Standard Model

Lectures given by Frank Krauss, Michaelmas Terms 2007 and 2008,
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Disclaimer:

This script is, to a large extent, influenced by a number of excellent books on quantum field theory, group theory, gauge theory, and particle physics. Specifically, the following books have been used:

- S. Coleman, “Aspects of symmetry”;
- J. F. Donoghue, E. Golowich, & B. R. Holstein, “Dynamics of the Standard Model”;
- H. Georgi, “Lie algebras and particle physics”;
- B. Hatfield, “Quantum field theory of point particles and strings”.
- H. F. Jones, “Groups, representations and physics”;
- T. Kugo, “Gauge theory”;
- P. Ramond, “Field theory, a modern primer”.

Contents

Notation

Throughout this lecture the following notation will be used:

1. Units:
   In general, natural units are used,
$$\hbar = c = 1\,. \tag{1}$$
   In some cases, where indicated, $\hbar$ is written out explicitly to indicate how the classical limit can be taken.

2. Four-vectors:
   Four-vectors in space-time are written as
$$x^\mu = (t,\,\vec{x}) = (x^0,\,x^1,\,x^2,\,x^3),\ \mu = 0,\,1,\,2,\,3\,, \tag{2}$$
   where the first coordinate, labelled as x^0, is the time coordinate. In general, in the framework of this lecture, greek indices $\mu,\ \nu,\ \dots$ will run from 0 to 3 whereas latin indices $i,\ j,\ \dots$ run from 1 to 3. In other words,
$$x^\mu = (t,\,\vec{x}) = (x^0,\,x^i)\,. \tag{3}$$
   The scalar product of two four-vectors is
$$x \cdot y = x^\mu y_\mu = x_\mu y^\mu = x^0 y^0 - \vec{x}\cdot\vec{y} = x^0 y^0 - x^i y^i\,, \tag{4}$$
   where the Einstein convention of summing over repeated Lorentz-indices is used unless stated otherwise.

3. Metric:
   The scalar product above can also be written as
$$x \cdot y = x^\mu y^\nu g_{\mu\nu} = x_\mu y_\nu g^{\mu\nu}\,, \tag{5}$$

where the Minkowski-metric is

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{6}$$

and satisfies

$$g_{\mu\nu} g^{\nu\rho} = g_{\mu\nu} g^{\rho\nu} = g_\mu^\rho := \delta_\mu^\rho \tag{7}$$

with the Kronecker-delta

$$\delta_\mu^\nu = \begin{cases} 1 & \text{for} \quad \mu = \rho \\ 0 & \text{for} \quad \mu \neq \rho \end{cases} \tag{8}$$

4. Co- and contravariant four-vectors:
   The Minkowski-metric above is used to raise and lower indices,

$$x_\mu = g_{\mu\nu} x^\nu = g_{\mu\rho} g^{\rho\nu} x_\nu = \delta_\mu^\nu x_\nu \,. \tag{9}$$

   Four-vectors with lower or upper indices, x_μ and x^μ are called covariant and contravariant, respectively. Their connection can be visualized employing their components. Obviuosly, covariant four-vectors are

$$x^\mu = (t,\, \vec{x}) \tag{10}$$

   and the corresponding contravariant four-vectors are given by

$$x_\mu = (t,\, -\vec{x})\,, \tag{11}$$

   i.e. they have negative space-components.

5. Transformation of tensors:
   In the same fashion, the indices of tensors can be lowered or raised,

$$T_{\mu\nu} = g_{\mu\rho} g_{\nu\sigma} T^{\rho\sigma} \,. \tag{12}$$

   Similarly, individual indices of tensors can be treated.

6. Totally anti-symmetric tensor:
   The totally anti-symmetric tensor $\epsilon^{\mu\nu\rho\sigma}$ is fixed by

$$\epsilon^{0123} = 1\,. \tag{13}$$

   Odd permutations of the indices result in the corresponding component to change its sign.

   Useful identities include:

$$\begin{aligned}
\epsilon^{\mu\nu\rho\sigma}\epsilon^{\mu'\nu'\rho'\sigma'} &= -\det\left[g^{\alpha\alpha'}\right]\,, \\
&\quad \text{where} \quad \alpha = \{\mu,\,\nu,\,\rho,\,\sigma\} \quad \text{and} \quad \alpha' = \{\mu',\,\nu',\,\rho',\,\sigma'\} \\
\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu}{}^{\nu'\rho'\sigma'} &= -\det\left[g^{\alpha\alpha'}\right]\,, \\
&\quad \text{where} \quad \alpha = \{\nu,\,\rho,\,\sigma\} \quad \text{and} \quad \alpha' = \{\nu',\,\rho',\,\sigma'\} \\
\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu}{}^{\rho'\sigma'} &= -2\left(g^{\rho\rho'}g^{\sigma\sigma'} - g^{\sigma\rho'}g^{\rho\sigma'}\right) \\
\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu\rho}{}^{\sigma'} &= -6g^{\sigma\sigma'} \\
\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu\rho\sigma} &= -24\,.
\end{aligned} \tag{14}$$

Chapter 1

Gauge symmetries and Lagrangians

In this chapter basic symmetry principles will be discussed, and their implications to the construction of theories and their Lagrangians obeying these symmetries.
In particular this chapter starts out with the basic property of fields in quantum field theory, namely their invariance under Lorentz transformations leading to the notion of the Lorentz group. The general form of its generators will be constructed, and, with their help, representations of this group will be classified before some of the representations such as, e.g. spinors, will be discussed in some more detail. Then the effect of symmetries in general will be highlighted and the implications of the Noether theorem will be summarized. Finally, some basic building blocks of Lagrange densities for fields prevalent in particle physics will be constructed. This chapter closes with a discussion of how the gauge principle comes into play when constructing Lagrange densities for gauge theories. This is exemplified in some detail for the simplest case, electromagnetism, which is connected to the group $U(1)$ as gauge group. The more involved case of $SU(N)$ gauge theories is only briefly highlighted.

1.1 Space-time symmetries: Lorentz group

1.1.1 Lorentz transformations

Recalling special relativity

The structure of space-time in standard quantum field theory and, thus, in gauge theories is usually given by the one of special relativity determined by the following principles:

1. Space and time are homogeneous,
2. space is isotropic,
3. in all inertial systems of reference the speed of light has the same value c (put to 1 in the framework of this lecture).

This has a number of consequences. Remember that the space-time distance of two events 1 and 2 is the same in two frames S and S' that move relative to each other with constant velocity v,

$$(x_1 - x_2)^2 = (x'_1 - x'_2)^2 \,. \tag{1.1}$$

Assuming the velocity v to point along an axis labelled by $\|$, the relation between points in the two systems reads

$$t' = \gamma \cdot (t - x_\| \cdot v)\,, \quad x'_\| = \gamma(x_\| - v \cdot t)\,, \quad x'_\perp = x_\perp \,. \tag{1.2}$$

(Remember, in any case, that this lecture uses natural units, i.e. $c \equiv 1$). In the equations above, Eq. (1.2), the boost γ is given by

$$\gamma = 1/\sqrt{1-v^2}\,. \tag{1.3}$$

The Lorentz transformation of Eq. (1.2) can be cast into a form similar to the following one for rotations, namely

$$\begin{aligned} t' &= t \\ x'_1 &= x_1 \cos\theta - x_2 \sin\theta \\ x'_2 &= x_1 \sin\theta + x_2 \cos\theta \\ x'_3 &= x_3 \,, \end{aligned} \tag{1.4}$$

where the rotation angle is θ and the rotation is around the z- (or 3-) axis. For rotations the spatial distance is constant,

$$\vec{x'}^2 = \vec{x}^2 \,. \tag{1.5}$$

In three dimensions, the set of all rotations forms a non-commutative or non-Abelian group under consecutive application called $SO(3)$, the special orthogonal group in three dimensions. In this framework, orthogonal denotes such groups that can be represented by real matrices and leave Euclidean distances invariant, i.e. have determinant equal to one; the group is special because its determinant equals plus one. In contrast, the inclusion of reflections around a plane leads to matrices with determinant equal to minus one.

Returning to Lorentz transformations and identifying the boost to be along the z-axis it is easy to check that Eq. (1.2) can be written as

$$\begin{aligned} x'_0 &= x_0 \cosh\xi - x_3 \sinh\xi \\ x'_1 &= x_1 \\ x'_2 &= x_2 \\ x'_3 &= -x_0 \sinh\xi + x_3 \cosh\xi \,, \end{aligned} \tag{1.6}$$

with $\gamma = \cosh\xi$, or, equivalently, $\tanh\xi = v$.

