QM3 workshop 4, **Problem 1** (Problems from D. Griffiths, Introduction to QM)
A hydrogen atom is placed in a time-dependent electric field $\boldsymbol{E} = E(t)\,\hat{\boldsymbol{z}}$.
We consider the ground state ($n = 1$) and the quadruply degenerate first excited states ($n = 2$).
(a) Calculate all four matrix elements H'_{ij} of the perturbation $H' = -eEz$ between the ground state ($n = 1$) and the quadruply degenerate first excited states ($n = 2$).
Note: Only one integral is nonzero; you can realise which one it is if you exploit oddness with respect to z.
(b) Show that $H'_{ii} = 0$ for all five states.
The eigenfunctions of the hydrogen atom are: ($m = -l, \ldots, l$; $l = 0, 1, \ldots, n-1$; $n = 1, 2, \ldots$)

$$\psi_{nlm}(r, \theta, \phi) = R_{nl}(r)\, Y_{lm}(\theta, \phi)$$

with

$$\begin{aligned} R_{10} &= \frac{2}{\sqrt{a^3}}\, e^{-r/a}, \\ R_{20} &= \frac{1}{\sqrt{2\,a^3}}\left(1 - \frac{r}{2a}\right) e^{-r/2a}, \\ R_{21} &= \frac{1}{2\sqrt{6\,a^3}}\,\frac{r}{a}\, e^{-r/2a} \end{aligned}$$

$$\begin{aligned} Y_{0,0}(\theta, \phi) &= \frac{1}{\sqrt{4\pi}} \\ Y_{1,0}(\theta, \phi) &= \sqrt{\frac{3}{4\pi}}\,\cos\theta \\ Y_{1,\pm 1}(\theta, \phi) &= \mp\sqrt{\frac{3}{8\pi}}\,\sin\theta\exp(\pm i\phi) \end{aligned}$$

$$\int_0^\infty r^k \exp(-\alpha r)\,\mathrm{d}r = k!/\alpha^{k+1}.$$

Problem 2: As a mechanism for downward transitions, spontaneous emission competes with thermally stimulated emission (i.e. by the blackbody radiation). Show that at room temperature, ($T = 300$K) thermal stimulation dominates for frequencies well below 5×10^{12}Hz, whereas spontaneous emission dominates for frequencies well above 5×10^{12}Hz. Which mechanism dominates for visible light?

The spontaneous emission rate is:

$$A = \frac{\omega^3 |\mathcal{P}|^2}{3\pi\,\epsilon_0 \hbar c^3}$$

Rate for emission stimulated by thermal (blackbody) radiation:

$$R = \frac{\pi}{3\epsilon_0 \hbar^2} |\mathcal{P}|^2 \rho(\omega), \quad \rho(\omega) = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\hbar\omega/k_B T} - 1}$$

Problem 3 Calculate the rate for spontaneous emission

$$A = \frac{\omega_0^3 |\mathcal{P}|^2}{3\,\pi\,\epsilon_0\,\hbar\,c^3}$$

and the lifetime, $\tau = 1/A$, for each of the four $n = 2$ states of hydrogen.

$\mathcal{P}$ is the matrix element of the dipole moment $q\boldsymbol{r}$ in the initial and final states $\mathcal{P} = \langle \psi_{\text{in}} | q\boldsymbol{r} | \psi_{\text{fi}} \rangle$. You will need to evaluate matrix elements of the form $\langle \psi_{100} | x | \psi_{200} \rangle$ and so on. Remember:

$$x = r \ \sin\theta \ \cos\phi, \ \ y = r \ \sin\theta \ \sin\phi, \ \ z = r \ \cos\theta.$$

The ground state energy of the H atom and the Bohr radius a are:

$$E_1 = -\frac{\hbar^2}{2ma^2}, \quad a = \frac{4\pi\epsilon_0\hbar^2}{me^2}$$

Problem 4 The Hamiltonian for a particle with charge q, mass m in a vector potential $\boldsymbol{A}$ is:

$$H = \frac{1}{2m}\left[\boldsymbol{p} - q\,\boldsymbol{A}(\boldsymbol{r})\right]^2. \tag{1}$$

In general, the commutator $[\boldsymbol{p}, \boldsymbol{A}(\boldsymbol{r})]$ does not vanish.
For vector operators, the commutator is defined: $[\boldsymbol{p}, \boldsymbol{A}(\boldsymbol{r})] = \boldsymbol{p} \cdot \boldsymbol{A}(\boldsymbol{r}) - \boldsymbol{A}(\boldsymbol{r}) \cdot \boldsymbol{p}$.
(a) Obtain the commutator $\left[\boldsymbol{p}, \boldsymbol{A}(\boldsymbol{r})\right]$.
(b) Expand the Hamiltonian (1). Explain why it is convenient to choose the gauge of $\boldsymbol{A}$ to satisfy $\boldsymbol{\nabla} \cdot \boldsymbol{A}(\boldsymbol{r}) = 0$.

[Hint: Use a test function ϕ to obtain the commutator and when you expand the Hamiltonian.]