Theoretical Physics II B – Quantum Mechanics

Lecture 18

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Solutions to previous control questions

17.1 To answer the first questions, realise that the state vector can be rewritten as

$$\begin{aligned}|\psi\rangle = |\uparrow\rangle^{(-)}|\uparrow\rangle^{(+)} &= |m_1 m_2; j_1 j_2\rangle = \left|\frac{1}{2}\frac{1}{2}; \frac{1}{2}\frac{1}{2}\right\rangle \\ &= |jm; j_1 j_2\rangle = \left|11; \frac{1}{2}\frac{1}{2}\right\rangle .\end{aligned}$$

(ab) For the case $A \to 0$, the Hamiltonian can be rewritten by the spin operator $\hat{S}_z$ of the combined state:

$$\hat{H} = \frac{eB}{mc}\left(\hat{S}_z^{(-)} + \hat{S}_z^{(+)}\right) = \frac{eB}{mc}\left(\hat{S}_{1,z} + \hat{S}_{2,z}\right) = \frac{eB}{mc}\hat{S}_z ,$$

and $|\psi\rangle$ obviously is an eigenket of the Hamiltonian, with eigenvalue (energy) $E^{(B)}_{|11\rangle} = \frac{eB\hbar}{mc}$.

In contrast, for the case $B \to 0$, the Hamiltonian is given by

$$\hat{H} = A\underline{\hat{S}}_1 \cdot \underline{\hat{S}}_2 = \frac{A}{2}\left(\underline{\hat{S}}^2 - \underline{\hat{S}}_1^2 - \underline{\hat{S}}_2^2\right) .$$

$|\psi\rangle$ is also an eigenstate of this operator, with eigenvalue (i.e. energy)

$$E^{(A)}_{|11\rangle} = \frac{A\hbar^2}{2}\left(2 - \frac{3}{4} - \frac{3}{4}\right) = \frac{A\hbar^2}{4} .$$

Therefore the overall energy expectation value is just

$$E^{(A)+(B)}_{|11\rangle} = \frac{A\hbar^2}{4} + \frac{eB\hbar}{mc} .$$

(c) Following the reasoning up to now, the expectation values read

$$\left\langle \underline{\hat{S}}_1 \cdot \underline{\hat{S}}_2 \right\rangle_S = -\frac{3\hbar^2}{4} \quad \text{and} \quad \left\langle \underline{\hat{S}}_1 \cdot \underline{\hat{S}}_2 \right\rangle_T = \frac{\hbar^2}{4} .$$

17.2 (a)

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Addition of angular momenta

- Assume two angular momentum operators $\underline{\hat{J}}_1$ and $\underline{\hat{J}}_2$ in two different subspaces, which satisfy the usual commutation relations

$$\left[\hat{J}_{1i},\,\hat{J}_{1j}\right] = i\hbar\epsilon_{ijk}\hat{J}_{1k}\;,\quad \left[\hat{J}_{2i},\,\hat{J}_{2j}\right] = i\hbar\epsilon_{ijk}\hat{J}_{2k}\;,\quad \text{and}\quad \left[\hat{J}_{1i},\,\hat{J}_{2j}\right] = 0\,.$$

- The total angular momentum is given by

$$\underline{\hat{J}} = \underline{\hat{J}}_1 \otimes \hat{\mathbf{1}}_2 + \hat{\mathbf{1}}_1 \otimes \underline{\hat{J}}_2 = \underline{\hat{J}}_1 + \underline{\hat{J}}_2\,.$$

- Rotations $\mathcal{R}(\underline{n},\,\phi)$ of the compound are described by

$$\hat{R}(\underline{n},\,\phi) = \exp\left(-\frac{i\underline{\hat{J}}_1\cdot\underline{n}\phi}{\hbar}\right)\otimes\exp\left(-\frac{i\underline{\hat{J}}_2\cdot\underline{n}\phi}{\hbar}\right)\,,$$

where it is important to use identical axes and angles in both subspaces for a uniquely and sensibly defined rotation.