Definition

Lorentz transformations are defined as linear transformations

$$x^\mu \;\longrightarrow\; x'^\mu = \Lambda^\mu_{\ \nu} x^\nu \tag{1.7}$$

of four-vectors that leave their norm invariant:

$$x^\mu \to x'^\mu = \Lambda^\mu_{\ \nu} x^\nu, \;\; x'^2 = x'^\mu x'_\mu =: x^2 \,. \tag{1.8}$$

Invariance of the norm implies that

$$x'^\mu x'^\nu g_{\mu\nu} = g_{\mu\nu} \Lambda^\mu_{\ \rho} \Lambda^\nu_{\ \sigma} x^\rho x^\sigma = x^\rho x^\sigma g_{\rho\sigma} \Longrightarrow \Lambda^\mu_{\ \rho} \Lambda^\nu_{\ \sigma} g_{\mu\nu} = g_{\rho\sigma} \,. \tag{1.9}$$

Multiplication of the second part of this equation with $g^{\rho\tau}$ leads to

$$\Lambda^\mu_{\ \rho} \Lambda^\nu_{\ \sigma} g_{\mu\nu} g^{\rho\tau} = \Lambda_\mu^{\ \tau} \Lambda^\mu_{\ \sigma} = g_{\rho\sigma} g^{\rho\tau} = \delta^\tau_\sigma \,. \tag{1.10}$$

This together with

$$\Lambda^{\mu}{}_{\rho} g_{\mu\nu} g^{\rho\tau} = \Lambda_{\nu}^{\ \tau} \tag{1.11}$$

immediately implies that

$$\Lambda_{\mu}^{\ \tau} = (\Lambda^{\mu}{}_{\tau})^{-1} = \left(\Lambda^{-1}\right)^{\tau}{}_{\mu}\,, \tag{1.12}$$

fixing the inverse of Λ. Using the inverse the Lorentz transformation of a covariant four-vector x_μ can thus be written as

$$x_\mu \;\longrightarrow\; x'_\mu = \Lambda_{\mu}^{\ \nu} x_\nu = x_\nu \left(\Lambda^{-1}\right)^{\nu}{}_{\mu}\,. \tag{1.13}$$

For the Lorentz transformation of an arbitrary tensor all its indices have to be transformed:

$$T'{}_{\mu_1\mu_2\ldots\mu_n}{}^{\nu_1\nu_2\ldots\nu_m} = \Lambda_{\mu_1}^{\ \rho_1}\Lambda_{\mu_2}^{\ \rho_2}\ldots\Lambda_{\mu_n}^{\ \rho_n}\;\Lambda^{\nu_1}_{\ \sigma_1}\Lambda^{\nu_2}_{\ \sigma_2}\ldots\Lambda^{\nu_m}_{\ \sigma_m}\;T_{\rho_1\rho_2\ldots\rho_n}{}^{\sigma_1\sigma_2\ldots\sigma_m}\,. \tag{1.14}$$

Classification of Lorentz transformations

There are only two constant, Lorentz invariant tensors, namely the metric tensor $g^{\mu\nu}$ and its co- and contravariant versions, especially the unity matrix, and the totally anti-symmetric tensor $\epsilon^{\mu\nu\rho\sigma}$. Lorentz invariance can be checked easily,

$$\begin{aligned} g'^{\mu\nu} &= \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma} g^{\rho\sigma} = \Lambda^{\mu\sigma}\Lambda^{\nu}{}_{\sigma} = g^{\mu\rho}\Lambda_{\rho}^{\ \sigma}\Lambda^{\nu}{}_{\sigma} = g^{\mu\rho}\delta^{\nu}_{\rho} = g^{\mu\nu}\,, \\ \epsilon'^{\mu\nu\rho\sigma} &= \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}\Lambda^{\rho}{}_{\gamma}\Lambda^{\sigma}{}_{\delta}\epsilon^{\alpha\beta\gamma\delta} = \epsilon^{\mu\nu\rho\sigma}\cdot\det\Lambda\,. \end{aligned} \tag{1.15}$$

The identities above can be easily constructed using Eq. (1.12) and the definition of the determinant. Due to Eq. (1.10) this determinant is fixed such that

$$|\det\Lambda| = 1 \;\longrightarrow\; \det\Lambda = \pm 1\,. \tag{1.16}$$

In other words, if the determinant equals plus one, the totally anti-symmetric tensor does not change its sign under the corresponding Lorentz transformations, otherwise it does. This in addition with the sign of the $\mu = \nu = 0$ component of the Lorentz transformation allows to classify any such transformation Λ. The $\mu = \nu = 0$ component of any Lorentz transformation Λ is given by

$$\left(\Lambda_0^{\ 0}\right)^2 = 1 + \sum_{i=1,2,3}\left(\Lambda_0^{\ i}\right)^2 \geq 1\,. \tag{1.17}$$

According to the determinant and the sign of the 00 component Lorentz transformations can be classified as:

1. Proper, orthochronous:
   Transformations connected with the identity or unity matrix **1**, given below. These transformations have $\det\Lambda = +1$ and $\Lambda_0^{\ 0} \geq 1$.

2. Improper, orthochronous:
   Transformations connected with the parity transformation matrix **P**, given below. These transformations have $\det\Lambda = -1$ and $\Lambda_0^{\ 0} \geq 1$.

3. Improper, non-orthochronous:
   Transformations connected with the time reversal matrix **T**, given below. These transformations have $\det\Lambda = -1$ and $\Lambda_0^{\ 0} \leq 1$.

4. Proper, non-orthochronous:
   Transformations connected with the combination of both, **PT**, given below. These transformations have $\det\Lambda = +1$ and $\Lambda_0^{\ 0} \leq 1$.

The matrices named above are

$$\mathbf{1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$\mathbf{T} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{PT} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{1.18}$$

For pseudo-tensors the Lorentz transformation, Eq. (1.14), reads

$$T'_{\mu_1\mu_2\ldots\mu_n}{}^{\nu_1\nu_2\ldots\nu_m} = (\det\Lambda)\cdot\Lambda_{\mu_1}^{\rho_1}\Lambda_{\mu_2}^{\rho_2}\ldots\Lambda_{\mu_n}^{\rho_n}\,\Lambda^{\nu_1}_{\sigma_1}\Lambda^{\nu_2}_{\sigma_2}\ldots\Lambda^{\nu_m}_{\sigma_m}\,T_{\rho_1\rho_2\ldots\rho_n}{}^{\sigma_1\sigma_2\ldots\sigma_m}, \tag{1.19}$$

i.e. , the determinant of the transformation has to be added. Its effect is that pseudo-tensors transform like ordinary tensors under Lorentz transformation with determinant equals to one, but they transform differently otherwise and gain an extra sign. Vectors having this behavior under Lorentz transformations are called axial vectors, an example is the angular momentum.
From the considerations so far it is obvious that all Lorentz indices of a term have to be contracted to ensure its Lorentz -invariance. In other words, such

terms have to be Lorentz scalars or pseudo-scalars, the latter emerge when the indices are contracted through a pseudo-tensor. This can be exemplified by terms of the form

$$\epsilon_{\mu\nu\rho\sigma}\, x^\mu x^\nu x^\rho x^\sigma\,, \tag{1.20}$$

which changes sign when Lorentz transformations are applied that are connected with either **P** or **T**.

Infinitesimal Lorentz transformations

Consider a Lorentz transformation that differs from the identity only by infinitesimal amounts,

$$\Lambda^\mu_{\ \nu} = \delta^\mu_\nu + \varepsilon^\mu_{\ \nu}\,, \tag{1.21}$$

where

$$|\varepsilon^\mu_\nu| \ll 1\,. \tag{1.22}$$

Plugging this into Eq. (1.10) and expanding up to first order in ε yields

$$g_{\mu\nu} + g_{\mu\sigma}\varepsilon^\sigma_{\ \nu} + g_{\tau\nu}\varepsilon^\tau_{\ \mu} = g_{\mu\nu}\,, \tag{1.23}$$

resulting in

$$\varepsilon_{\mu\sigma} + \varepsilon_{\sigma\mu} = 0\,. \tag{1.24}$$

This proves that the tensor ε is an anti-symmetric tensor which can be expressed through six independent parameters (for the six off-diagonal elements forming either the upper or the lower triangle).
Applying such an infinitesimal Lorentz transformation looks like

$$x'^\mu = \Lambda^\mu_{\ \nu} x^\nu = x^\mu + \varepsilon^\mu_{\ \nu} x^\nu := \left(1 - \frac{i}{2}\varepsilon^{\rho\sigma} M_{\rho\sigma}\right)^\mu_{\ \nu} x^\nu\,, \tag{1.25}$$

where the six totally antisymmetric 4×4 matrices $M_{\rho\sigma} = -M_{\rho\sigma}$ have been introduced. They read

$$(M_{\rho\sigma})^\mu_{\ \nu} = i\left(\delta^\mu_\rho g_{\sigma\nu} - \delta^\mu_\sigma g_{\rho\nu}\right)\,. \tag{1.26}$$

The commutator of two such matrices obeys the following relation

$$[M_{\mu\nu}\,, M_{\rho\sigma}] = -i\left(g_{\mu\rho}M_{\nu\sigma} + g_{\nu\sigma}M_{\mu\rho} - g_{\mu\sigma}M_{\nu\rho} - g_{\nu\rho}M_{\mu\sigma}\right)\,. \tag{1.27}$$

Casting Eq. (1.25) into the finite version of a Lorentz transformation,

$$\hat{\Lambda} = \exp\left(-\frac{i}{2}\varepsilon^{\mu\nu}\hat{\mathcal{M}}_{\mu\nu}\right) \approx 1 - \frac{i}{2}\varepsilon^{\mu\nu}\hat{\mathcal{M}}_{\mu\nu}\,. \tag{1.28}$$

the six matrices $\hat{\mathcal{M}}_{\rho\sigma}$ are recognized as the generators of the Lorentz group $SO(3,\,1)$. The specific form in Eq. (1.26) is also the one for the generators $\hat{\mathcal{M}}_{\rho\sigma}$ when acting on four-vectors. When acting on other entities or fields, the $\hat{\mathcal{M}}_{\mu\nu}$ might have another form. This specific form depends on the representation and on what they act on. In any case, of course, these generators obey the same commutation relations above, Eq. (1.27). In what follows now, a more general form of the generators $\hat{\mathcal{M}}_{\mu\nu}$ will be reconstructed.
It is easy to check that the operators

$$\hat{L}_{\mu\nu} := i(x_\mu\partial_\nu - x_\nu\partial_\mu) \tag{1.29}$$

also satisfy the same Lie algebra, Eq. (1.27). Therefore, the $\hat{\mathcal{M}}_{\mu\nu}$ can be expressed as

$$\hat{\mathcal{M}}_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu) + \hat{S}_{\mu\nu}\,, \tag{1.30}$$

where the hermitian (spin-) operator $\hat{S}_{\mu\nu}$ obeys the same Lie-algebra, Eq. (1.27), and commutes with the $\hat{L}_{\mu\nu}$. The last identity will be discussed later on, in connection with the transformation properties of various fields. For now it should be sufficient to mention that Eq. (1.30) states that the generators of the Lorentz group can be decomposed into two parts. One part, $\hat{L}_{\mu\nu}$, acts on the spatial dependence of the fields, the other part, $\hat{S}_{\mu\nu}$ acts on its internal symmetries, such as spin.

