Introduction to parton showers, matching and merging

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Before we begin

Parton showers are an **active field of research**, though we have the experience of over four decades of development.

Many issues are currently actively debated and developed. In many cases, there is no final answer yet.

I am an author of the **SHERPA** Monte-Carlo event generator. Although I endevour to be agnostic, this will invariably influence my point of view and choice of examples to some extent.

Many thanks to S. Höche for letting me steal many plots/sketches/illustrations from his lectures in the MCnet School '21.

What to expect

- A basic understanding of what a parton shower is, its features and its limitations.
- The underlying concepts of matching and merging, used in most theory predictions for collider experiments today.
- The background that allows you to follow the discussions in the past, present, and (hopefully) future parton shower literature.

What not to expect

- All the latest and greatest plots, as well as a survey of all possible algorithms. This could fill the entire time of the school.

Literature

1. R. K. Ellis, W. J. Stirling, B. R. Webber
   QCD and Collider Physics
   Cambridge University Press, 2003
2. R. D. Field
   Applications of Perturbative QCD
   Addison-Wesley, 1995
3. M. E. Peskin, D. V. Schroeder
   An Introduction to Quantum Field Theory
   Westview Press, 1995
4. T. Sjöstrand, S. Mrenna, P. Z. Skands
   PYTHIA 6.4 Physics and Manual
   JHEP 05 (2006) 026
5. S. Höche,
   Introduction to parton-shower event generators
   TASI lectures, 2014

Overview of lectures

1) Introduction to parton showers
   - approximate higher-order corrections
   - building a parton shower

2) Improving parton showers
   - assessing the properties of a parton shower
   - NLL accuracy and beyond

3) Matching and merging
   - matching
   - merging

Introduction to parton showers

Approximate higher-order corrections

Leading order cross section

- hadron collider cross section for production of system Y (think $Y = \ell^+\ell^-$, $t\bar{t}$, W^+W^-, dijets, ...)

$$\mathrm{d}\sigma_{pp\to Y+X} = \sum_{a,b\in\{q,g\}} \mathrm{d}x_a \mathrm{d}x_b \; f_a(x_a,\mu_F^2) f_b(x_b,\mu_F^2)\; \mathrm{d}\Phi_n \, \frac{\mathrm{d}\hat{\sigma}_{ab\to Y+X}(\Phi,\mu_F^2)}{\mathrm{d}\Phi_n}$$

- PDFs $f_i(x_i,\mu_F^2)$, n-particle phase space element $\mathrm{d}\Phi_n$
- partonic cross section at LO

$$\mathrm{d}\hat{\sigma}_{ab\to Y+X} \propto \left|\mathcal{M}^{\text{tree}}_{ab\to Y}\right|^2$$

Note: every cross-section is inclusive in some additional particles. The leading order cross section does not contain them explicitly. Higher-order corrections must allow additional radiation.

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Approximate NLO corrections

- partonic cross section at NLO

$$\mathrm{d}\hat{\sigma}_{ab\to n+X} \propto \underbrace{\left|\mathcal{M}^{\text{tree}}_{ab\to n}\right|^2}_{\text{Born}} + \underbrace{2\mathcal{R}e\left\{\mathcal{M}^{\text{loop}}_{ab\to n}\mathcal{M}^{\text{tree}\,*}_{ab\to n}\right\}}_{\text{virtual corr.}} + \underbrace{\left|\mathcal{M}^{\text{tree}}_{ab\to n+1}\right|^2}_{\text{real corr.}}$$

  real and virtual correction separately diverging
  (infrared singularities caused by soft or collinear parton emission)
  sum is finite due to Kinoshita-Lee-Nauenberg (KLN) theorem

- infrared limit is universal, depends only on external states, construct

$$\mathrm{d}\hat{\sigma}^{\text{approx}}_{n+1} = \mathrm{d}\hat{\sigma}_n \otimes \sum_{i,k} \mathrm{d}V_{ik}$$

  some splitting function V_{ik}, $ik \to ijk$

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Approximate NLO corrections

Collinear approximation

- collinear splitting function $F_{ab}(z,\phi)$, $a \to bj$,
  emission phase space parametrised thorugh (t,z,ϕ)

$$\mathrm{d}V_{ak} \to \frac{\mathrm{d}t}{t}\,\mathrm{d}z\,\frac{\mathrm{d}\phi}{2\pi}\frac{\alpha_s}{2\pi}\,F_{ab}(z,\phi) \xrightarrow{\phi\text{ av.}} \frac{\mathrm{d}t}{t}\,\mathrm{d}z\,\frac{\alpha_s}{2\pi}\,P_{ab}(z)$$

