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problem-one

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Problem 1

a) i) $\int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\phi \; Y_{0,0}^*(\theta,\phi)\, Y_{0,0}(\theta,\phi)$

$$= \frac{1}{4\pi}\, 2\pi \left[-\cos\theta\right]_0^{\pi} = 1$$

ii) $\int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\phi \; Y_{1,1}^*(\theta,\phi)\, Y_{1,1}(\theta,\phi)$

$$= \frac{3}{8\pi} \underbrace{\int_0^{\pi} d\theta \, \sin^3\theta}_{4/3} \int_0^{2\pi} d\phi$$

$u = \cos\theta$

$du = -\sin\theta \, d\theta$

$\sin^2\theta = 1 - \cos^2\theta = 1 - u^2$

$$= \frac{3}{4} \int_{-1}^{1} du \, (1 - u^2)$$

$$= \frac{3}{4} \left[u - \frac{1}{3} u^3 \right]_{-1}^{1}$$

$$= \frac{3}{4} \left[\frac{2}{3} - \left(-1 + \frac{1}{3}\right)\right]$$

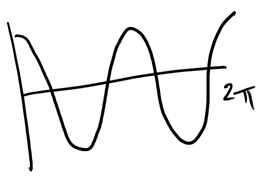
$$= 3/4 \left[2/3 + 2/3 \right]$$

$$= 1$$

b) i) $\int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\phi \; Y_{1,-1}^*(\theta,\phi)\, Y_{1,1}(\theta,\phi)$

$$= \int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\phi \left(+ \sqrt{\frac{3}{8\pi}} \sin\theta \, e^{i\phi} \right)\left(- \sqrt{\frac{3}{8\pi}} \sin\theta \, e^{i\phi} \right)$$

$$= -\frac{3}{8\pi} \underbrace{\int_0^{\pi} d\theta \, \sin^3\theta}_{4/3} \int_0^{2\pi} d\phi \; e^{2i\phi}$$



$= 0$

OR

4/3

OR

$$= -\frac{1}{2\pi}\underbrace{\left[\frac{1}{2i}e^{2i\phi}\right]_0^{2\pi}}$$

$$e^{in\pi} = \begin{cases} -1 & n \in \text{odd} \\ 1 & n \in \text{even} \end{cases}$$

$$e^{i4\pi} - e^{0} = 1 - 1 = 0$$

ii) $$\int_0^{\pi} d\theta \, \sin\theta \int_0^{2\pi} d\phi \left(-\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\,\cancel{\exp(-i\phi)}\right)\left(-\sqrt{\frac{3}{8\pi}}\sin\theta\,\cancel{e^{i\phi}}\right)$$

$$\sim \int_0^{\pi} d\theta \, \sin^3\theta \, \cos\theta$$

$$= \int_{-1}^{1} du \left(u - u^3\right) \quad = 0 \quad \text{since odd.}$$

OR

$$= \left[\frac{u^2}{2} - \frac{u^4}{4}\right]_{-1}^{1} = \frac{1}{2} - \frac{1}{4} - \left(\frac{1}{2} - \frac{1}{4}\right) = 0$$

c) $$\cos\theta \sin\theta \cos\phi \qquad \cos\phi = \frac{e^{i\phi} + e^{-i\phi}}{2}$$

$$= \frac{1}{2}\left[\cos\theta\sin\theta\, e^{i\phi} + \;{}'' \; e^{-i\phi}\right]$$

$$= \frac{1}{2}\left[-\sqrt{\frac{8\pi}{15}}\,Y_{2,1} + \sqrt{\frac{8\pi}{15}}\,Y_{2,-1}\right]$$

problem-two

Friday, October 5, 2018 12:15 PM

a) i) $\int_{-\infty}^{\infty} dV \, |\psi_{100}(r)|^2$

n.b. can use $\int_0^\infty r^k e^{-\alpha r} dr = \frac{k!}{\alpha^{k+1}}$

$$= \underbrace{\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta}_{} \int_0^\infty r^2 dr \quad 4 \, e^{-2r} \, \underbrace{|Y_{0,0}|^2}_{=1}$$

$(fg)' = f'g + fg'$

$$= 4\int_0^\infty dr \, r^2 e^{-2r}$$

$f = r^2 \quad g' = e^{-2r}$

$f' = 2r \quad g = -\frac{1}{2} e^{-2r}$

$$\overset{IBP}{=} 4\left\{ \cancel{\left[-\frac{r^2}{2} e^{-2r}\right]_0^\infty} + \int_0^\infty dr \, r \, e^{-2r} \right\}$$

$$\overset{IBP}{=} 4\left\{ \cancel{\left[r\left(-\frac{1}{2}\right) e^{-2r}\right]} + \frac{1}{2}\int_0^\infty dr \, e^{-2r} \right\}$$

$$= 2\left(-\frac{1}{2}\right)\left[e^{-2r}\right]_0^\infty$$

$$= 1$$

ii) $\int dV \, \psi_{100} \, \psi_{200}^*$

$$= 1 \times \frac{2}{\sqrt{2}} \int_0^\infty r^2 dr \left(1 - \frac{r}{2}\right) e^{-\frac{3}{2}r}$$