Construction of the generators

To see the form of the $M_{\mu\nu}$, Eq. (1.26), let us step back to the rotation group in three dimensions, $SO(3)$. The generators of this (and - in principle - any other) group can be obtained through a cook-book recipe by differentiating matrices related to finite transformations and taking the limit of the finite parameter going to zero, cf. Eq. (A.9). The generators are then a minimal set of basis-matrices. In the case of SO(3) we have

$$\hat{\mathcal{R}}_j := i\left.\frac{d\mathcal{R}_j(\phi_j)}{d\phi_j}\right|_{\phi_j\to 0}\,, \tag{1.31}$$

where the $\hat{\mathcal{R}}$ denote the generators and $j = 1, 2, 3$ labels the axis around which the finite rotation of angle ϕ_j is performed. The generators are

$$\begin{aligned}
\hat{\mathcal{R}}_1 &= i \left[\frac{d}{d\phi_1}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\phi_1 & -\sin\phi_1 \\ 0 & 0 & \sin\phi_1 & \cos\phi_1 \end{pmatrix}\right]_{\phi_1\to 0} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \\
\hat{\mathcal{R}}_2 &= i \left[\frac{d}{d\phi_2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi_2 & 0 & \sin\phi_2 \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\phi_2 & 0 & \cos\phi_2 \end{pmatrix}\right]_{\phi_2\to 0} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \\
\hat{\mathcal{R}}_3 &= i \left[\frac{d}{d\phi_3}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi_3 & -\sin\phi_3 & 0 \\ 0 & \sin\phi_3 & \cos\phi_3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right]_{\phi_3\to 0} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} .
\end{aligned} \tag{1.32}$$

But apart from the three rotations the Lorentz group also contains three more transformations related to boosts along the three spatial axes. Their generators $\hat{\mathcal{B}}_i$ are given by

$$\begin{aligned}
\hat{\mathcal{B}}_1 &= i \left[\frac{d}{d\xi_1}\begin{pmatrix} \cosh\xi_1 & -\sinh\xi_1 & 0 & 0 \\ -\sinh\xi_1 & \cosh\xi_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right]_{\xi_1\to 0} = \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
\hat{\mathcal{B}}_2 &= i \left[\frac{d}{d\xi_2}\begin{pmatrix} \cosh\xi_2 & 0 & -\sinh\xi_2 & 0 \\ 0 & 1 & 0 & 0 \\ -\sinh\xi_2 & 0 & \cosh\xi_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right]_{\xi_2\to 0} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
\hat{\mathcal{B}}_3 &= i \left[\frac{d}{d\xi_3}\begin{pmatrix} \cosh\xi_3 & 0 & 0 & -\sinh\xi_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\xi_3 & 0 & 0 & \cosh\xi_3 \end{pmatrix}\right]_{\xi_3\to 0} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} .
\end{aligned} \tag{1.33}$$

The commutation relations can now be checked explicitly; for instance we have

$$\begin{aligned} \left[\hat{\mathcal{B}}_3 \,, \hat{\mathcal{R}}_1\right] &= i\hat{\mathcal{B}}_2 \\ \left[\hat{\mathcal{B}}_1 \,, \hat{\mathcal{B}}_2\right] &= i\hat{\mathcal{R}}_3 \,. \end{aligned} \tag{1.34}$$

The full algebra can be summarized as

$$\begin{aligned} \left[\hat{\mathcal{R}}_i \,, \hat{\mathcal{R}}_j\right] &= i\epsilon_{ijk}\hat{\mathcal{R}}_k \\ \left[\hat{\mathcal{R}}_i \,, \hat{\mathcal{B}}_j\right] &= i\epsilon_{ijk}\hat{\mathcal{B}}_k \\ \left[\hat{\mathcal{B}}_i \,, \hat{\mathcal{B}}_j\right] &= -i\epsilon_{ijk}\hat{\mathcal{R}}_k \,. \end{aligned} \tag{1.35}$$

Taken together it can be stated that the generators of rotations close on themselves, the generators of boosts behave like vectors under rotations, and that the boosts do not close on themselves, as the last equation shows. These generators are identical to the ones in Eq. (1.26) and can be identified as

$$\hat{\mathcal{R}}_i = \epsilon_{ijk} M_{jk} \,, \quad \hat{\mathcal{B}}_i = M_{i0} \,. \tag{1.36}$$

The algebraic structure of Eq. (1.35)can be greatly simplified by considering the following operators:

$$\hat{\mathcal{X}}_i^{\pm} = \frac{1}{2}\left[\hat{\mathcal{R}}_i \pm i\hat{\mathcal{B}}_i\right] \,. \tag{1.37}$$

It is easy to check that in terms of these operators the commutation relations of Eq. (1.35) are

$$\begin{aligned} \left[\hat{\mathcal{X}}_i^+ \,, \hat{\mathcal{X}}_j^+\right] &= i\epsilon_{ijk}\hat{\mathcal{X}}_k^+ \\ \left[\hat{\mathcal{X}}_i^- \,, \hat{\mathcal{X}}_j^-\right] &= i\epsilon_{ijk}\hat{\mathcal{X}}_k^- \\ \left[\hat{\mathcal{X}}_i^+ \,, \hat{\mathcal{X}}_j^-\right] &= 0 \,. \end{aligned} \tag{1.38}$$

Obviously, the effect of the transformation Eq. (1.37) is to split the algebra of Eq. (1.38) into two independent $SU(2)$ algebras [1]. Having at hand this

[1] A similar thing happens in fact when considering $SO(4)$ instead of $SO(3,\,1)$, which also splits into two $SU(2)$ algebras. The crucial difference, however, lies in the fact that in the case of the $SO(4)$ there is no imaginary unit i involved in the transformation analogous to Eq. (1.37). That's why $SO(4)$ is isomorphic to $SU(2) \otimes SU(2)$, whereas $SO(3,\,1)$ is isomorphic to $SL(2,\,C)$

decomposition, the representation of $SO(3,\,1)$ is quite straightforward. Irreducible representations are connected to the irreducible representations of the two $SU(2)$'s. Each of the two $SU(2)$ has one Casimir operator, namely

$$\mathcal{X}_i^+\mathcal{X}_i^+ \quad \text{with eigenvalues } j_1(j_1+1)$$

and

$$\mathcal{X}_i^-\mathcal{X}_i^- \quad \text{with eigenvalues } j_2(j_2+1)\,, \tag{1.39}$$

where, in both cases $j_{1,2} = 0,\, 1/2,\, 1,\, \ldots$. Using, as usual, the eigenvalues of the Casimir operators to label representations, the representations of $SO(3,\,1)$ can be identified through

$$(2j_1+1,\, 2j_2+1)\,. \tag{1.40}$$

Apart from the trivial representation $(0,\,0)$ we will encounter in the following the two Weyl-representations $(0,\,\frac{1}{2})$ and $(\frac{1}{2},\,0)$, which in particle physics are used to describe the neutrino, the Dirac-representation $(\frac{1}{2},\,0)\oplus(0,\,\frac{1}{2})$, which is used for, e.g., the electron, and the defining representation $(\frac{1}{2},\,\frac{1}{2})$ providing the transformations of four-vectors.

The Poincare group

Apart from Lorentz invariance, there is another symmetry of space-time that is realized for many systems, and, especially, for the field theories that will be studied during this lecture. This is the symmetry under space-time translations,

$$x^\mu \longrightarrow x'^\mu = x^\mu + a^\mu\,. \tag{1.41}$$

Therefore the general space-time invariance group is the ten-parameter (six from the Lorentz piece, four from translations) Poincare group, under which

$$x^\mu \longrightarrow x'^\mu = \Lambda^\mu_{\ \nu}x^\mu + a^\mu\,. \tag{1.42}$$

Simple considerations show that the translations do not commute with the Lorentz transformations - just imagine what happens to a translated system being rotated and translated again and interchange the order of the two last transformations. The generators for the translations are the momenta, just consider small changes

$$\delta x^\mu := \varepsilon^\rho g^\mu_\rho = \varepsilon^\rho \partial_\rho x^\mu = i\varepsilon^\rho \hat{p}_\rho x^\mu\,, \tag{1.43}$$

where the $\hat{p}_\mu$ are the momentum operators which in fact act as the generators of translation. They commute with each other,

$$[\hat{p}_\mu, \, \hat{p}_\nu] = 0\,, \tag{1.44}$$

but, of course, not with the generators of the Lorentz group,

$$\left[\hat{M}_{\mu\nu}\,, \hat{p}_\rho\right] = -ig_{\mu\rho}\hat{p}_\nu + ig_{\nu\rho}\hat{p}_\mu\,. \tag{1.45}$$

To construct Casimir operators of this group one has, again, to look for operators that commute with all generators. Such an operator is given by the length of the four-momentum squared, $p_\rho p^\rho$ that is obviously invariant under Lorentz transformations (and therefore commutes with their generators) and that commutes with $\hat{p}_\mu$. The other Casimir operator is constructed by the Pauli-Lubanski four-vector,

$$\hat{W}^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\,\hat{p}_\nu\hat{M}_{\rho\sigma}\,. \tag{1.46}$$

It is easy to check that it commutes with $\hat{p}_\kappa$ and that it transforms like a four-vector under Lorentz transformations. The latter property means that it's length is therefore invariant, leaving the operator $W^\mu W_\mu$ a Casimir operator. The representations of the Poincare group fall into three classes:

1. $P^\rho P_\rho \equiv m^2 > 0$ is a real number. Then, eigenvalues of $W^\mu W_\mu$ are given by $-m^2\,s(s+1)$, where s is the spin of the particle and has discrete values $0,\, 1/2,\, 1,\, \dots$. States within this representation are then characterized by the continuous value of the momentum (three degrees of freedom, since $P^\rho P_\rho \equiv m^2$) and by the third component s_3 of the spin with $s_3 = -s,\, -s+1,\, \dots,\, s$. Massive particles have thus $2s+1$ spin degrees of freedom.

