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problem-one

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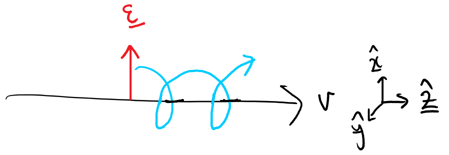

$$\underline{\mathcal{E}}(t) = \frac{\mathcal{E}_0}{2}\left[\hat{\underline{\varepsilon}}\, e^{-i\omega t} + \hat{\underline{\varepsilon}}^{*} e^{i\omega t}\right] \qquad \hat{\underline{\varepsilon}}_{R,L} = \frac{\hat{\underline{x}} \mp i\hat{\underline{y}}}{\sqrt{2}}$$

$$\hat{\underline{\varepsilon}}_R \equiv \hat{\underline{\varepsilon}}, \quad \left(\hat{\underline{\varepsilon}}_L \equiv \hat{\underline{\varepsilon}}^{*}\right)$$ – choose right circularly polarized field.

$$\underline{r} \equiv x\hat{\underline{x}} + y\hat{\underline{y}} + z\hat{\underline{z}}$$

$$\underline{r}\cdot\hat{\underline{\varepsilon}} = \left(\underline{r}\cdot\hat{\underline{x}} - i\,\underline{r}\cdot\hat{\underline{y}}\right)/\sqrt{2} = \frac{x - iy}{\sqrt{2}}$$

a) i) $-\underline{\nabla} H'(\underline{r},t) = q\,\underline{\mathcal{E}}(t)$

$$\Rightarrow \quad H'(\underline{r},t) = \underbrace{-q}_{=e}\,\underline{r}\cdot\underline{\mathcal{E}}(t)$$

$$= \frac{e\,\mathcal{E}_0}{2}\left[\underline{r}\cdot\hat{\underline{\varepsilon}}\, e^{-i\omega t} + \text{c.c.}\right]$$

$$\Rightarrow \quad H'(\underline{r},t) = \frac{e\,\mathcal{E}_0}{2\sqrt{2}}\left[(x-iy)e^{-i\omega t} + (x+iy)e^{i\omega t}\right]$$

ii) $\langle n_b l_b m_b |\; H' \;| n_a l_a m_a\rangle$

$$= \frac{e\,\mathcal{E}_0}{2\sqrt{2}}\left[e^{-i\omega t}\underbrace{\langle n_b l_b m_b|(x-iy)|n_a l_a m_a\rangle}_{\text{matrix el.}} + e^{i\omega t}\underbrace{\langle n_b l_b m_b|(x+iy)|n_a l_a m_a\rangle}\right]$$

for non-zero, need one of these non-zero.

Now, $[L_z, x \pm iy] = \pm\hbar\,(x \pm iy)$ & $L_z|nlm\rangle = \hbar m\,|nlm\rangle$ $\Big\}$ (*)

$\langle nlm| L_z = \hbar m \langle nlm|$

so write:

$$\langle n_b l_b m_b | [L_z, x \pm iy] | n_a l_a m_a \rangle = \pm \hbar \langle b | (x \pm iy) | a \rangle$$

n.b. $|n_a l_a m_a\rangle =: |a\rangle$

$$\text{LHS} = \langle b | \{ L_z (x \pm iy) - (x \pm iy) L_z \} | a \rangle$$

$$= \hbar \{ (m_b - m_a) \langle b | (x \pm iy) | a \rangle \}$$

$(*)$

Thus, $\hbar (m_b - m_a \mp 1) \langle b | (x \pm iy) | a \rangle = 0$ ← ie LHS − RHS = 0

one must vanish for equality to hold.

If we want $\langle b | (x \pm iy) | a \rangle \neq 0$, then we need $(m_b - m_a \mp 1) = 0$

So either $m_b - m_a - 1 = 0$ or $m_b - m_a + 1 = 0$

$$\underline{\underline{m_b = m_a + 1}} \qquad \underline{\underline{m_b = m_a - 1}}$$

problem-two

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particle: mass m

initially in state a w/ wf. $\psi_a(\underline{r})$

$\exists$ another state b $(\psi_b(\underline{r}))$ – 2-level sys.

after $t=0$, static potential: $H'(t) = \begin{cases} 0 & t<0 \\ V(\underline{r}) & t \geq 0 \end{cases}$

$$\Rightarrow \psi(\underline{r}, t>0) = \sum_{i \in \{a,b\}} c_i(t) \underbrace{\psi_i(\underline{r})\, e^{-iE_i t/\hbar}}_{=:\, \Psi_i(\underline{r},t)} \quad \text{— } (*)$$

$**$

Aside 1

Recall sinusoidal perturbation: $V(\underline{r}) \cos(\omega t)$ $\quad [\hbar\omega_0 = E_b - E_a]$

