# Theoretical Physics II B – Quantum Mechanics

## Lecture 13

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1. Eigenvalues and eigenstates of angular momentum

# Solutions to previous control questions

12.1 (a) Use the definition of the Levi-Civita tensor to obtain

$$\epsilon_{ijk} = \begin{cases} 1 & : \{ijk\} = \{123,\, 231,\, 312\} \\ -1 & : \{ijk\} = \{132,\, 213,\, 321\} \\ 0 & : \text{else.} \end{cases}$$

and therefore

$$G_1 = i\hbar \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad G_2 = i\hbar \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad G_3 = i\hbar \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Note that they can be identified with $\hat{G}_{1,2,3} = \hat{R}_{x,y,z}(i\hbar)$.

# Solutions to previous control questions

(b) Obviously, the commutators could be calculated directly or by just looking up the commutator relations from lecture 10. Here, we do something else instead; using

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

we find, e.g.

$$\begin{aligned}
(\hat{G})_{jk} &= \left[\hat{G}_1,\ \hat{G}_2\right] = -\hbar^2(\epsilon_{1jl}\epsilon_{2lk} - \epsilon_{2jl}\epsilon_{1lk}) \\
&= -\hbar^2(\delta_{1k}\delta_{j2} - \delta_{12}\delta_{jk} - \delta_{2k}\delta_{j1} + \delta_{21}\delta_{jk}) \\
&= \hbar^2\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = i\hbar\hat{G}_3\,,
\end{aligned}$$

and similarly for the other pairs, such that

$$\left[\hat{G}_i,\ \hat{G}_j\right] = i\hbar\epsilon_{ijk}\hat{G}_k\,.$$

# Solutions to previous control questions

12.2 Use $\hat{L}_i = \epsilon_{ijk}\hat{x}_j\hat{p}_k$ yields

$$\begin{aligned}\left[\hat{L}_1,\,\hat{L}_2\right] &= [\hat{x}_2\hat{p}_3 - \hat{p}_2\hat{x}_3,\,\hat{x}_3\hat{p}_1 - \hat{p}_3\hat{x}_1] \\ &= [\hat{x}_2\hat{p}_3,\,\hat{x}_3\hat{p}_1] - [\hat{p}_2\hat{x}_3,\,\hat{x}_3\hat{p}_1] - [\hat{x}_2\hat{p}_3,\,\hat{p}_3\hat{x}_1] + [\hat{p}_2\hat{x}_3,\,\hat{p}_3\hat{x}_1] \\ &= [\hat{x}_2\hat{p}_3,\,\hat{x}_3\hat{p}_1] + [\hat{p}_2\hat{x}_3,\,\hat{p}_3\hat{x}_1] = \hat{x}_2\,[\hat{p}_3,\,\hat{x}_3]\,\hat{p}_1 + \hat{p}_2\,[\hat{x}_3,\,\hat{p}_3]\,\hat{x}_1 \\ &= -i\hbar\,(\hat{x}_2\hat{p}_1 - \hat{p}_2\hat{x}_1) = i\hbar\,[\hat{x}_1,\,\hat{p}_2] = i\hbar\hat{L}_3\,.\end{aligned}$$

and similarly for the other commutators such that indeed

$$\left[\hat{L}_i,\,\hat{L}_j\right] = i\hbar\epsilon_{ijk}\hat{L}_k\,.$$

Note that on the way we used that commuting operators can be pushed outside the commutaotr bracket and that $[\hat{x}_i,\,\hat{p}_j] = i\hbar\delta_{ij}$.

# Reminder: properties of $\underline{\hat{J}}$

- Remember, from lecture 10, that we have identified a generalised form of angular momentum as the generator for rotations.
- In particular, introducing three operators $\hat{J}_{x,y,z}$ forming a vector $\underline{\hat{J}}$, rotations around an axis $\underline{n}$ with an infinitesimal angle $\mathrm{d}\phi$ have been written as

$$\hat{R}(\mathrm{d}\underline{\phi}) = 1 - \frac{i\underline{\hat{J}}\cdot\underline{n}}{\hbar}\mathrm{d}\phi\,.$$