2. $P^\rho P_\rho = 0$, and, therefore, also $W^\mu W_\mu = 0$. Since $P_\rho W^\rho = 0$, both vectors are proportional, the constant of proportionality is the helicity which is equal to $\pm s$. Thus, massless particles with spin $s > 0$ have two spin degrees of freedom.

3. A case seemingly not realized in nature: $P^\rho P_\rho = 0$ but continuous spin.

1.1.2 Fields and representations of the Lorentz group

Behavior of fields under coordinate changes

As stated before, Lorentz transformations connect two systems S and S' with each other that have constant relative velocity. In other words, two observers O and O' in the two systems use the coordinates x and x' to characterize events. The connection of these coordinates is given by Eq. (1.7),

$$x^\mu \longrightarrow x'^\mu = \Lambda^\mu_\nu x^\nu \,.$$

For scalar function, both observers should measure the same value at points connected by the Lorentz transformation,

$$\phi'(x') = \phi(x)\,. \tag{1.47}$$

and for vector fields, being equipped with Lorentz indices, they have

$$A'_\mu(x') = \Lambda_\mu^{\ \nu}\, A_\nu(x)\,. \tag{1.48}$$

In general, arbitrary N-dimensional fields $\Phi_i^{(N)}$ transform like

$$\Phi'^{(N)}_i(x') = \mathcal{D}^{(N)}(\Lambda)_i^j\, \Phi_j^{(N)}(x)\,, \tag{1.49}$$

where $\mathcal{D}^{(N)}(\Lambda)$ is an $N \times N$ matrix. It should be stressed that this matrix acts on both the coordinates and the functional dependence of the field on them. In fact, as will be discussed more explicitly for the individual fields, this matrix is connected with the operator $\hat{\mathcal{M}}$ of Eq. (1.30).
Of course, this game of transforming systems can be played with three systems as well. Then coordinates are connected through

$$x''_\mu = {\Lambda_2}_\mu^{\ \nu}\, x'_\nu = {\Lambda_2}_\mu^{\ \nu}\, {\Lambda_1}_\nu^{\ \rho}\, x_\rho = [\Lambda_2\Lambda_1]_\mu^{\ \rho}\, x_\rho\,, \tag{1.50}$$

and, similarly, for scalar fields

$$\phi''(x'') = \mathcal{D}^{(0)}(\Lambda_2)\,\phi'(x') = \mathcal{D}^{(0)}(\Lambda_2)\,\mathcal{D}^{(0)}(\Lambda_1)\,\phi(x) = \mathcal{D}^{(0)}(\Lambda_2\Lambda_1)\,\phi(x)\,. \tag{1.51}$$

Those more familiar with group theory will immediately recognize the matrices $\mathcal{D}$ to be representations of the Lorentz group. This establishes an intimate connection of all representations of a symmetry group on the fields with all kinds of fields that are allowed in the corresponding Lagrangian of the theory.

Returning to the considerations that lead to the explicit construction of the generators of the Lorentz group and their naming, Eq. (1.40), the dimension of the representation can be read off to be

$$\dim\left(\mathcal{D}^{(N)}\right) = (2j_1+1)\cdot(2j_2+1)\,, \tag{1.52}$$

and, accordingly, fields $\phi^{(j_1,j_2)}$ in this representation have to have the same dimension. Recalling the representation of Λ with help of the generators of the Lorentz group, Eq. (1.28), it becomes obvious that any finite Lorentz transformation

$$\hat{\Lambda} = \exp\left(-\frac{i}{2}\varepsilon^{\mu\nu}\hat{\mathcal{M}}_{\mu\nu}\right)$$

can be represented through the corresponding matrix representation of $\hat{\mathcal{M}}_{\mu\nu}$

$$\mathcal{D}^{(N)}(\Lambda) = \exp\left[-\frac{i}{2}\varepsilon^{\mu\nu}\mathcal{D}^{(N)}\left(\hat{\mathcal{M}}_{\mu\nu}\right)\right]\,. \tag{1.53}$$

Of course, similar statements hold true for infinitesimal Lorentz transformations.

$SL(2,C)$ spinors

As is well known, spins being representations of $SU(2)$ can be only half-integer or integer. Therefore, as already mentioned above, the simplest non-trivial representations of the Lorentz group, i.e. those with minimal dimension, are the two Weyl representations, either $(0,\,\frac{1}{2})$ or $(\frac{1}{2},\,0)$.
For the former case, $(0,\,\frac{1}{2})$, the field ξ_α has two components labelled by $\alpha = 1,\,2$. Being of the form $(0,\,\frac{1}{2})$ we can read off the representations of the generators $\hat{\mathcal{X}}^{\pm}$, namely

$$\mathcal{D}(\hat{\mathcal{X}}^+) = 0\,,\quad \mathcal{D}(\hat{\mathcal{X}}^-) = \frac{1}{2}\vec{\sigma}\,, \tag{1.54}$$

where the σ_i are the Pauli-matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,,\quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\,,\quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,. \tag{1.55}$$

Returning to these generators being written by the generators of rotations and boosts, this representation reads

$$\mathcal{D}(\hat{\mathcal{R}}) = \frac{1}{2}\,\vec{\sigma}\,, \quad \mathcal{D}(\hat{\mathcal{B}}) = \frac{i}{2}\,\vec{\sigma}\,. \tag{1.56}$$

Therefore, a finite Lorentz transformation parameterized by $\varepsilon_{\mu\nu}$, cf. Eq. (1.28), reads

$$\hat{\Lambda} = \exp\left(i\vec{\vartheta}\cdot\hat{\mathcal{R}} + i\vec{\omega}\cdot\hat{\mathcal{B}}\right)\,, \tag{1.57}$$

where

$$\vec{\vartheta} = -\left(\varepsilon_{23},\, \varepsilon_{31},\, \varepsilon_{12}\right)^T\,, \quad \vec{\omega} = -\left(\varepsilon_{01},\, \varepsilon_{02},\, \varepsilon_{03}\right)^T\,. \tag{1.58}$$

Hence, for two-component Spin-1/2 fields belonging to the representation $(0,\, \frac{1}{2})$ the representation $\mathcal{D}(\Lambda)$ has two indices for the two components α and β and reads

$$a_\alpha^{\ \beta} \equiv \mathcal{D}^{(0,\,1/2)}(\Lambda)_\alpha^{\ \beta} = \left[\exp\left(\frac{i}{2}\vec{\vartheta}\cdot\vec{\sigma} + \frac{i}{2}\vec{\omega}\cdot\vec{\sigma}\right)\right]_\alpha^{\ \beta}\,. \tag{1.59}$$

It acts on the spinors like

$$\xi'_\alpha(x') = \mathcal{D}^{(0,\,1/2)}(\Lambda)_\alpha^{\ \beta}\xi_\beta(x)\,. \tag{1.60}$$

Spinors of this representation are also known as left-handed two-component Weyl-spinors.

To construct the two-component spinors in the $(\frac{1}{2},\, 0)$ representation, the spinor of the $(0,\, \frac{1}{2})$ representation have to be complex conjugated. Such complex conjugated spinors are written with a dot as

$$(\xi_\alpha)^* := \eta_{\dot{\alpha}} \tag{1.61}$$

and transform according to

$$\eta'_{\dot{\alpha}}(x') = \left(\mathcal{D}^{(0,\,1/2)}(\Lambda)^*\right)_{\dot{\alpha}}^{\ \dot{\beta}}\ \eta_{\dot{\beta}}(x)\,. \tag{1.62}$$

These spinors are called right-handed two-component Weyl-spinors. It should be stressed that in the complex conjugation of the transformation matrix only its individual components are complex conjugated.

$$a^{*\ \dot{\beta}}_{\ \dot{\alpha}} = \left(\mathcal{D}^{(1/2,\,0)}(\Lambda)^*\right)_{\dot{\alpha}}^{\ \dot{\beta}} = \left[\exp\left(-\frac{i}{2}\vec{\vartheta}\cdot\vec{\sigma} + \frac{i}{2}\vec{\omega}\cdot\vec{\sigma}\right)\right]_{\dot{\alpha}}^{\ \dot{\beta}}\,. \tag{1.63}$$

Finally, the representation in terms of the $\hat{\mathcal{X}}^\pm$ operators is given by

$$\mathcal{D}(\hat{\mathcal{X}}^+) = -\frac{1}{2}\vec{\sigma}^*\,, \quad \mathcal{D}(\hat{\mathcal{X}}^-) = 0\,, \tag{1.64}$$

Calculating with $SL(2, C)$ spinors

To calculate with the spinors introduced above, first of all ways have to be discussed for indices to be lowered or raised. For the two component spinors this happens through the $SL(2, C)$ invariant antisymmetric tensor

$$\epsilon^{\alpha\beta} = \epsilon_{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = \epsilon_{\dot{\alpha}\dot{\beta}} = i\sigma_2\,, \tag{1.65}$$

the analogon to $g_{\mu\nu}$ with

$$\epsilon_{\alpha\beta}\epsilon^{\alpha\beta} = \epsilon_{\dot{\alpha}\dot{\beta}}\epsilon^{\dot{\alpha}\dot{\beta}} = -2 \tag{1.66}$$