1st order TD PT: $\quad c_a^{(1)}(t) = 1$

$$c_b^{(1)}(t) = -\frac{V_{ba}}{2\hbar}\left[\frac{e^{i(\omega_0+\omega)t} - 1}{\omega_0+\omega} + \frac{e^{i(\omega_0-\omega)t} - 1}{\omega_0-\omega}\right]$$

$$\simeq -i\frac{V_{ba}}{\hbar} e^{i(\omega_0-\omega)t/2} \frac{\sin\left[\frac{1}{2}(\omega_0-\omega)t\right]}{\omega_0-\omega} \quad \text{— } (\dagger)$$

transition probability

$\&$ $\quad P_{a\to b} = |c_b^{(1)}(t)|^2$

i.e. recall; $\quad P_{a\to b} = |\langle\psi_b| H' |\psi_a\rangle|^2$

$\&$ $\quad c_i = \langle\psi_i| H' |\psi_a\rangle$

final state $\quad$ initial state

$$\Rightarrow \quad P_{a\to b} = |c_b^{(1)}(t)|^2 \quad \text{to } 1^{st} \text{ order}$$

& also, $$\Psi(\underline{r},t) = \sum_{i\in\{a,b\}} \langle \Psi_i | H' | \Psi_a \rangle \, \Psi_i(\underline{r},t)$$ — cf. (*)

So,

$$P_{a\to b} \simeq \left| -i \frac{V_{ba}}{\hbar} e^{i(\omega_0-\omega)t/2} \frac{\sin\left[\frac{1}{2}(\omega_0-\omega)t\right]}{\omega_0-\omega} \right|^2$$

$$= \frac{|V_{ba}|^2}{\hbar^2} \frac{\sin^2\left(\frac{1}{2}(\omega_0-\omega)t\right)}{(\omega_0-\omega)^2}$$

But what about our static (after $t=0$) system?

end aside !

Consider $0 \leq t' \leq t$ times $\Rightarrow$ T.I.

a) $c_a^{(1)}(t) = 1$

$$c_b^{(1)}(t) = -\frac{V_{ba}}{2\hbar}\left[\frac{e^{i(\omega_0+\omega)t}-1}{\omega_0+\omega} + \frac{e^{i(\omega_0-\omega)t}-1}{\omega_0-\omega}\right]$$

Still true for TI, except

time ind. $\Rightarrow$ $\omega = 0$

$$\Rightarrow c_b^{(1)}(t) = -\frac{V_{ba}}{2\hbar\omega_0}\left(e^{i\omega_0 t} - 1 + e^{i\omega_0 t} - 1\right)$$

$$= -\frac{2V_{ba}i}{\hbar\omega_0}\left(\frac{e^{\frac{i\omega_0 t}{2}} - e^{-\frac{i\omega_0 t}{2}}}{2i \; e^{-\frac{i\omega_0 t}{2}}}\right)$$

i.e. the $\simeq$ in (†)

TI: for TD, could neglect one of these, but can't for TI $\Rightarrow$ TI = 2×TD here.

$$\Rightarrow \quad c_b^{(1)}(t) = -\frac{2V_{ba}i}{\hbar\omega_0} \sin\left(\frac{\omega_0}{2}t\right) e^{i\omega_0 t/2}$$

$$\Rightarrow P_{a\to b} = |c_b^{(1)}(t)|^2 = \frac{4|V_{ba}|^2}{\;} \sin^2\left(\frac{\omega_0 t}{2}\right)$$

← so $P_{TI} = 4 P_{TD}$

$$\Rightarrow P_{a\to b} = |c_b^{(1)}(t)|^2 = \frac{4|V_{ba}|^2}{\hbar^2 \omega_0^2} \sin^2\left(\frac{\omega_0 t}{2}\right)$$ ← so $P_{TI} = 4 P_{TD}$

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b) $\psi_i(\underline{r}) = \frac{1}{\sqrt{V}} e^{i\underline{k}'\cdot\underline{r}}$ — particle has definite wavevector $\underline{k}'$ initially
⇒ plane wave wf.

$$\underline{j}_i(\underline{r}) = -\frac{i\hbar}{2m}\left[\psi_i^*(\underline{r})\,\underline{\nabla}\psi_i(\underline{r}) - \psi_i(\underline{r})\underline{\nabla}\psi_i^*(\underline{r})\right]$$ — probability current.

$$\psi_i^*(\underline{r}) = \frac{1}{\sqrt{V}} e^{-i\underline{k}'\cdot\underline{r}} \qquad \underline{\nabla}\psi_i(\underline{r}) = \frac{1}{\sqrt{V}} i\underline{k}' e^{i\underline{k}'\cdot\underline{r}} = i\underline{k}'\psi_i(\underline{r})$$

$$|\psi_i(\underline{r})|^2 = \frac{1}{V} \qquad \underline{\nabla}\psi_i^*(\underline{r}) = -i\underline{k}'\psi_i^*(\underline{r})$$

$$\Rightarrow \underline{j}_i(\underline{r}) = -\frac{i\hbar}{2m}\left[i\underline{k}'|\psi_i(\underline{r})|^2 + i\underline{k}'|\psi_i(\underline{r})|^2\right]$$