- azimuthal average: $F_{ab}(z,\phi) \to P_{ab}(z)$
  Altarelli-Parisi splitting functions
- azimuthally averaged collinear limit of $n+1$ matrix element
- dropped spin-correlations in splitting,
  $\to \mathrm{d}V_{ak}$ is purely multiplicative factor

Approximate NLO corrections

Soft approximation

- limit of soft gluon emission

$$\mathrm{d}V_{ik} \to \omega \mathrm{d}\omega \, \frac{\mathrm{d}\Omega}{2\pi} \, \frac{\alpha_s}{2\pi} \, C_{ik} \, \frac{p_i \cdot p_k}{p_i \cdot q \; p_k \cdot q}$$

- kinematics decsribed by Eikonal
- colour factor in general matrix valued, but

$$C_{ik} = -\mathbf{T}_i \mathbf{T}_k \xrightarrow{\text{large-}N_c} \left\{ \begin{array}{ll} \mathbf{T}_i^2 + \mathcal{O}(1/N_c^2) & \text{for } i = q \\ \frac{1}{2}\mathbf{T}_i^2 + \mathcal{O}(1/N_c^2) & \text{for } i = g \end{array} \right\} \equiv C_i$$

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Approximate NLO corrections

- partial-fractioning the Eikonal

$$\frac{p_i\cdot p_k}{p_i\cdot q\, p_k\cdot q} \to \frac{1}{p_i\cdot q}\frac{p_i\cdot p_k}{(p_i+p_k)q} + \frac{1}{p_k\cdot q}\frac{p_i\cdot p_k}{(p_i+p_k)q}$$

  The first term contains the soft singularity associated with the region collinear to p_i, while the second that collinear to p_k.

- with this, we get

$$\mathrm{d}V_{ik} \to \mathrm{d}V_i = \omega\mathrm{d}\omega\,\frac{\mathrm{d}\Omega}{2\pi}\,\frac{\alpha_s}{2\pi}\,C_i\,W^i_{ijk}$$

  a real-number-valued multiplicative factor of the soft gluon-emission correction in the large-N_c limit

- combine with coll. limit to soft-collinear (dipole) splitting functions

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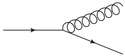
Higher-order corrections and parton branchings

The heuristic view

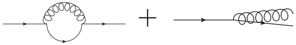
Radiative corrections as a branching process

- parton branchings are IR divergent, introduce a resolution parameter to regulate the branching process t_{res}



- resolvable, $t > t_c$, finite

- include



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- **Assumption:** corrections from resolvable and unresolvable branchings add up to zero, true for divergent leading logarithms (KLN theorem), amounts to saying that integrated higher-order corrections vanish

$\Rightarrow$ parton branchings can be interpreted probabilistically, either a parton branches resolvably with a probability given by the resolvable branching process or it does not

$\rightarrow$ same as nuclear decay

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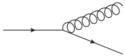

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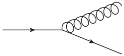

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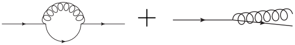

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Radiative corrections as a branching process

- The consequence is Poisson statistics
  - let the branching probability be λ
  - assume indistinguishable particles $\to$ naïve probability for n emissions

$$P_{\text{naïve}}(n,\lambda) = \frac{\lambda^n}{n!}$$

  - probability conservation (unitarity) implies a no-emission probability

$$P(n,\lambda) = \frac{\lambda^n}{n!}\exp\{-\lambda\} \qquad \longrightarrow \qquad \sum_{n=0}^{\infty} P(n,\lambda) = 1$$