Two bases

- There are essentially two options to choose base kets:

  A: Eigenkets of $\underline{\hat{J}}_1^2$, $\underline{\hat{J}}_2^2$, $\hat{J}_{1z}$, and $\hat{J}_{2z}$: $|j_1 j_2;\ m_1 m_2\rangle$:

$$\begin{aligned} \underline{\hat{J}}_i^2 \, |j_1 j_2;\ m_1 m_2\rangle &= j_i(j_i+1)\hbar^2 \, |j_1 j_2;\ m_1 m_2\rangle \\ \hat{J}_{iz} \, |j_1 j_2;\ m_1 m_2\rangle &= m_i \hbar \, |j_1 j_2;\ m_1 m_2\rangle \ . \end{aligned}$$

  B: Eigenkets of $\underline{\hat{J}}^2$, $\underline{\hat{J}}_1^2$, $\underline{\hat{J}}_2^2$, and $\hat{J}_z$: $|jm;\ j_1 j_2\rangle$. Noting that because off

$$\underline{\hat{J}}^2 = \underline{\hat{J}}_1^2 + \underline{\hat{J}}_2^2 + 2\hat{J}_{1z}\hat{J}_{2z} + \hat{J}_{1+}\hat{J}_{2-} + \hat{J}_{1-}\hat{J}_{2+} \quad \longrightarrow \quad \left[\underline{\hat{J}}^2,\, \underline{\hat{J}}_i^2\right] = 0$$

$$\begin{aligned} \underline{\hat{J}}^2 \, |jm;\ j_1 j_2\rangle &= j(j+1)\hbar^2 \, |j_1 j_2;\ j_1 j_2\rangle \\ \underline{\hat{J}}_i^2 \, |jm;\ j_1 j_2\rangle &= j_i(j_i+1)\hbar^2 \, |j_1 j_2;\ j_1 j_2\rangle \\ \hat{J}_z \, |jm;\ j_1 j_2\rangle &= m\hbar \, |jm;\ j_1 j_2\rangle \ . \end{aligned}$$

Base transformation

- When going from one base to the other, it is important to realise that although

$$\left[\hat{J}_i,\, \hat{J}_j\right] = i\hbar\epsilon_{ijk}\hat{J}_k \quad \text{and} \quad \left[\underline{\hat{J}}^2,\, \hat{J}_z\right] = 0$$

we still have

$$\left[\underline{\hat{J}}^2,\, \hat{J}_{1z}\right] \neq 0 \quad \text{and} \quad \left[\underline{\hat{J}}^2,\, \hat{J}_{2z}\right] \neq 0\,,$$

which implies that we can neither add $\underline{\hat{J}}^2$ to set A nor the $\hat{J}_{iz}$ to set B. This means that these two sets indeed have the maximal number of independent quantum numbers – eigenvalues of mutually compatible operators – and thus serve as viable bases.

Defining Clebsch-Gordan coefficients

- When going from one base to another, we can write the identity

$$|jm; j_1 j_2\rangle = \sum_{m_1 m_2} |m_1 m_2; j_1 j_2\rangle \langle m_1 m_2; j_1 j_2 | jm; j_1 j_2\rangle$$

where we used to completeness of the states $|m_1 m_2; j_1 j_2\rangle$:

$$\hat{\mathbf{1}} = \sum_{m_1 m_2} |m_1 m_2; j_1 j_2\rangle \langle m_1 m_2; j_1 j_2|$$

- The matrix elements $\langle m_1 m_2; j_1 j_2 | jm; j_1 j_2\rangle$ are the **Clebsch-Gordan coefficients**.

Properties of Clebsch-Gordan coefficients

1. CG coefficients vanish if $m \neq m_1 + m_2$:
   Since $\hat{J}_z = \hat{J}_{1z} + \hat{J}_{2z}$, $\left(\hat{J}_z - \hat{J}_{1z} - \hat{J}_{2z}\right)|jm; j_1 j_2\rangle = 0$. Multiplying from the left with $\langle m_1 m_2; j_1 j_2|$ yields the desired result:

$$\begin{aligned}&\left\langle m_1 m_2; j_1 j_2 \middle| \left(\hat{J}_z - \hat{J}_{1z} - \hat{J}_{2z}\right) \middle| jm; j_1 j_2 \right\rangle \\ = \quad &(m - m_1 - m_2) \left\langle m_1 m_2; j_1 j_2 | jm; j_1 j_2 \right\rangle = 0\,.\end{aligned}$$