For the Pauli matrices we have

$$\sigma_2\sigma_i\sigma_2 = -\sigma_i^* \tag{1.67}$$

Due to its antisymmetry the following relations hold true

$$\xi_\alpha = -\epsilon_{\alpha\beta}\xi^\beta = \xi^\beta\epsilon_{\beta\alpha}\,, \quad \eta_{\dot{\alpha}} = -\epsilon_{\dot{\alpha}\dot{\beta}}\eta^{\dot{\beta}} = \eta^{\dot{\beta}}\epsilon_{\dot{\beta}\dot{\alpha}}\,. \tag{1.68}$$

Scalars can be constructed through

$$\xi^\alpha\Xi_\alpha = \epsilon^{\alpha\beta}\xi_\beta\Xi_\alpha = -\xi_\beta\Xi^\beta\,, \quad \eta^{\dot{\alpha}}H_{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\eta_{\dot{\beta}}H_{\dot{\alpha}} = -\eta_{\dot{\beta}}H^{\dot{\beta}}\,. \tag{1.69}$$

That these terms are in fact scalars can be tested by using the following property of ϵ when applied on general 2×2 matrices M:

$$\epsilon M^T\epsilon^T = \det M\cdot M^{-1} \tag{1.70}$$

Consider, for instance, the first product,

$$\epsilon^{\alpha\beta}\xi_\beta\Xi_\alpha = \xi^T\epsilon^T\Xi \tag{1.71}$$

and apply a Lorentz transformation, here again abbreviated as a,

$$\begin{aligned}(\xi^T\epsilon^T\Xi)' &= \xi^T a^T\epsilon^T a\Xi = \xi^T\epsilon^T\epsilon a^T\epsilon^T a\Xi = \xi^T\epsilon^T\left[\det(a)a^{-1}\right]a\Xi \\ &= \det(a)\,\xi^T\epsilon^T\Xi = \xi^T\epsilon^T\Xi\,.\end{aligned} \tag{1.72}$$

Four-vectors and vector fields

Four-vectors and corresponding fields are in the $(\frac{1}{2},\,\frac{1}{2})$ representation which can be constructed from the Weyl representations through

$$\left(0,\,\frac{1}{2}\right)\times\left(\frac{1}{2},\,0\right)=\left(\frac{1}{2},\,\frac{1}{2}\right)\,. \tag{1.73}$$

To reconstruct the connection of the indices α and $\dot\beta$ of the two Weyl factors with the index μ of the four-vector the following vectors of Pauli matrices are employed

$$\left(\sigma_\mu\right)_{\alpha\dot\beta}=(\mathbf{1},\,\vec\sigma)\;,\quad\left(\bar\sigma_\mu\right)^{\dot\alpha\beta}=\left(\sigma_\mu\right)_{\alpha\dot\beta}\,\epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}=(\mathbf{1},\,-\vec\sigma)\;. \tag{1.74}$$

Again, to lower and raise indices $\epsilon_{\alpha\beta}$ from Eq. (1.65) is employed. Simple consideration show that

$$\left(\bar\sigma_\mu\right)_{\dot\alpha\beta}=\epsilon_{\dot\alpha\dot\gamma}\epsilon_{\beta\delta}\left(\bar\sigma_\mu\right)^{\dot\gamma\delta} \tag{1.75}$$

and that

$$\left(\bar\sigma_\mu\right)_{\dot\alpha\beta}=\left[\left(\sigma_\mu\right)_{\alpha\dot\beta}\right]^*\,, \tag{1.76}$$

the complex conjugated of σ_μ. Furthermore, the following relations hold

$$\mathrm{Tr}\left[\bar\sigma^\mu\sigma_\nu\right]=2\delta^\mu_\nu\,,\quad\left(\sigma_\mu\right)_{\alpha\dot\beta}\left(\bar\sigma^\mu\right)^{\dot\gamma\delta}=2\delta^\delta_\alpha\delta^{\dot\gamma}_{\dot\beta}\,. \tag{1.77}$$

Also,

$$\begin{aligned}\left(\sigma_\mu\bar\sigma_\nu+\sigma_\nu\bar\sigma_\mu\right)_\alpha^{\;\beta}&=2g_{\mu\nu}\delta_\alpha^{\;\beta}\\ \left(\bar\sigma_\mu\sigma_\nu+\bar\sigma_\nu\sigma_\mu\right)^{\dot\alpha}_{\;\dot\beta}&=2g_{\mu\nu}\delta^{\dot\alpha}_{\;\dot\beta}\,.\end{aligned} \tag{1.78}$$

With help of Eq. (1.59) the behavior of the σ_μ under Lorentz transformations can be determined, namely

$$\sigma_\mu\Lambda^\mu_\nu=\left[\mathcal{D}^{(0,\,1/2)}(\Lambda)\right]\cdot\sigma_\nu\cdot\left[\mathcal{D}^{(0,\,1/2)}(\Lambda)\right]^\dagger=a\cdot\sigma_\nu\cdot a^\dagger\,. \tag{1.79}$$

Therefore, for

$$\mathbf{x}:=x_\mu\sigma^\mu=\begin{pmatrix}x_0+x_3 & x_1-ix_2\\ x_1+ix_2 & x_0-x_3\end{pmatrix} \tag{1.80}$$

$$\mathbf{x}'=\left[\mathcal{D}^{(0,\,1/2)}(\Lambda)\right]\cdot\mathbf{x}\cdot\left[\mathcal{D}^{(0,\,1/2)}(\Lambda)\right]^\dagger\,. \tag{1.81}$$

For this representation of four-vectors, we have also the following relations

$$x^\dagger = x \quad \text{and} \quad \det x = x^2 = x^\mu x^\nu g_{\mu\nu}\,. \tag{1.82}$$

Having this in mind covariant four vectors V_μ can be constructed out of two spinors ξ_α and $\eta_{\dot\beta}$,

$$V_\mu = \xi^\alpha (\sigma_\mu)_{\alpha\dot\beta}\eta^{\dot\beta} = \xi_\alpha(\epsilon^T\sigma_\mu\epsilon)^{\alpha\dot\beta}\eta_{\dot\beta} = \xi_\alpha(\bar\sigma_\mu^T)^{\alpha\dot\beta}\eta_{\dot\beta}\,. \tag{1.83}$$

Employing the form of Lorentz transformations of the spinors in Eqs. (1.60) and (1.62) and using the identity of Eq. (1.70) it is easy to check that the V_μ constructed in Eq. (1.83) in fact has the correct transformation behavior, in other words that it is covariant. To see this consider

$$V'_\mu = \xi^T a^T \epsilon^T \sigma_\mu \epsilon a^* \eta = \xi^T \epsilon^T a^{-1}\sigma_\mu {a^\dagger}^{-1}\epsilon\,\eta\,, \tag{1.84}$$

where

$$\epsilon a^T \epsilon^T = a^{-1}\,, \quad \text{and} \quad \epsilon a^* \epsilon^T = {a^\dagger}^{-1}$$

has been used in accordance with Eq. (1.70). Recalling Eq. (1.79) in the form

$$a^{-1}\sigma_\mu {a^\dagger}^{-1} = \Lambda_\mu^{\ \nu}\sigma_\nu \tag{1.85}$$

finally

$$V'_\mu = \xi^T\epsilon^T\Lambda_\mu^{\ \nu}\sigma_\nu\epsilon\,\eta = \xi_\alpha\left(\epsilon^T\Lambda_\mu^{\ \nu}\sigma_\nu\epsilon\right)^{\alpha\dot\beta}\eta_{\dot\beta}\,, \tag{1.86}$$

i.e. V_μ in fact is a covariant entity with the correct transformation behavior. This allows to translate Lorentz indices into spinor indices and vice versa. In doing so, a general rule emerges: Each Lorentz index is transformed into two spinor indices through a product with appropriate combinations of σ^μ. According to Eq. (1.80), the four vector V^μ and its mixed spinor representation are connected through

$$V_{\alpha\dot\beta} = V^\mu\left(\sigma_\mu\right)_{\alpha\dot\beta}\,, \quad V^\mu = \frac{1}{2}\left(\bar\sigma^\mu\right)^{\dot\beta\alpha}V_{\alpha\dot\beta}\,. \tag{1.87}$$

The explicit form of the V^μ, and especially the factor 1/2, can be understood by considering

$$V^\mu = \frac{1}{2}\left(\bar\sigma^\mu\right)^{\dot\beta\alpha}V_{\alpha\dot\beta} = \frac{1}{2}\left(\bar\sigma^\mu\right)^{\dot\beta\alpha}V^\nu\left(\sigma_\nu\right)_{\alpha\dot\beta} = \frac{1}{2}\mathrm{Tr}\left(\bar\sigma^\mu\sigma_\nu\right)V^\nu = \delta^\mu_\nu V^\nu = V^\mu\,. \tag{1.88}$$

Antisymmetric tensors of rank two

As shown in the previous section, any Lorentz index can be translated into two spinor indices. Therefore, for any tensor of rank two,

$$X^{\mu\nu} \longrightarrow X_{\alpha\beta\dot{\alpha}\dot{\beta}} = X^{\mu\nu} \left(\sigma_\mu\right)_{\alpha\dot{\alpha}} \left(\sigma_\nu\right)_{\beta\dot{\beta}} \, . \tag{1.89}$$