- introduce Sudakov form factor $\Delta = \exp\{-\lambda\}$

Radiative corrections as a branching process

- branching probability for parton state at scale Q^2 in collinear limit in terms of resolution variable t

$$\lambda \to \int_t^{Q^2} \mathrm{d}\bar{t}\, \frac{\mathrm{d}}{\mathrm{d}\bar{t}} \left[\frac{\sigma_{n+1}(\bar{t})}{\sigma_n} \right] \approx \sum_{\text{jets}} \int_t^{Q^2} \mathrm{d}\bar{t} \int \mathrm{d}z\; \frac{\alpha_s}{2\pi \bar{t}} P(z)$$

- Altarelli-Parisi splitting functions $P(z)$, spin- and colour dependent

$$P_{qq}(z) = C_F \left[\frac{2z}{1-z} + (1-z) \right] \qquad P_{gq}(z) = T_R \left[z^2 + (1-z)^2 \right]$$

$$P_{gg}(z) = C_A \left[\frac{2z}{1-z} + z(1-z) \right] + (z \leftrightarrow 1-z)$$

- branching process conserves momentum, colour, and on-shellness

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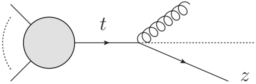

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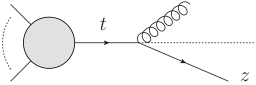

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The improved large-N_c approximation

Colour flow

- quark propagator in fundamental representation δ_{ij} contains $N_c = 3$ colour states
- gluon propagator in adjoint representation δ^{ab} contains $N_c^2 - 1 = 8$ colour states

using completeness relations

$$\underbrace{\delta^{ab}}_{\text{standard}} = 2\,\mathrm{Tr}(T^a T^b) = 2\,T^a_{ij} T^b_{ji} = T^a_{ij}\underbrace{2\,\delta_{ik}\delta_{jl}}_{\text{colour flow}}T^b_{lk}$$

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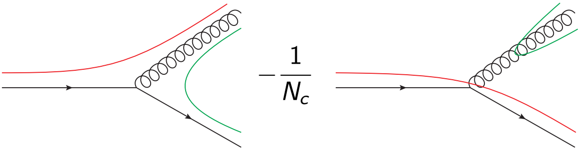
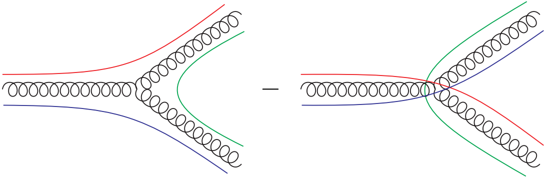
Colour flow

- Quark-gluon vertex $T^a_{ij} T^a_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{jk} - \frac{1}{N_c}\delta_{ij}\delta_{kl}\right)$



- Gluon-gluon vertex $f^{abc}\, T^a_{ij} T^b_{kl} T^c_{mn} = \delta_{il}\delta_{kn}\delta_{mj} - \delta_{in}\delta_{ml}\delta_{kj}$



The improved large-N_c approximation

- leading colour approximation

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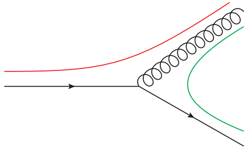
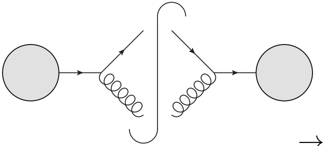


- this overestimates the colour charge of the quark:
  Consider process $q \to qg$ attached to some larger diagram $|\mathcal{M}|^2$



$$\propto \quad T^a_{ij}\,T^a_{jk} = C_F\,\delta_{ik}\ (\text{QCD},\ N_c = 3)$$

$$\to \tfrac{1}{2}\,\delta_{il}\delta_{jm}\delta_{mj}\delta_{lk} = \frac{C_A}{2}\,\delta_{ik}\ (N_c \to \infty)$$

- improved large-N_c approx.: keep colour charge of quarks at C_F

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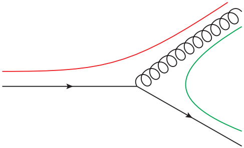

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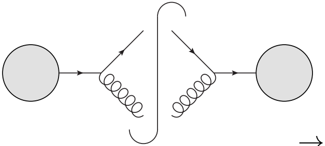

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Monte-Carlo methods for parton showers
–
The veto algorithm