$$= \frac{\hbar\underline{k}'}{m}|\psi_i(\underline{r})|^2$$

$$\Rightarrow \underline{j}_i(\underline{k}') = \frac{\hbar\underline{k}'}{Vm}$$

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Aside 2 under perturbation, state transitions to bunch of plane waves, $\sum_i \psi_f(\underline{k}_i, \underline{r})$

one such plane wave

$\hookrightarrow \psi_f(\underline{k},\underline{r}) = \frac{1}{\sqrt{V}} e^{i\underline{k}\cdot\underline{r}}$ $\qquad g(k)\,dk\,\frac{d\Omega}{4\pi} = \frac{Vk^2\,dk\,d\Omega}{8\pi^3}$ $\qquad E = \frac{\hbar^2k^2}{2m}$ — (*)

$$\Downarrow$$

$$g(k)\,dk\,\frac{d\Omega}{4\pi} = \rho(E)\,dE \qquad dE = \frac{\hbar^2}{2m}2k\,dk = \frac{\hbar^2 k\,dk}{m}$$

$$\Rightarrow \quad \frac{V k^2 \cancel{dk}\, d\Omega}{8\pi^3} = \rho(\epsilon) \frac{\hbar^2 \cancel{k}\, \cancel{dk}}{m}$$

$$k = \frac{\sqrt{2mE}}{\hbar}$$

$$\Rightarrow \quad \rho(\epsilon) = \frac{V k m\, d\Omega}{8\pi^3 \hbar^2}$$

$$\rho(\epsilon) = \frac{V m\, d\Omega \sqrt{2mE}}{8\pi^3 \hbar^3}$$

density of states ρ in final states ψ_f w/ energy (*) travelling in solid angle $d\Omega$

end aside 2

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recall: Fermi Golden Rule: rate at which p's scattered into $d\Omega$:

$$R_{i \to d\Omega} = \frac{\pi |V_{if}|^2}{2\hbar} \rho(E_f) \quad \leftarrow \text{TD}$$

c) In (b) we saw that V was $\times 2$ for TI vs TD

so $|V|^2$ is $\times 4$

ie. $$R_{i \to d\Omega} = \frac{2\pi |V_{if}|^2}{\hbar} \rho(E_f) \quad \leftarrow \text{TI} \quad (\text{ie our system at } t>0)$$

where $$V_{if} = \langle \psi_i | \underset{H' = V(\underline{r})}{V} | \psi_f \rangle = \frac{1}{V} \int dV\, e^{-i\underline{k}' \cdot \underline{r}}\, V(\underline{r})\, e^{i \underline{k} \cdot \underline{r}}$$

$$= \frac{1}{V} \int dV\, e^{i(\underline{k} - \underline{k}') \cdot \underline{r}}\, V(\underline{r})$$

from Aside 2, $$\rho(E_f) = \frac{V\, d\Omega \sqrt{2m^3 E_f}}{(2\pi\hbar)^3}$$

$$\Rightarrow R_{i\to d\Omega} = \frac{2\pi}{\hbar}\frac{1}{V^{\cancel{2}}}\left|\int dV\, e^{i(\underline{k}-\underline{k}')\cdot\underline{r}}\, V(\underline{r})\right|^2 \frac{\cancel{V}}{8\pi^3\hbar^3}\sqrt{2m^3E_f}\, d\Omega$$

ie $d^3\underline{r}$

$$\Rightarrow R_{i\to d\Omega} = \frac{\sqrt{2m^3E_f}}{4\pi^2\hbar^4 V}\left|\int dV\, e^{i(\underline{k}-\underline{k}')\cdot\underline{r}}\, V(\underline{r})\right|^2 d\Omega \quad — \quad (\dagger\dagger)$$

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$$E = \frac{\hbar^2k^2}{2m} \Downarrow k = \frac{\sqrt{2mE}}{\hbar}$$

Born approximation

Find differential cross-section

(d) $$\frac{d\sigma}{d\Omega} = \frac{R_{i\to d\Omega}}{j_i\, d\Omega}$$

recall: $$j_i(k') = \frac{\hbar k'}{Vm} = \frac{\hbar}{Vm}\frac{\sqrt{2mE_f}}{\hbar} = \frac{1}{V}\sqrt{\frac{2E_f}{m}} \quad \text{from (6)}$$

& $R_{i\to d\Omega}$ given by $(\dagger\dagger)$ in (c)

n.b. probability to scatter from $\psi_i \to \psi_f \neq 0$ only for $E_i = E_f$ ($\Leftrightarrow k' = k$)
ie conservation of energy

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{\cancel{V}}{\cancel{d\Omega}}\sqrt{\frac{m}{\cancel{2}\cancel{E_f}}}\frac{\sqrt{\cancel{2}m^3\cancel{E_f}}}{4\pi^2\hbar^4\cancel{V}}\left|\int dV\, e^{i(\underline{k}-\underline{k}')\cdot\underline{r}}\, V(\underline{r})\right|^2 \cancel{d\Omega}$$

$$= \left|\frac{m}{2\pi\hbar^2}\int dV\, e^{i(\underline{k}-\underline{k}')\cdot\underline{r}}\, V(\underline{r})\right|^2$$

$\boxtimes$