Denoting symmetrization and anti-symmetrization in spinor indices by () and by [], respectively, the tensor $X_{\alpha\beta\dot{\alpha}\dot{\beta}}$ can be decomposed as

$$\begin{aligned} X_{\alpha\beta\dot{\alpha}\dot{\beta}} &= X_{(\alpha\beta)(\dot{\alpha}\dot{\beta})} + X_{[\alpha\beta][\dot{\alpha}\dot{\beta}]} + X_{(\alpha\beta)[\dot{\alpha}\dot{\beta}]} + X_{[\alpha\beta](\dot{\alpha}\dot{\beta})} \\ &= X_{(\alpha\beta)(\dot{\alpha}\dot{\beta})} + \epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}X + \epsilon_{\dot{\alpha}\dot{\beta}}X_{(\alpha\beta)} + \epsilon_{\alpha\beta}X_{(\dot{\alpha}\dot{\beta})} \, . \end{aligned} \tag{1.90}$$

The irreducible components of $X_{\alpha\beta\dot{\alpha}\dot{\beta}}$ are given by

$$\begin{aligned} X &= \frac{1}{4}\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}X_{\alpha\beta\dot{\alpha}\dot{\beta}} \\ X_{(\alpha\beta)} &= -\frac{1}{2}\epsilon^{\dot{\alpha}\dot{\beta}}X_{(\alpha\beta)\dot{\alpha}\dot{\beta}} \\ X_{(\dot{\alpha}\dot{\beta})} &= -\frac{1}{2}\epsilon^{\alpha\beta}X_{\alpha\beta(\dot{\alpha}\dot{\beta})} \, . \end{aligned} \tag{1.91}$$

The pre-factors of $1/4$ and $-1/2$ can be deduced by applying Eq. (1.66). If $X^{\mu\nu}$ is a symmetric tensor, i.e. if

$$X^{\mu\nu} = X^{\nu\mu} \tag{1.92}$$

then, obviously, the last two terms of Eq. (1.90) vanish. If $X^{\mu\nu}$ is also traceless, i.e. if $\mathrm{Tr}(X) = X^\mu_\mu = 0$, then also the second term of this decomposition is zero. In contrast, for any antisymmetric tensor

$$X^{\mu\nu} = -X^{\nu\mu} \tag{1.93}$$

only the last two terms yield a non-vanishing result. Phrased in other words, any antisymmetric tensor is equivalent to pairs of symmetric spinors. To construct a mapping similar to the one encountered for four vectors the following matrices will be introduced:

$$\begin{aligned} \left(\sigma^{\mu\nu}\right)_\alpha^{\ \beta} &= \frac{i}{2}\left(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu\right)_\alpha^{\ \beta} \\ \left(\bar{\sigma}^{\mu\nu}\right)^{\dot{\alpha}}_{\ \dot{\beta}} &= \frac{i}{2}\left(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu\right)^{\dot{\alpha}}_{\ \dot{\beta}} \, , \end{aligned} \tag{1.94}$$

where the form Eq. (1.74) for the σ and $\bar{\sigma}$ has been used. Again, to raise and lower spinor indices, ϵ is used, yielding

$$
\begin{aligned}
(\sigma^{\mu\nu})_{\alpha\beta} &= \epsilon_{\beta\gamma}\,(\sigma^{\mu\nu})_{\alpha}^{\ \gamma} = (\sigma^{\mu\nu})_{\beta\alpha} \\
(\bar{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}} &= \epsilon_{\dot{\alpha}\dot{\gamma}}\,(\bar{\sigma}^{\mu\nu})^{\dot{\gamma}}_{\ \dot{\beta}} = (\bar{\sigma}^{\mu\nu})_{\dot{\beta}\dot{\alpha}}\,.
\end{aligned}
\tag{1.95}
$$

Furthermore,

$$
\sigma^{\mu\nu} = +\frac{i}{2}\epsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma}\,, \quad \bar{\sigma}^{\mu\nu} = -\frac{i}{2}\epsilon^{\mu\nu\rho\sigma}\bar{\sigma}_{\rho\sigma}\,, \tag{1.96}
$$

$$
(\sigma^{\mu\nu}\epsilon)_{\alpha\beta} = \left[\left(\epsilon^T\bar{\sigma}^{\mu\nu}\epsilon\right)_{\dot{\alpha}\dot{\beta}}\right]^* \tag{1.97}
$$

and

$$
\begin{aligned}
\frac{1}{2}\mathrm{Tr}\left[\sigma^{\mu\nu}\sigma_{\rho\sigma}\right] &= \delta^{\mu\nu}_{\rho\sigma} + i\epsilon^{\mu\nu}_{\rho\sigma} \\
\frac{1}{2}\mathrm{Tr}\left[\bar{\sigma}^{\mu\nu}\bar{\sigma}_{\rho\sigma}\right] &= \delta^{\mu\nu}_{\rho\sigma} - i\epsilon^{\mu\nu}_{\rho\sigma}
\end{aligned}
\tag{1.98}
$$

hold true. The tensors of the last identities are given by

$$
\delta^{\mu\nu}_{\rho\sigma} = \delta^{\mu}_{\rho}\delta^{\nu}_{\sigma} - \delta^{\mu}_{\sigma}\delta^{\nu}_{\rho}\,, \quad \text{and} \quad \epsilon^{\mu\nu}_{\rho\sigma} = g_{\rho\kappa}g_{\sigma\lambda}\epsilon^{\mu\nu\kappa\lambda}\,. \tag{1.99}
$$

Then, for antisymmetric tensors, the spinor decomposition of Eq. (1.90) reads

$$
X_{\alpha\beta\dot{\alpha}\dot{\beta}} = X^{\mu\nu}\,(\sigma_\mu)_{\alpha\dot{\alpha}}\,(\sigma_\nu)_{\beta\dot{\beta}} = \epsilon_{\alpha\beta}X_{(\dot{\alpha}\dot{\beta})} + \epsilon_{\dot{\alpha}\dot{\beta}}X_{(\alpha\beta)}\,. \tag{1.100}
$$

Multiplying the second part of this equation with $\epsilon^{\alpha\beta}$, and using Eq. (1.66) and the fact that, due to symmetry properties in the spinor indices, $\epsilon^{\alpha\beta}X_{(\alpha\beta)} = 0$, yields

$$
-2X_{(\dot{\alpha}\dot{\beta})} = X^{\mu\nu}\,(\sigma_\mu)_{\alpha\dot{\alpha}}\,(\sigma_\nu)_{\beta\dot{\beta}}\,\epsilon^{\alpha\beta}\,. \tag{1.101}
$$

Multiplying both sides with $\epsilon^{(\dot{\gamma}\dot{\alpha})}$, and transforming σ_μ into $\bar{\sigma}_\mu$ results in

$$
\begin{aligned}
-2X_{(\dot{\alpha}\dot{\beta})}\epsilon^{\dot{\gamma}\dot{\alpha}} &= X^{\mu\nu}\,(\bar{\sigma}_\mu)^{\dot{\gamma}\beta}\,(\sigma_\nu)_{\beta\dot{\beta}} \\
&= \frac{1}{2}X^{\mu\nu}\left[(\bar{\sigma}_\mu\sigma_\nu - \bar{\sigma}_\nu\sigma_\mu) + (\bar{\sigma}_\mu\sigma_\nu + \bar{\sigma}_\nu\sigma_\mu)\right]^{\dot{\gamma}}_{\ \dot{\beta}} \\
&= -iX^{\mu\nu}\,(\bar{\sigma}_{\mu\nu})^{\dot{\gamma}}_{\ \dot{\beta}}\,,
\end{aligned}
\tag{1.102}
$$

where the second term in the square brackets vanishes after applying Eq. (1.78) due to its symmetry in the Lorentz indices. For the other component similar considerations apply and finally the two spinor tensors $X_{(\dot\alpha\dot\beta)}$ and $X_{(\alpha\beta)}$ are given by

$$X_{(\alpha\beta)} = \tfrac{i}{2}\left(\sigma_{\mu\nu}\epsilon\right)_{\alpha\beta} X^{\mu\nu} \quad X_{(\dot\alpha\dot\beta)} = -\tfrac{i}{2}\left(\epsilon^T\bar\sigma_{\mu\nu}\right)_{\dot\alpha\dot\beta} X^{\mu\nu}$$

$$X^{\mu\nu} = \frac{i}{4}\left(\epsilon\sigma^{\mu\nu}\right)^{\alpha\beta} X_{(\alpha\beta)} - \frac{i}{4}\left(\bar\sigma^{\mu\nu}\epsilon^T\right)^{\dot\alpha\dot\beta} X_{(\dot\alpha\dot\beta)}\,. \tag{1.103}$$

Introducing the notion of (anti-) self-dual tensors

$$F^{\pm\mu\nu} = \pm i\epsilon^{\mu\nu\rho\sigma}F^{\pm}_{\rho\sigma}\,, \tag{1.104}$$

where the minus-sign corresponds to anti-self-dual tensors, it is easy to check that self-dual tensors correspond to the un-dotted spinor representation whereas the dotted spinor representation is for the anti-self-dual tensors. It is worth noting that real anti-symmetric tensors of rank two, like for instance the electromagnetic field strength tensor, can always be decomposed into a self-dual and an anti-self-dual component. Defining

$$\tilde F^{\mu\nu} = i\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} \tag{1.105}$$

this decomposition reads

$$F^{\pm}_{\mu\nu} = \frac{1}{2}\left(F_{\mu\nu} \pm \tilde F_{\mu\nu}\right)\,. \tag{1.106}$$

It follows that the corresponding spinor representations are mutually complex conjugated,

$$\bar F_{\dot\alpha\dot\beta} = \left(F_{\alpha\beta}\right)^* \tag{1.107}$$

and have also six parameters. Going back to the $\sigma_{\mu\nu}$, it can be checked explicitly that

$$\left(\sigma_{23},\,\sigma_{31},\,\sigma_{12}\right)^T = \vec\sigma \quad \text{and} \quad \left(\sigma_{10},\,\sigma_{20},\,\sigma_{30}\right)^T = \vec\sigma\,. \tag{1.108}$$