Monte-Carlo methods: Poisson distributions

- assume branching process described by $g(t)$
- branching can happen only if it has not happened already, must account for survival probability $\leftrightarrow$ Poisson distribution

$$\mathcal{G}(t) = g(t)\Delta(t,t_0) \qquad \text{where} \qquad \Delta(t,t_0) = \exp\left\{-\int_t^{t_0} \mathrm{d}t'\, g(t')\right\}$$

- if $G(t)$ is known, then we also know the integral of $\mathcal{G}(t)$

$$\int_t^{t_0} \mathrm{d}t' \mathcal{G}(t') = \int_t^{t_0} \mathrm{d}t' \, \frac{\mathrm{d}\Delta(t',t_0)}{\mathrm{d}t'} = 1 - \Delta(t,t_0)$$

- can generate events by requiring $1 - \Delta(t,t_0) = 1 - R$ ($R \in [0,1]$)

$$t = G^{-1}\Big[G(t_0) + \log R\Big]$$

Veto algorithm – importance sampling for Poisson dists

Parton shower branching probability $f(t) \propto \frac{\alpha_s(t)}{t}\,P(z)$
Problem: we do not know $F(t)$

Solution: veto algorithm

1. find overestimate $g(t) \geq f(t)\;\forall t \in [t_c, t_0]$, generate event according to

$$\mathcal{G}(t) = g(t)\,\exp\left\{-\int_t^{t_0} \mathrm{d}t'\,g(t')\right\}$$

2. accept with $w(t) = f(t)/g(t)$
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Veto algorithm – importance sampling for Poisson dists

Does this give the correct distribution?

- probability for immediate acceptance of emission at scale t

$$\frac{f(t)}{g(t)}\,g(t)\exp\left\{-\int_t^{t_0}\mathrm{d}t'\,g(t')\right\}$$

- probability for acceptance after one rejection

$$\frac{f(t)}{g(t)}\,g(t)\int_t^{t_0}\mathrm{d}t_1\exp\left\{-\int_t^{t_1}\mathrm{d}t'\,g(t')\right\}\left(1-\frac{f(t_1)}{g(t_1)}\right)g(t_1)\exp\left\{-\int_{t_1}^{t_0}\mathrm{d}t'\,g(t')\right\}$$

- For n rejections we obtain n nested integrals $\int_t^{t_0}\int_{t_1}^{t_0}\dots\int_{t_{n-1}}^{t_0}$
- disentangling yields $1/n!$, summing over all possible rejections gives

$$f(t)\exp\left\{-\int_t^{t_0}\mathrm{d}t'\,g(t')\right\}\sum_{n=0}^{\infty}\frac{1}{n!}\left[\int_t^{t_0}\mathrm{d}t'\,\left[g(t')-f(t')\right]\right]^n$$

$$=f(t)\exp\left\{-\int_t^{t_0}\mathrm{d}t'\,f(t')\right\}$$

Veto algorithm – importance sampling for Poisson dists

Does this give the correct distribution?

- probability for immediate acceptance of emission at scale t

$$\frac{f(t)}{g(t)}\, g(t) \exp\left\{-\int_t^{t_0} \mathrm{d}t'\, g(t')\right\}$$

- probability for acceptance after one rejection

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Veto algorithm

What have we achieved?

- we have generated a parton branching (real resolved emission) according to

$$f(t)\,\Delta(t,t_0) \qquad \text{with} \qquad \Delta(t,t_0) = \exp\left\{-\int_t^{t_0} \mathrm{d}t'\, f(t')\right\}$$

$$f(t) \equiv f(t,z) = \frac{\alpha_s}{2\pi\, t}\, P(z)$$

- the no-branching probability implies a virtual correction (including unresolved real emissions) of $\Delta(t_c,t_0)$

  Note: The Sudakov form factor Δ resums logs to all orders

$$\mathrm{d}\hat{\sigma}^{\text{approx}}_{\text{NLO}} = \mathrm{d}\hat{\sigma}_n \left[\Delta(t_c,t_0) + \int_{t_c}^{t_0} \mathrm{d}t\, f(t)\,\Delta(t,t_0)\right]$$

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Veto algorithm – iteration

1) consider a set of n partons at scale t_0, which evolve collectively
Sudakovs factorise, schematically

$$\Delta(t,t_0)=\prod_{i=1}^{n}\Delta_i(t,t_0)\,,\qquad \Delta_i(t,t_0)=\prod_{j=q,g}\Delta_{i\to j}(t,t_0)$$