- The three generators enjoy the commutation relation

$$\left[\hat{J}_i,\,\hat{J}_j\right] = i\hbar\epsilon_{ijk}\hat{J}_k\,.$$

# Invariant operator $\underline{\hat{J}}^2$

- Define an operator

$$\underline{\hat{J}}^2 = \hat{J}_x\hat{J}_x + \hat{J}_y\hat{J}_y + \hat{J}_z\hat{J}_z$$

which commutes with any of the three $\hat{J}_i$:

$$\left[\underline{\hat{J}}^2,\, \hat{J}_k\right] = 0 \;\; \forall\; k \in \{1,\, 2,\, 3\}\,.$$

- Because all $\hat{J}_k$ commute with $\underline{\hat{J}}^2$, but do not commute with each other, one can pick one to be diagonalised together with $\underline{\hat{J}}^2$. By *convention*, $\hat{J}_z$ is picked.

# Joint eigenvalues and eigenvectors of $\hat{\underline{J}}^2$ and $\hat{J}_z$

- We define eigenvalues $a$ and $b$ of $\hat{\underline{J}}^2$ and $\hat{J}_z$, respectively:

$$\begin{aligned} \hat{\underline{J}}^2\,|a,\,b\rangle &= a\,|a,\,b\rangle \\ \hat{J}_z\,|a,\,b\rangle &= b\,|a,\,b\rangle \end{aligned}$$

- Of course we denote the **joint** eigenkets with **both** eigenvalues, and we also note that they will be demanded to span a basis for all kets in rotation space.

# Ladder operators

- Introduce ladder operators (remember the harmonic oscillator!)

$$\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y$$

with commutation relations

$$\left[\hat{J}_+,\, \hat{J}_-\right] = 2\hbar\hat{J}_z\,, \quad \left[\hat{J}_z,\, \hat{J}_\pm\right] = \pm\hbar\hat{J}_\pm \quad \text{and} \quad \left[\hat{J}_\pm,\, \underline{\hat{J}}^2\right] = 0\,.$$

- Obviously,

$$\begin{aligned}
\hat{J}_z\left(\hat{J}_\pm\,|a,\,b\rangle\right) &= \left(\hat{J}_\pm\hat{J}_z + \left[\hat{J}_z\,,\hat{J}_\pm\right]\right)|a,\,b\rangle = (b \pm \hbar)\,\hat{J}_\pm\,|a,\,b\rangle \\
\underline{\hat{J}}^2\left(\hat{J}_\pm\,|a,\,b\rangle\right) &= \hat{J}_\pm\underline{\hat{J}}^2\,|a,\,b\rangle = a\hat{J}_\pm\,|a,\,b\rangle\ .
\end{aligned}$$

- So, applying the ladder operator to an $\hat{J}_z$-eigenket results in another $\hat{J}_z$-eigenket, but with corresponding $\hat{J}_z$-eigenvalue shifted by $\hbar$, while the $\hat{\underline{J}}^2$-eigenvalue remains the same:

$$\hat{J}_\pm \left|a,\, b\right\rangle = c_\pm \left|a,\, b \pm \hbar\right\rangle$$

- So for each set of eigenvalues and eigenkets with respect to $\hat{J}_z$ but to the same $\hat{\underline{J}}^2$-eigenvalue, this is very similar to the action of the ladder operators on the eigenkets of the number operator $\hat{N}$ for the harmonic oscillator!
- This means that there will only be a limited set of such different $\hat{J}_z$ states for each value $a$.

# Eigenvalues of $\hat{\underline{J}}^2$ and $\hat{J}_z$

- Explicit calculation shows that

$$\hat{\underline{J}}^2 - \hat{J}_z^2 = \frac{1}{2}\left(\hat{J}_+\hat{J}_- + \hat{J}_-\hat{J}_+\right) = \frac{1}{2}\left(\hat{J}_+\hat{J}_+^\dagger + \hat{J}_+^\dagger\hat{J}_+\right) .$$

- $\hat{J}_+\hat{J}_+^\dagger$ and $\hat{J}_+^\dagger\hat{J}_+$ must have non-negative expectation values, because

$$\langle a,\, b|\,\hat{J}_+^\dagger = \left(\hat{J}_+\,|a,\, b\rangle\right)^\dagger \quad \text{and} \quad \langle a,\, b|\,\hat{J}_+ = \left(\hat{J}_+^\dagger\,|a,\, b\rangle\right)^\dagger .$$

- As a consequence,

$$\left\langle a,\, b\left|\left(\hat{\underline{J}}^2 - \hat{J}_z^2\right)\right| a,\, b\right\rangle \geq 0\,,$$

implying that $a \geq b^2$, and therefore, there must be a maximal, $a$-dependent $b$, which will be deonted as $b_{\rm max}$.