Therefore, identifying entries in the (0, 1/2) and (1/2, 0) representations of the Lorentz algebra for dotted and un-dotted spinors, Eqs. (1.59) and (1.63), respectively, Eq. (1.53) reads for spinors

$$\begin{aligned} a_\alpha^{\ \beta} &= \mathcal{D}^{(0,\,1/2)}(\Lambda)_\alpha^{\ \beta} = \left[\exp\left(\frac{i}{4}\varepsilon^{\mu\nu}\sigma_{\mu\nu}\right)\right]_\alpha^{\ \beta}\,,\\ a^{*\ \dot\beta}_{\ \dot\alpha} &= \left(\mathcal{D}^{(1/2,\,0)}(\Lambda)^*\right)_{\dot\alpha}^{\ \dot\beta} = \left[\exp\left(-\frac{i}{4}\varepsilon^{\mu\nu}\bar\sigma_{\mu\nu}\right)\right]_{\dot\alpha}^{\ \dot\beta}\,. \end{aligned} \tag{1.109}$$

1.2 Noether's theorem

1.2.1 Stating the theorem

So far, Lorentz transformation and their properties have been discussed. Lorentz invariance as well as other symmetries are guiding principles for the construction of theories and their Lagrange densities (also dubbed Lagrangians). In general, symmetries are connected to invariance of the action integral. Noether's theorem states that each symmetry in the action is connected to a conserved quantity. In the following action integrals of fields are considered. The corresponding Lagrangians can be understood as functions of these fields ϕ_i and their first derivative,

$$S[\phi] = \int d^4x \mathcal{L}(\phi_i(x), \partial_\mu \phi_i(x))\,, \tag{1.110}$$

where $d^4x = dx^0dx^1dx^2dx^3$ and $\partial_\mu = \partial/\partial^\mu$. In classical theory the action S has definite units of angular momentum, thus, in quantum theory it is in corresponding units of $\hbar$, in the notation of this lecture, $\hbar \equiv 1$. In other words, in the notation employed here, the action is "dimensionless", and, therefore, the Lagrangian has dimension of inverse length to the fourth.
As usual the equations of motion can be obtained through the extremal principle as Euler-Lagrange equations,

$$\begin{aligned}
\delta S &= \int d^4x \delta\mathcal{L} = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi_i(x)}\delta\phi_i(x) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i(x))}\delta(\partial_\mu\phi_i(x))\right] \\
&= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi_i(x)} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i(x))}\right)\right]\delta\phi_i(x) + \\
&\qquad \int d^4x \left[\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i(x))}\right)\delta\phi_i(x)\right]\,,
\end{aligned} \tag{1.111}$$

where the transition from the first to the second line holds true only, if x does not change as well in the variation. Then

$$\delta\left(\partial_\mu\phi_i(x)\right) = \partial_\mu\left(\delta\phi_i(x)\right)\,. \tag{1.112}$$

The last term of the result of Eq. (1.111) is a surface term that vanishes when demanding that the variation of the fields vanishes on the surface of the four-dimensional integration region. By requiring S to be stationary under such variations of the field(s) $\phi_i(x)$ the Euler-Lagrange equations for $\phi_i(x)$ can be obtained,

$$\frac{\partial \mathcal{L}}{\partial \phi_i(x)} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i(x))} \right) = 0\,, \tag{1.113}$$

the classical field equations. They can be identified as the functional derivative of S with respect to $\phi_i(x)$. Note that dropping the surface terms results in an invariance of these equations under the transformation

$$\mathcal{L}' = \mathcal{L} + \partial_\mu L^\mu\,, \tag{1.114}$$

i.e. under adding a total derivative of a vector.
Consider now invariance of the action under infinitesimal transformations

$$\phi_i(x) \longrightarrow \phi_i'(x) = \phi_i(x) + \varepsilon G_i\left[\phi(x)\right]\,, \tag{1.115}$$

where $\phi(x)$ denotes all fields. In this case, the variation of the Lagrangian is

$$\delta\mathcal{L} = \mathcal{L}(\phi_i'(x), \partial_\mu \phi_i'(x)) - \mathcal{L}(\phi_i(x), \partial_\mu \phi_i(x)) = \varepsilon \partial_\mu X^\mu(\phi_i(x))\,. \tag{1.116}$$

On the other hand, using Eq. (1.115), the Lagrangian changes like

$$\begin{aligned} \delta\mathcal{L} &= \varepsilon \frac{\partial \mathcal{L}}{\partial \phi_i} G_i(\phi) + \varepsilon \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \partial_\mu G_i(\phi) \\ &= \varepsilon \left[G_i(\phi)\, \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\, \partial_\mu G_i(\phi) \right]\,, \end{aligned} \tag{1.117}$$

where the Euler-Lagrange equation, Eq. (1.113), and the independence of the variation parameter from space-time have been used. Equating both equations,

$$\partial_\mu X^\mu(\phi_i(x)) = \left[G_i(\phi)\, \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\, \partial_\mu G_i(\phi) \right] \tag{1.118}$$

the conserved current

$$j^\mu(x) = G_i(\phi) \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right) - X^\mu(\phi)\,, \tag{1.119}$$

fulfilling

$$\partial_\mu j^\mu(\phi) = 0 \tag{1.120}$$

can be read-off. This entity is called the Noether-current. Integrating over the spatial components, it leads to the conserved charge corresponding to it,

$$Q = \int d^3\vec{x} j^{\mu=0}(x) \ , \quad \frac{dQ}{dt} = 0 \,. \tag{1.121}$$

The important thing about this charge generates the infinitesimal transformations above, Eq. (1.115),

$$[iQ,\, \phi_i(x)] = G_i[\phi(x)] \,. \tag{1.122}$$

1.2.2 First consequences of Noether's theorem

Energy-momentum tensor

Consider linear translations under which the fields transform as

$$\phi_i'(x') = \phi_i(x) \,. \tag{1.123}$$

Demanding that the fields stay unchanged at the same points,

$$\delta_L \phi_i(x) = \phi_i'(x) - \phi_i(x) = \varepsilon^\rho \partial_\rho \phi_i(x) \tag{1.124}$$

for infinitesimal translations. If the Lagrangian does not depend explicitly on space-time,

$$\partial_L \mathcal{L} = \mathcal{L}(\phi_i'(x), \partial_\mu \phi_i'(x)) - \mathcal{L}(\phi_i(x), \partial_\mu \phi_i(x)) = \varepsilon^\rho \partial_\rho \mathcal{L} \,. \tag{1.125}$$

Hence,

$$G_{\rho,\,i}(\phi(x)) = \partial_\rho \phi_i(x) \,, \quad X_\rho^\mu = \delta_\rho^\mu \mathcal{L}(\phi, \partial_\mu \phi) \tag{1.126}$$

and the Noether-current reads

$$j_\rho^{\ \mu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \partial_\rho \phi_i - \delta_\rho^\mu \mathcal{L}(\phi, \partial_\mu \phi) = T_\rho^{\ \mu} \,. \tag{1.127}$$

This is the energy-momentum tensor.

Angular momentum tensor

Turning to infinitesimal Lorentz transformations applied on the fields, and reconciling Eqs. (1.29,1.30),

$$\phi'_i(x') = \left[1 - \frac{i}{2}\varepsilon^{\rho\sigma}\hat{S}_{\rho\sigma}\right]_i^{\ j} \phi_j(x)\,. \tag{1.128}$$

But, after Taylor expansion, also

$$\phi'_i(x') = \left[1 + \frac{1}{2}\varepsilon^{\rho\sigma}\left(x_\sigma\partial_\rho - x_\rho\partial_\sigma\right)\right]\phi'_i(x) = \left[1 + \frac{i}{2}\varepsilon^{\rho\sigma}\hat{L}_{\sigma\rho}\right]\phi'_i(x)\,. \tag{1.129}$$

Therefore, up to first order in the small parameter ε,

$$\delta_L\phi_i(x) = \phi'_i(x) - \phi_i(x) = -\frac{i}{2}\varepsilon^{\rho\sigma}\left[\left(\hat{L}_{\rho\sigma}\right)\delta_i^{\ j} + \left(\hat{S}_{\rho\sigma}\right)_i^{\ j}\right]\phi_j(x) \tag{1.130}$$

But since in a Lorentz invariant theory Lagrangians are scalars, its variation reads

$$\delta_L\mathcal{L} = \frac{i}{2}\varepsilon^{\rho\sigma}\hat{L}_{\sigma\rho}\mathcal{L}\,. \tag{1.131}$$

Therefore, the entities constituting the Noether current are

$$\begin{aligned} G_{\rho\sigma,\,i}(\phi) &= -i\left[\left(\hat{L}_{\rho\sigma}\right)\delta_i^{\ j} + \left(\hat{S}_{\rho\sigma}\right)_i^{\ j}\right]\phi_j \\ X^\mu_{\rho\sigma}(\phi) &= \left(x_\rho\delta^\mu_\sigma - x_\sigma\delta^\mu_\rho\right)\mathcal{L} \end{aligned} \tag{1.132}$$

and the current reads

$$j_{\rho\sigma}^{\ \ \mu}(\phi) = x_\rho T_\sigma^{\ \mu} - x_\sigma T_\rho^{\ \mu} - i\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\left(\hat{S}_{\rho\sigma}\right)_i^{\ j}\phi_j \equiv \mathcal{M}_{\rho\sigma}^{\ \ \mu}\,, \tag{1.133}$$

the angular momentum density.
Taken together, the conserved currents that correspond to the two conserved currents of Eqs. (1.127) and (1.133) are the generators of the Poincare group, P_ρ and $M_{\rho\sigma}$,

$$P_\rho = \int d^3x T_\rho^{\ 0} \quad \text{and} \quad M_{\rho\sigma} = \int d^3x M_{\rho\sigma}^{\ \ 0}\,. \tag{1.134}$$

Non-uniqueness of the Noether current

The demand to have a conserved current according to

$$\partial_\mu j^\mu = 0 \tag{1.135}$$

does not fix the current completely. In fact, if $f^{\mu\nu}$ is a totally antisymmetric tensor of rank two, then, also

$$\tilde{j}^\mu = j^\mu + \partial_\nu f^{\mu\nu} \tag{1.136}$$

is a conserved current. This is important when considering massless theories leading notoriously to massless one-particle states resulting in non-convergence of the spatial integral leading to the conserved charge. Then, by choosing and appropriate tensor, one can make these contributions vanish.