Properties of Clebsch-Gordan coefficients

2. CG coefficients vanish unless $|j_1 - j_2| \leq j \leq j_1 + j_2$:
   You can check the plausibility of this by comparing the dimensionalities of the spaces spanned in different notations, which must be the same.
   In $m_1 m_2$-notation $N = (2j_1 + 1)(2j_2 + 1)$. Compare this with the jm notation; assuming without any loss of generality $j_1 \geq j_2$, we find

$$\begin{aligned}\sum_{j=j_1-j_2}^{j_1+j_2}(2j+1) &= \frac{1}{2}\left\{[2(j_1-j_2)+1]+[2(j_1+j_2)+1]\right\}(2j_2+1)\\ &= (2j_1+1)(2j_2+1) = N\,.\end{aligned}$$

Properties of Clebsch-Gordan coefficients

3. CG coefficients form a unitary matrix, choosing the matrix elements to be real by convention, means that the matrices in fact are orthogonal. Therefore

$$\begin{aligned}
\sum_{jm} \langle m_1 m_2; j_1 j_2 | jm; j_1 j_2\rangle \langle j_1 j_2; m_1' m_2' | jm; j_1 j_2\rangle &= \delta_{m_1 m_1'} \delta_{m_2 m_2'} \\
\sum_{m_1 m_2} \langle m_1 m_2; j_1 j_2 | jm; j_1 j_2\rangle \langle m_1 m_2; j_1 j_2 | j' m'; j_1 j_2\rangle &= \delta_{jj'} \delta_{mm'} \,.
\end{aligned}$$

4. Setting $j' = j$ and $m = m_1 + m_2 = m'$, we find

$$\sum_{m_1 m_2} |\langle m_1 m_2; j_1 j_2 | j(m_1 + m_2); j_1 j_2\rangle|^2 = 1 \,,$$

providing the normalisation condition for the $|jm; j_1 j_2\rangle$

Recursion relations for the Clebsch-Gordans

(Not examinable.)

- Consider

$$\begin{aligned}\hat{J}_\pm\,|jm;j_1j_2\rangle &= \sqrt{(j\mp m)(j\pm m+1)}\;|j(m\pm 1);j_1j_2\rangle\\ &= \sum_{m_1'm_2'}\left[\left(\hat{J}_{1\pm}+\hat{J}_{2\pm}\right)|j_1j_2;m_1'm_2'\rangle\,\langle j_1j_2;m_1'm_2'|jm;j_1j_2\rangle\right]\end{aligned}$$

  and multiply it from the left with $\langle m_1m_2;j_1j_2|$. This means that the right hand side vanishes unless $m_1' = m_1\pm 1$ and $m_2' = m_2$ or $m_1' = m_1$ and $m_2' = m_2\pm 1$.
- This shifting of the m-values by the ladder operators will result in a new condition on the *emerging* CG coefficients:
  For them, $m\pm 1 = m_1+m_2$ is neccessary to guarantee their existence.

- This leads to the recursion relation

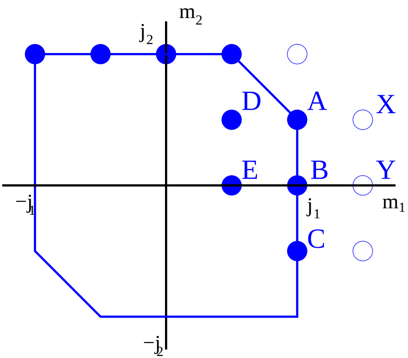
$$
\begin{aligned}
&\langle j1j2; m_1 m_2 | j(m\pm 1); j_1 j_2\rangle \\
&= \sqrt{\frac{(j_1\pm m_1)(j_1\mp m_1+1)}{(j\mp m)(j\pm m+1)}}\;\langle j1j2; (m_1\mp 1)m_2 | jm; j_1 j_2\rangle \\
&\quad + \sqrt{\frac{(j_2\pm m_2)(j_2\mp m_2+1)}{(j\mp m)(j\pm m+1)}}\;\langle j1j2; m_1(m_2\mp 1) | jm; j_1 j_2\rangle \;.
\end{aligned}
$$