- For this $b_{\max}$ we copy a trick from the harmonic oscillator, namely

$$\hat{J}_+ |a,\, b_{\max}\rangle = 0 \quad \text{and, of course} \quad \hat{J}_-\hat{J}_+ |a,\, b_{\max}\rangle = 0\,.$$

Re-expressing this through $\underline{\hat{J}}^2$ and $\hbar\hat{J}_z$ yields

$$\hat{J}_-\hat{J}_+ = \hat{J}_x^2 + \hat{J}_y^2 - i\left(\hat{J}_y\hat{J}_x - \hat{J}_x\hat{J}_y\right) = \underline{\hat{J}}^2 - \hat{J}_z^2 - \hbar\hat{J}_z\,,$$

implying that

$$\left(\underline{\hat{J}}^2 - \hat{J}_z^2 - \hbar\hat{J}_z\right)|a,\, b_{\max}\rangle = \left(a - b_{\max}^2 - \hbar b_{\max}\right)|a,\, b_{\max}\rangle = 0\,.$$

- Therefore

$$a = b_{\max}\left(b_{\max} + \hbar\right)$$

- Similarly, there must also be a $b_{\rm min}$ such that

$$\hat{J}_- \left|a,\, b_{\rm min}\right\rangle = 0 \quad \text{and, of course} \quad \hat{J}_+\hat{J}_- \left|a,\, b_{\rm min}\right\rangle = 0\,.$$

- Repeating the same steps as before results in

$$a = b_{\rm min}\left(b_{\rm min} - \hbar\right) \quad \text{and} \quad b_{\rm min} = -b_{\rm max}$$

- Therefore, for positive $b_{\rm max}$ we have a costraint for b:

$$b\in\left[-b_{\rm max},\,b_{\rm max}\right].$$

- Since the ladder operators add integer multiples to the $b$ value, there must be $n$ steps between $b_{\rm min}$ and $b_{\rm max}$,

$$b_{\rm max}=b_{\rm min}+n\hbar\quad\longleftrightarrow\quad b_{\rm max}=\frac{n\hbar}{2}\equiv j\hbar\,.$$

- This means that for $m\in\{-j,\,-j+1,\,-j+2,\,dots,\,j-1,\,j\}$:

$$a=j(j+1)\hbar^2\quad\text{and}\quad b=m\hbar$$

# Eigenkets $|j,\, m\rangle$ and their matrix elements

- Labelling the eigenkets with $j$ and $m$ yields

$$\begin{aligned} \underline{\hat{J}}^2 |j,\, m\rangle &= j(j+1)\hbar^2 |j,\, m\rangle \\ \hat{J}_z |j,\, m\rangle &= m\hbar |j,\, m\rangle \end{aligned}$$

  with $j$ being an integer or half-integer.

- Since these states form an orthonormal base,

$$\begin{aligned} \langle j',\, m'| \underline{\hat{J}}^2 |j,\, m\rangle &= j(j+1)\hbar^2 \delta_{jj'}\delta_{mm'} \\ \langle j',\, m'| \hat{J}_z |j,\, m\rangle &= m\hbar \delta_{jj'}\delta_{mm'} \end{aligned}$$

- Use the equations for the ladder operators

$$\hat{J}_\pm |j,\, m\rangle = c_{\pm,jm} |j,\, m+1\rangle$$

to obtain, for $c_{+,jm}$,

$$\begin{aligned} |c_{+,jm}|^2 \langle j,\, m|j,\, m\rangle &= \left\langle j,\, m \left| \hat{J}_+^\dagger \hat{J}_+ \right| j,\, m \right\rangle \\ &= \left\langle j,\, m \left| \left[ \underline{\hat{J}}^2 - \hat{J}_z^2 - \hbar \hat{J}_z \right] \right| j,\, m \right\rangle \\ &= \hbar^2 \left[ j(j+1) - m(m+1) \right] \langle j,\, m|j,\, m\rangle \end{aligned}$$

and therefore

$$|c_{+,jm}|^2 = \hbar^2 \left[ j(j+1) - m(m+1) \right] = \hbar^2 (j-m)(j+m+1)\,.$$