2) find new scale t where next branching occurs using veto algorithm
  - generate t using overestimate $g(t)=\sum g_{ab}(t)$
  - select on splitting function according to their contribution to the sum
  - accept splitting with weight $w_{ab}=f_{ab}/g_{ab}$

3) construct splitting kinematics and update event record

4) continue until $t<t_c$, t_c infrared cut-off

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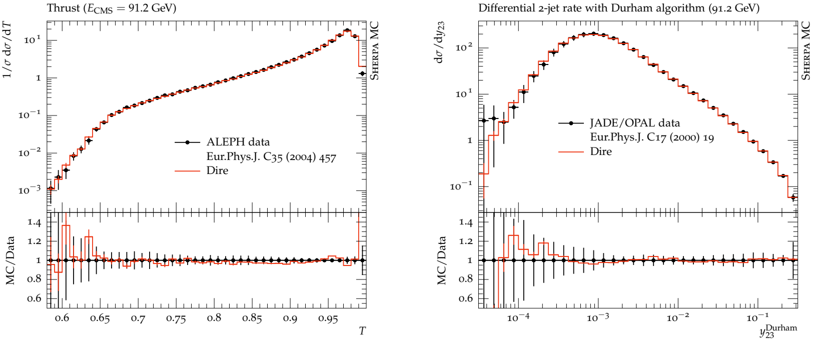
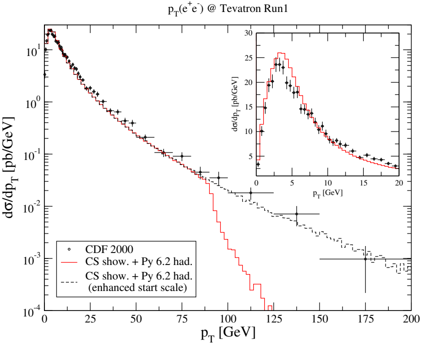


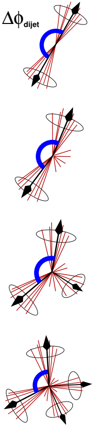
- Thrust and Durham $2 \to 3$-jet rate in $e^+e^- \to$ hadrons
- hadronisation region to the right (left) in left (right) plot

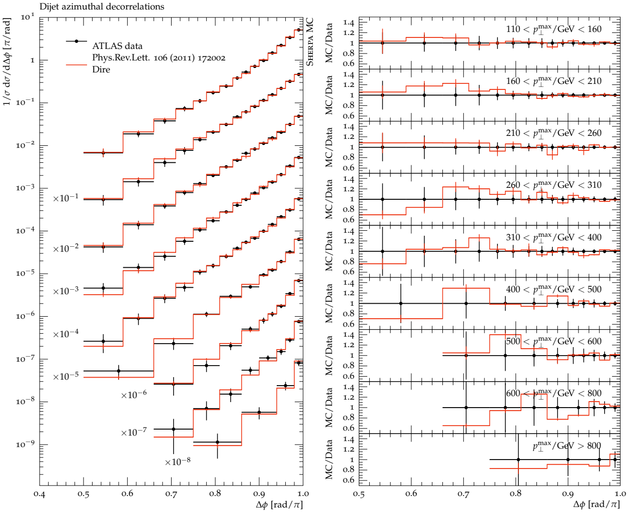
Effects of the parton shower



- Drell-Yan lepton pair production at Tevatron
- if hard cross section computed at leading order, then parton shower is only source of transverse momentum
- starting scale of evolution chosen as $Q^2 = m_W^2$

Effects of the parton shower





Recap

This lecture:

- parton showers encode approximate higher-order corrections
  $\rightarrow$ build upon universal soft-collinear approximation
  (Altarelli-Parisi splitting functions, large-N_c, spin-averaged)
- implemented as a statistical branching process, ordered in evolution variable t (k_T^2, $\tilde{q}^2$, etc.)
- produce resolved final state up to scale $t_\mathrm{res} \approx \Lambda_\mathrm{QCD}$
  $\rightarrow$ further evolution needs hadrons as degrees of freedom

Next lectures:

- limitations of parton showers and how to overcome them