- To use them, realise that there are a number of “boundary” conditions applying to the CGs in the $m_1 m_2$ plane and hence to the CGs enetering the recursion relations, namely

$$|m_1| \le j_1\;,\quad |m_2| \le j_2\;,\quad \text{and}\quad -j \le |m_1+m_2| \le j\,.$$

This allows to choose values where either the first or second term on the r.h.s. vanish and navigate accordingly.

- Start at point A, with $m_1 = j_1$.
- Use the $\hat{J}_-$ relation, yielding one term X with $m_1 \to m_1 + 1$, which is forbidden, and one term B with $m_2 \to m_2 - 1$, which is allowed.
- Therefore the CG for B can be obtained from A alone.
- Now form the $\hat{J}_+$ triangle corresponding to A, B, and D, which results in the CG for D.
- This strategy can be repeaeted, with triangles built by the ladder operators, and using the boundaries.



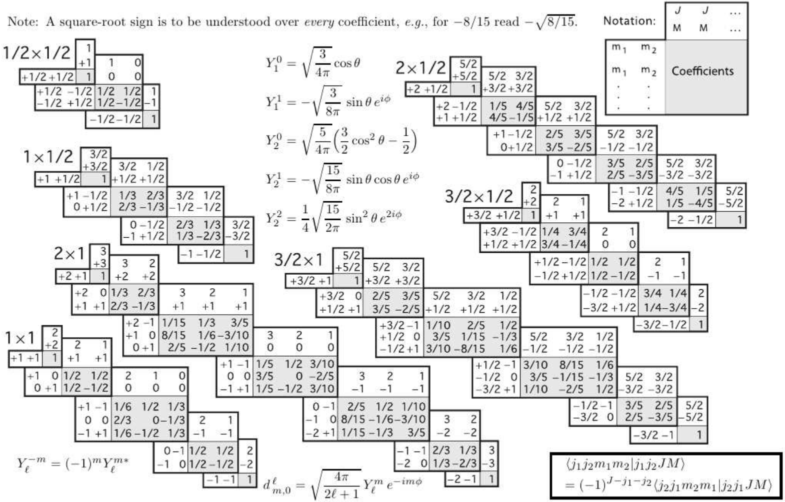
Concrete example: spin-$\frac{1}{2}$ particle

- Consider a spin-$\frac{1}{2}$ particle in a situation where orbital angular momentum matters.
- Then $j_1 = l$ and $m_1 = m_l$ are integer numbers and $j_2 = s = \frac{1}{2}$ and $m_2 = \pm\frac{1}{2}$. The allowed values of j are given by $j = l \pm \frac{1}{2} > 0$.
- Using the CG recursion relations allows to define two-component spin-angular functions:

$$\mathcal{Y}_l^{j=l\pm\frac{1}{2};\,m}(\theta,\,\phi) = \frac{1}{\sqrt{2l+1}} \left(\begin{array}{c} \pm\sqrt{l \pm m + \frac{1}{2}}\ Y_{l(m-\frac{1}{2})}(\theta\,,\phi) \\ \sqrt{l \mp m + \frac{1}{2}}\ Y_{l(m+\frac{1}{2})}(\theta\,,\phi) \end{array} \right)$$

  Note that, of course, $m = m_l + m_s$ takes an half-integer value.
- The functions above, are by construction, eigenfunctions of $\underline{\hat{J}}^2$, $\underline{\hat{L}}^2$, $\underline{\hat{S}}^2$ and $\hat{J}_z$. They are also eigenfunctions of the spin-orbital term $\underline{\hat{L}}\cdot\underline{\hat{S}} = \frac{1}{2}(\underline{\hat{J}}^2 - \underline{\hat{L}}^2 - \underline{\hat{S}}^2)$.

Explicit coefficients from the PDG



Learning outcomes

- Clebsch-Gordan coefficients and their properties.
- Formal theory of spin addition.