} \sin\theta/2 \\ -\cos\theta/2\, e^{i\phi} \end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix} \cos\theta\sin\theta/2 - \sin\theta\cos\theta/2 \\ (\sin\theta\sin\theta/2 + \cos\theta\cos\theta/2)e^{i\phi} \end{pmatrix}$$

$$= -\frac{\hbar}{2}\begin{pmatrix} \sin\theta/2 \\ e^{i\phi}\cos\theta/2 \end{pmatrix}$$

$$\lambda_{\downarrow} = -\frac{\hbar}{2}$$

$$\begin{pmatrix} \cos\theta/2 & \sin\theta/2\, e^{i\phi} \end{pmatrix}^{*}\begin{pmatrix} \sin\theta/2 \\ -\cos\theta/2\, e^{i\phi} \end{pmatrix}$$

$= \sin\theta/2 \cos\theta/2 - \sin\theta/2 \cos\theta/2$

$= 0$

$$\begin{pmatrix} \cos\theta/2 & \sin\theta/2 \, e^{i\phi} \end{pmatrix}^* \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \, e^{i\phi} \end{pmatrix}$$

$= \cos^2\theta/2 + \sin^2\theta/2 \quad = 1$

||ly for other

b) $S_x = \frac{\hbar}{2}\sigma_x$

need $\theta = \frac{\pi}{2}, \ \phi = 0$ for $\underline{n = x}$

$$\chi_x^{\uparrow} = \begin{pmatrix} \cos(\pi/4) \\ \sin(\pi/4) \, e^{i0} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\chi_x^{\downarrow} = \begin{pmatrix} \sin(\pi/4) \\ -\cos\left(\frac{\pi}{4}\right) e^{i0} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\underline{n=z}$: $\theta=0, (\phi=0)$

$\chi_z^{\uparrow} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\chi_z^{\downarrow} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \neq 0$ etc.

c) $S_y = \frac{\hbar}{2}\sigma_y$ $\left(\theta = \frac{\pi}{2}, \phi = \frac{\pi}{2}\right)$

$\chi_x^{\downarrow}$

so $\langle \chi_x^{\downarrow} | S_y | \chi_x^{\downarrow} \rangle$

$= \frac{\hbar}{4} \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\sim \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} i \\ i \end{pmatrix}$

$= \quad i - i = 0$

$\Rightarrow$ equal probability to be $\pm \frac{\hbar}{2}$

d) initial is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

say want final $\chi_{\hat{n}}^{\uparrow}(\theta, \phi)$

$$P_{\uparrow} = \left| \begin{pmatrix} \cos\theta/2 & \sin\theta/2 \, e^{i\phi} \end{pmatrix}^{*} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right|^2$$

$$= \cos^2 \theta/2$$

$$\Rightarrow P_{\downarrow} = \sin^2 \theta/2 .$$

problem-three

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$$H = -\underline{B}\cdot\underline{M}_s \quad ; \quad \underline{M}_s = \frac{\hbar}{2}\gamma\,\underline{\sigma} \,, \quad \underline{S} = \frac{\hbar}{2}\underline{\sigma}$$

$$\therefore\ H = -\frac{\hbar\gamma}{2}\,\underline{B}\cdot\underline{\sigma} \quad ; \quad \underline{B} = B\hat{\underline{z}}$$

$$\therefore\ H = -\frac{\hbar}{2}\gamma B\sigma_z \quad ; \quad \omega = -\gamma B$$

$$\therefore\ H = \frac{\hbar}{2}\omega\sigma_z = \frac{\hbar}{2}\omega\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$\Rightarrow$ eigenstates: $e_\uparrow = \begin{pmatrix}1\\0\end{pmatrix}$, $e_\downarrow = \begin{pmatrix}0\\1\end{pmatrix}$

with energies: $E_\uparrow = \frac{\hbar\omega}{2}$, $E_\downarrow = -\frac{\hbar\omega}{2}$

$$\chi_n^\uparrow(t=0) = \chi_n^\uparrow = \cos\theta/2\begin{pmatrix}1\\0\end{pmatrix} + \sin\theta/2\, e^{i\phi}\begin{pmatrix}0\\1\end{pmatrix}$$

$$\Rightarrow \chi_n^\uparrow(t) = \cos\theta/2\begin{pmatrix}1\\0\end{pmatrix} e^{-iE_\uparrow t/\hbar} + \sin\theta/2\, e^{i\phi}\begin{pmatrix}0\\1\end{pmatrix} e^{-iE_\downarrow t/\hbar}$$

$$= \begin{pmatrix}\cos\theta/2\; e^{-i\omega t/2}\\ \sin\theta/2\; e^{i\phi}\, e^{i\omega t/2}\end{pmatrix}$$

$$= e^{-i\omega t/2} \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \; e^{i[\phi+\omega t]} \end{pmatrix}$$

problem-four

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$$S^2 = \underline{S} \cdot \underline{S}$$

$$= \left| \frac{\hbar}{2} \left(\sigma_x + \sigma_y + \sigma_z \right) \right|^2$$

$$= \frac{\hbar^2}{4} \underbrace{\left(\sigma_x^2 + \sigma_y^2 + \sigma_z^2 \right)}$$

$$\sigma_i = \mathbb{1}$$

$$= \frac{3\hbar^2}{4}$$