$f = r^3 \quad g' = e^{-3/2 r}$

$f' = 3r^2 \quad g = -\frac{2}{3} e^{-3/2 r}$

$$\sim \int dr \, r^2 e^{-3/2 r} - \underbrace{\frac{1}{2}\int dr \, r^3 \, e^{-3/2 r}}_{}$$

$$\overset{IBP}{=} 0 + \frac{1}{\cancel{2}} \int dr \, \cancel{3} \, r^2 \, e \left(+\frac{\cancel{2}}{\cancel{3}}\right) e^{-\frac{3}{2}r}$$

$= 0$

$-iHt/\hbar \qquad -\frac{i}{\hbar} E_{1s} t$

$= 0$

b) $\quad e^{-iHt/\hbar} = e^{-\frac{i}{\hbar} E_{1s} t}$

c) stationary state $\Rightarrow$ eigenstate of H $\Rightarrow$ well-defined E

1s, 2s have diff. E, so no.

d)
$$P = |\langle \psi(r, t=0) \,|\, \psi_{100}(r) \rangle|^2$$

$$= \left| \int dV \, \frac{1}{\sqrt{2}} e^{-r/2} |Y_{00}|^2 \, 2 e^{-r} \right|^2$$

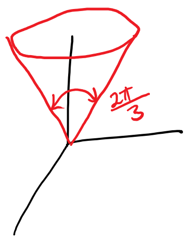
$$= \left| \int_0^\infty r^2 \, dr \, \sqrt{2} \quad e^{-\frac{3}{2} r} \right|^2$$

$$= \left| \sqrt{2} \int r^2 e^{-\frac{3}{2} r} \, dr \right|^2$$

$$= \left| \sqrt{2} \, \frac{2!}{(3/2)^3} \right|^2 = \left| 2\sqrt{2} \left(\frac{2}{3} \right)^3 \right|^2 = \left| \frac{16\sqrt{2}}{27} \right|^2 = \frac{2^9}{3^6} < 1 \quad ✓$$

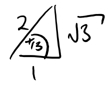
problem-three

Saturday, October 6, 2018 5:22 PM



$$\Psi = R(r)\, Y_{1,0}(\theta,\phi) \qquad |\Psi|^2 = |R|^2 \, |Y_{1,0}|^2$$

$$P = \int_0^{2\pi/6} \left|\sqrt{\frac{3}{4\pi}}\cos\theta\right|^2 \sin\theta \, d\theta \int_0^{2\pi} d\phi \int_0^{\infty} r^2 |R(r)|^2$$



$$= \frac{3}{2} \int_0^{\pi/3} \cos^2\theta \sin\theta \, d\theta$$

$u = \cos\theta$

$du = -\sin\theta \, d\theta$

$$= -\frac{3}{2} \int_1^{1/2} u^2 \, du$$

$$= -\frac{1}{2} \left[u^3 \right]_1^{1/2}$$

$$= -\frac{1}{2} \left(\frac{1}{8} - 1 \right)$$

$$= -\frac{1}{2} \left(-\frac{7}{8} \right)$$

$$= \frac{7}{16}$$

problem-four

Saturday, October 6, 2018 5:42 PM

a) i) $-\frac{1}{\sqrt{2}}\left(Y_{1,1} - Y_{1,-1}\right)$

$$= -\frac{1}{\sqrt{2}}\left(-\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi} - \sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\phi}\right)$$

$$= \frac{1}{2}\sqrt{\frac{3}{\pi}}\sin\theta\,\frac{e^{i\phi}+e^{-i\phi}}{2}$$

$$= \frac{1}{2}\sqrt{\frac{3}{\pi}}\sin\theta\cos\phi$$

$$= \sqrt{\frac{3}{4\pi}}\frac{x}{r}$$

ii) $\frac{i}{\sqrt{2}}\left(Y_{11} + Y_{1-1}\right)$

$$= \frac{i}{\sqrt{2}}\left(-\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi} + \sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\phi}\right)$$

$$= -\frac{i}{2}\sqrt{\frac{3}{\pi}}\, i\,\frac{e^{i\phi}-e^{-i\phi}}{2i}\sin\theta$$

$$= \sqrt{\frac{3}{4\pi}}\sin\theta\sin\phi$$

$$= \sqrt{\frac{3}{4\pi}}\frac{y}{r}$$

iii) $Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta = \sqrt{\frac{3}{4\pi}}\frac{z}{r}$

b) $L_x(a)$

$$i\hbar\left(-\sin\phi\,\partial_\theta - \cot\theta\cos\phi\,\partial_\phi\right)\sqrt{\frac{3}{4\pi}}\sin\theta\cos\phi$$

$$= -i\hbar\left(-\sin\phi\,\partial_\theta - \cot\theta\cos\phi\,\partial_\phi\right)\sqrt{\frac{3}{4\pi}}\,\sin\theta\cos\phi$$

$$= i\hbar\sqrt{\frac{3}{4\pi}}\left[\sin\phi\cos\phi\cos\theta - \cos\theta\cos\phi\sin\phi\right]$$

