Standard Model

Problems 1

1. **Complex scalar fields**

   - The Klein-Gordon Lagrangian for a free complex field reads

     $$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^*)(\partial^\mu \phi) - \frac{m^2}{2}\phi^*\phi\,.$$

     Derive the Equations of Motion (E.o.M.) for the independent fields ϕ and ϕ^* and convince yourself that they indeed are consistent, i.e. one is the complex conjugate of the other.
   - What happens if you add a potential of the form

     $$\mathcal{U} = \frac{\lambda}{4}(\phi^*\phi)^2\,?$$

   - The Lagrangian (including the potential above) is invariant under global phase (gauge) transformations of the form

     $$\phi \to \phi' = \exp(i\theta)\phi \quad \text{and} \quad \phi^* \to \phi'^* = \exp(-i\theta)\phi^*\,.$$

     Impose local gauge invariance to construct the full gauge-invariant Lagrangian. What is the current density j^μ? Check that it fulfills the continuity equation. Construct a charge operator and calculate its commutator with ϕ and ϕ^*.
   - Check the behaviour of the Klein-Gordon Lagrangian under infinitesimal gauge transformations, i.e. calculate $\delta\mathcal{L}$ when an infinitesimal phase transformation with $\delta\theta$ is applied to the fields. Using Noether's theorem, can you reconstruct the conserved current above?

2. **Non-abelian gauge theories**
   Consider the Lagrangian for a non-abelian gauge theory, interacting with fermions (group indices written out, i and j for fundamental, a, b, c for the adjoint representation)

   $$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a,\mu\nu} + \bar{\psi}_i\left(\partial_\mu\delta_{ij} - igT^a_{ij}A^a_\mu\right)\gamma^\mu\psi_j\,.$$

   - Convince yourself that this Lagrangian is invariant under local gauge transformations

     $$\begin{aligned} \psi_i \to \psi'_i &= U_{ij}(x)\psi_j := \exp(i\theta^a(x)T^a_{ij})\psi_j \\ A_\mu = T^a A^a_\mu(x) \to A'_\mu &= U(x)A_\mu U^\dagger(x) + \frac{i}{g}\left[\partial_\mu U(x)\right]U^\dagger(x)\,. \end{aligned}$$

     Don't forget the kinetic term!
   - Can you derive the Feynman rules? You should have two propagators and three vertices, if you ignore Fadeev-Popov ghosts. Try to relate the fermion-boson with the three-boson coupling in terms of relative coupling strength.

- Try to quantise the theory in the path integral formalism in Lorentz-gauge; introduce Fadeev-Popv ghosts. Can you think of a ghost-free gauge?

3. **Spontaneous symmetry breaking and the Higgs mechanism**
Consider the $\lambda\phi^4$ Lagrangian of a complex scalar field

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi^*)(\partial^\mu\phi) + \frac{\mu^2}{2}\phi^*\phi - \frac{\lambda}{4}(\phi^*\phi)^2 \,.$$

It has a global $U(1)$ symmetry.

- Replace the global with a local $U(1)$ symmetry and construct the emerging Lagrangian. Can you construct its Feynman rules?
- Rewrite the Lagrangian with real scalar fields $\phi_{1,2}$:

$$\phi = \phi_1 + i\phi_2 \quad \text{and} \quad \phi^* = \phi_1 - i\phi_2$$

and choose a vacuum expectation value for ϕ_1 alone, i.e.

$$\phi_1 \to \eta = \phi_1 - v \quad \text{and} \quad \phi_2 \to \zeta = \phi_2 \,,$$

where v is a function of μ and λ. Determine v and construct the Lagrangian. What is the mass of the A field now?
- Looking closely, you will see terms of the kind

$$(\partial_\mu\zeta)A^\mu \quad \text{and} \quad [\zeta(\partial_\mu\eta) - \eta(\partial_\mu\zeta)]\,A^\mu \,.$$

Can you represent them in terms of Feynman rules and interpret them?
- To avoid this, use a different linear combination of ϕ_1 and ϕ_2 to construct real and imaginary parts, namely

$$\begin{aligned}\phi &= (\cos\theta + i\sin\theta)(\phi_1 + i\phi_2) \\ &= (\phi_1\cos\theta - \phi_2\sin\theta) + i(\phi_1\sin\theta + \phi_2\cos\theta) = \psi_1 + i\psi_2 \,.\end{aligned}$$

Rewrite the Lagrangian in terms of the fields $\psi_{1,2}$. Can you interpret the operation above?
- If you again apply the vev to the real field component only, you can write the Lagrangian once more in terms of (modified) fields η and ζ. Picking

$$\theta = -\tan^{-1}\frac{\phi_2}{\phi_1}$$

should result in vanishing ζ.