- Choosing $c_{+,jm}$ to be real and positive, we arrive at

$$\hat{J}_+ \left|j,\, m\right\rangle = \sqrt{(j-m)(j+m+1)}\,\hbar \left|j,\, m+1\right\rangle$$

and, similarly,

$$\hat{J}_- \left|j,\, m\right\rangle = \sqrt{(j+m)(j-m+1)}\,\hbar \left|j,\, m-1\right\rangle$$

leading to

$$\left\langle j',\, m' \left| \hat{J}_\pm \right| j,\, m \right\rangle = \sqrt{(j\mp m)(j\pm m+1)}\,\hbar\delta_{j'j}\delta_{m'\,m\pm 1}\,.$$

# Learning outcomes

- $\left[\hat{\underline{J}}^2,\,\hat{J}_k\right] = 0$ and construction of eigenstates as simultaneous eigenstates of $\hat{\underline{J}}^2$ and $\hat{J}_z$:
$$\hat{\underline{J}}^2\,|j,\,m\rangle = \hbar^2 j(j+1)\,|j,\,m\rangle \text{ and } \hat{J}_z\,|j,\,m\rangle = m\hbar\,|j,\,m\rangle.$$
- Ladder operators $\hat{J}_\pm$ and their commutators:
$$\left[\hat{J}_+,\,\hat{J}_-\right] = 2\hbar\hat{J}_z,\ \left[\hat{J}_\pm,\,\hat{J}_z\right] = \pm\hbar\hat{J}_\pm,\ \left[\hat{J}_\pm,\,\hat{\underline{J}}^2\right] = 0.$$
- Range of eigenvalues:
  - $j$ (all positive integers or half-integers) and
  - $m \in \{-j,\,-j+1,\,\ldots,\,j-1,\,j\}$.

# Control questions

13.1 (a) Check the following commutator relations from the lecture:

$$\begin{aligned}
\left[\hat{\underline{J}}^2,\,\hat{J}_k\right] &= 0\,,\\
\left[\hat{J}_+,\,\hat{J}_-\right] &= 2\hbar\hat{J}_z\,,\\
\left[\hat{J}_\pm,\,\hat{J}_z\right] &= \left[\hat{J}_z,\,\hat{J}_\pm\right] = \mp\hbar\hat{J}_\pm\,,\\
\left[\hat{J}_\pm,\,\hat{\underline{J}}^2\right] &= 0\,.
\end{aligned}$$

(b) Check that

$$\hat{\underline{J}}^2 = \hat{J}_z^2 + \hat{J}_+\hat{J}_- - \hbar\hat{J}_z$$

and derive $|c_{-,jm}|^2$ from

$$\hat{J}_-\,|j,\,m\rangle = c_{-,jm}\,|j,\,m-1\rangle$$

13.2 The algebra obtained in this lecture is also valid for the orbital angular momentum, i.e. for the identification $\hat{J} \to \hat{L}$.
Suppose a spin-less particle in a spherically symmetric potential is known to be in an eigenstate of $\hat{\underline{L}}^2$ and $\hat{L}_z$, $|\psi\rangle = |lm\rangle$ with eigenvalues $\hbar^2 l(l+1)$ and $\hbar m$, respectively. Show that the expectation values with respect to this state satisfy

$$\langle \mathcal{L}_x \rangle_{|lm\rangle} = \langle \mathcal{L}_y \rangle_{|lm\rangle} = 0 \quad \text{and} \quad \left\langle \mathcal{L}_x^2 \right\rangle_{|lm\rangle} = \left\langle \mathcal{L}_y^2 \right\rangle_{|lm\rangle} = \frac{l(l+1) - m^2}{2}\,\hbar^2\,.$$

Hint: Express the operators $\hat{L}_{x,y}$ through the ladder operators $\hat{L}_\pm$.