$= 0$ = e.va

L_y (b)

$$= -i\hbar\left(\cos\phi\,\partial_\theta - \cot\theta\sin\phi\,\partial_\phi\right)\sqrt{\frac{3}{4\pi}}\,\sin\theta\sin\phi$$

$$= i\hbar\sqrt{\frac{3}{4\pi}}\left(-\cos\phi\sin\phi\cos\theta + \cos\theta\sin\phi\cos\phi\right)$$

$= 0$

L_z (c)

$$= -i\hbar\sqrt{\frac{3}{4\pi}}\,\partial_\phi\cos\theta$$

$= 0$

c) 3: X : which projection of $\underline{L}$ (ie. e.f$\underline{n}$s of L_x, L_y, L_z ...) = convention

2: X : normalisation <u>useful only</u>, not essential

1: ✓ : we have a L.I. set → basis for $l=1$ states (from lin. comb. of Y's)

problem-five

Saturday, October 6, 2018 6:00 PM

a) $\int \psi^*_{211}(\underline{r})\, H'(r)\, \psi_{200}\, d^3r$

$$= \int r^2 dr\, d\phi\, \sin\theta\, d\theta \left(-V_0 \frac{e^{-3r}}{r}\right) \frac{1}{\sqrt{2}} \left(1 - \frac{r}{2}\right) e^{-r/2}\, Y^*_{0,0}\, \frac{1}{\sqrt{24}}\, r\, e^{-r/2}\, Y_{1,1}$$

$$= \frac{-V_0}{4\sqrt{3}} \int d\Omega\, \underbrace{Y^*_{0,0}\, Y_{1,1}}_{= \delta_{10}\, \delta_{10}\ =\ 0} \cdots$$

b) i) $\int \psi^*_{200}\, H'\, \psi_{200}\, dV$

$$= \int r^2 \sin\theta\, d\theta\, d\phi\, dr\, (-V_0) \frac{e^{-3r}}{r}\, \frac{1}{2} \left(1 - \frac{r}{2}\right)^2 e^{-r} |Y_{0,0}|^2$$

$$= -\frac{V_0}{2} \int dr\, r\, e^{-4r} \left(1 + \frac{r^2}{4} - r\right)$$

$$= -\frac{V_0}{2} \left(\int dr\, r e^{-4r} + \frac{1}{4} \int dr\, r^3 e^{-4r} - \int dr\, r^2 e^{-4r} \right)$$

$$= -\frac{V_0}{2} \left(\frac{1}{4^2} + \frac{1}{4} \frac{6}{4^4} - \frac{2}{4^3} \right)$$

$$= -\frac{V_0}{2^5} \left[1 + \frac{6}{4^3} - \frac{2}{4} \right]$$

$$= 1 + \frac{6}{64} - \frac{1}{2} = \frac{1}{2} + \frac{3}{32} = \frac{19}{32}$$

$$= -\frac{19 V_0}{2^{10}}$$

ii)

$$\int \psi_{21m}^{*} \, H' \, \psi_{21m} \, d^3 r$$

$$= \int dr \, r^2 \, \frac{1}{24} r^2 e^{-r} \left(-V_0 \frac{e^{-3r}}{r} \right)$$

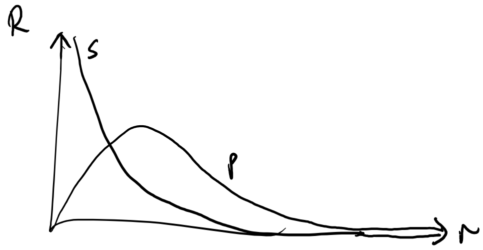
$$= \frac{1}{24} \int dr \, r^3 e^{-4r}$$

$$= \frac{1}{2^3 \cdot 3} \frac{6}{4^4} (-V_0)$$

$$= \frac{-V_0}{2^{10}}$$

c) H' weaker on $\overset{p}{\infty}$ than $\overset{s}{\bigcirc}$.

since $H' \sim e^{-3r}$



d) ψ_{2lm} are all L.I. ($l = 1, 2$ for s, p) ← degenerate states

($m = (l-1) \ldots -(l-1)$)

d) $\psi_{2\ell m}$ are all L.I. $(\ell = 1, 2$ for $s, p)$
$(m = (\ell-1), \ldots, -(\ell-1))$

recall: $E_n^{(1)} = \langle n^{(0)} | H' | n^{(0)} \rangle$

since Y orthog.

off-diagonals elements of matrix are zero $\Rightarrow$ (e) no mixing.

& $E^{(1)}_{\ell=0} = -19 \frac{V_0}{1024}$

$E^{(1)}_{\ell=1} = -\frac{V_0}{1024}$

corrections (1st order)

$\rightarrow$ splitting of deg. en. level