1.3 Actions for field theories

1.3.1 General considerations

When discussing Noether's theorem the general form of the action has been given as, c.f. Eq. (1.110),

$$S[\phi] = \int d^4x \mathcal{L}(\phi_i(x), \partial_\mu \phi_i(x)) \,. \tag{1.137}$$

This integral has the following properties:

1. Locality:
   This is obvious when realizing that only one space-time coordinate appears and is integrated over. This immediately leads to local interactions.

2. $\mathcal{L}$ is real (in field theory: Hermite-an):
   In classical mechanics this means that the energy is a real quantity, in field theory, this boils down to having a stable and "meaningful" time evolution of probabilities.

3. $\mathcal{L}$ depends only on fields and their derivatives (with up to second derivative w.r.t. time):

In field theory, higher derivatives would lead either to states with negative norm (i.e. negative probability, not an appealing concept), or to particles moving with superluminal velocities (i.e. tachyons, usually considered to be a sign of an unhealthy theory).

4. Poincare invariance:
   In other words, $\mathcal{L}$ must not depend explicitly on x.

5. Renormalizability:
   In theories living in four dimensional this means that the Lagrangian must have dimension four or less in every single term.

1.3.2 Scalar theories

Neutral scalar fields

Starting with a single real field, the most general Lagrangian is of the form

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - V(\phi)\,, \tag{1.138}$$

where the first time is the kinetic term and the second term is the potential term, giving rise to interactions etc.. The plus sign in front of the kinetic term reflects the demand to have positive energies. Expanding the potential in powers of ϕ is is obvious that any term linear in ϕ can be absorbed in a redefinition $\phi(x) \to \phi(x) + a$. Thus,

$$V(\phi) = \frac{1}{2}\mu^2\phi^2 + \frac{g}{3!}\phi^3 + \frac{\lambda}{4!}\phi^4 \dots\,. \tag{1.139}$$

Considering physical units in field theory, space and time both have dimensions -1, mass and energy have dimension $+1$ (in units of energy). Since the action, in the units used, has dimension equal to zero, the Lagrangian has the inverse dimension of $d^n x$ in an n-dimensional space, i.e. $+n$. By inspection of the kinetic term, it can be seen that the real scalar field has dimension

$$\dim[\phi(x)] = \frac{n-2}{2} \xrightarrow{n\to 4} 1\,. \tag{1.140}$$

Renormalizability disallows mass dimensions larger than n for the individual terms in the Lagrangian, therefore, in four dimensions, the expansion of the potential stops at the quartic term. Demanding symmetry under reflections of ϕ, $\phi \to -\phi$, such a potential in four dimensions is of the form

$$V(\phi) = \frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4!}\phi^4 \,. \tag{1.141}$$

This potential, together with the kinetic term for the ϕ defines a very simple field theory, called the $\lambda\phi^4$-theory. This theory plays a significant role in the mechanism of spontaneous symmetry breaking. Its Euler-Lagrange equation reads

$$\left(\partial_\mu\partial^\mu + \mu^2\right)\phi = -\frac{\lambda}{3!}\phi^3 \,. \tag{1.142}$$

In field theory, the term $\sim \mu^2$ in the potential of Eq. (1.141) is called the mass term, the other term constitutes the interaction term.

Charged scalar fields

The $\lambda\phi^4$ theory can also be formulated as a theory for a complex scalar field. Then its Lagrangian is given by

$$\mathcal{L} = (\partial_\mu\phi)(\partial^\mu\phi^*) - \mu^2\phi\phi^* - \frac{\lambda}{2}(\phi\phi^*)^2 \,. \tag{1.143}$$

Apart from some simple factors this theory has another difference with respect to the neutral (real) field theory: It also enjoys a $U(1)$ symmetry. In other words, under transformations

$$\begin{aligned} \phi(x) \quad &\longrightarrow \quad \phi'(x) \;\; = e^{i\theta}\;\phi(x) \\ \phi^*(x) \quad &\longrightarrow \quad \phi^{*\prime}(x) = e^{-i\theta}\;\phi^*(x) \end{aligned} \tag{1.144}$$

the Lagrangian remains invariant. This symmetry, basically invariance under re-phasing of the fields, has a continuous group structure that can be easily identified as the simple $U(1)$ with θ being the parameter of the phase transformation. The infinitesimal version of this is

$$\delta\phi = \phi'(x) - \phi(x) = i\theta\phi(x) \quad \text{and} \quad \delta\phi^* = \phi^{*\prime}(x) - \phi^*(x) = -i\theta\phi^*(x) \tag{1.145}$$

leading to the Noether current

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}i\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}(-i\phi^*) = -i\left(\phi^*\partial_\mu\phi - \partial_\mu\phi^*\cdot\phi\right) = -i\phi^*\overset{\leftrightarrow}{\partial}\phi \,. \tag{1.146}$$

The corresponding charge is

$$Q = \int d^3x j^0(x) = i \int d^3x \left(\dot{\phi}^*\phi - \phi^*\dot{\phi}\right) . \tag{1.147}$$

As mentioned already before, the commutators of this charge with the fields generates the infinitesimal transformation. Thus,

$$[iQ,\, \phi(x)] = i\phi(x) \qquad \text{and} \quad [iQ,\, \phi^*(x)] = i\phi^*(x)\,. \tag{1.148}$$

The commutators above immediately show that the fields ϕ and ϕ^* carry conserved charges ± 1, respectively. This is in striking contrast to real fields (it is impossible to apply a phase transition of this kind to a single real field) and, therefore, real fields are neutral.

$O(N)$-scalar fields

Consider now an N-dimensional scalar field $\Phi = (\phi_1,\, \phi_2,\, \ldots,\, \phi_n)^T$. Its Lagrangian in a $\lambda\phi^4$ type of theory reads

$$\mathcal{L} = \frac{1}{2}\sum_{i=1}^{n}\left[(\partial_\mu\phi_i)(\partial^\mu\phi_i) - \mu^2\phi_i^2\right] - \left[\frac{\lambda}{8}\sum_{i=1}^{n}\phi_i^2\right]^2 . \tag{1.149}$$

The pictorial interpretation of this model is of the fields ϕ_i being an N-dimensional vector Φ, and the theory then enjoys an $O(N)$-symmetry under rotations of Φ. The case $N = 2$ can be simply connected to the charged fields discussed before. Just rewrite

$$\phi = \frac{\phi_1 + i\phi_2}{\sqrt{2}} \quad \text{and} \quad \phi^* = \frac{\phi_1 - i\phi_2}{\sqrt{2}}\,. \tag{1.150}$$

In other words, any scalar charged (complex) field can be represented by two scalar neutral (real) fields that enjoy an $O(2)$ symmetry. Re-phasing of such a model amounts to rotations in the ϕ_1-ϕ_2 plane in which the charged field ϕ and its complex conjugate ϕ^* are constructed.

1.3.3 Theories with spinors

$SL(2,\, C)$ spinors (two-component spinors)

Having discussed scalar fields a Lagrange density will be constructed for free two-component spinors which represent the simplest possible non-trivial fields with respect to the Lorentz-transformation. Such a Lagrangian consists of two parts, a kinetic and a mass term. To build the kinetic term, the connection between four-vectors and spinor indices has to be employed. For the derivative, it reads, cf. Eq. (1.87)

$$\partial_\mu \longrightarrow \partial_{\alpha\dot{\beta}} = (\sigma_\mu)_{\alpha\dot{\beta}}\, \partial_\mu \,. \tag{1.151}$$

Therefore, the simplest Lorentz scalar containing at least one derivative and two spinors reads

$$\eta^{*\alpha}\, (\sigma_\mu)_{\alpha\dot{\beta}}\, \partial_\mu \eta^{\dot{\beta}} = \eta^\dagger \sigma^\mu \partial_\mu \eta \,, \tag{1.152}$$

where the definition of the dotted spinor by means of the complex conjugate one has been plugged in,

$$\left(\eta^{\dot{\alpha}}\right)^* = (\eta^*)^{\dot{\alpha}} \quad \text{and} \quad (\xi_\alpha)^* = (\xi^*)_\alpha \,. \tag{1.153}$$

To guaranteed that the first guess, Eq. (1.152), becomes a real quantity, its Hermitian conjugate

$$\left(\partial_\mu \eta^\dagger\right) \sigma^\mu \eta \tag{1.154}$$

has to come into the game. But the sum of both terms yields a total derivative which, of course, vanishes such that the action is zero. But subtracting and multiplying with i provides a meaningful result,

$$\mathcal{L}_L^{\text{kin.}} = \frac{i}{2}\left[\eta^\dagger \sigma^\mu \partial_\mu \eta - \left(\partial_\mu \eta^\dagger\right) \sigma^\mu \eta\right] = \frac{i}{2} \eta^\dagger \sigma^\mu \overset{\leftrightarrow}{\partial_\mu}\, \eta \,, \tag{1.155}$$

giving rise to the most simple kinetic term for dotted spinors η. In the same fashion, a kinetic term for un-dotted spinors can be constructed and reads

$$\mathcal{L}_R^{\text{kin.}} = \frac{i}{2}\left[\xi^*_{\dot{\beta}}\, (\bar{\sigma}^\mu)^{\dot{\beta}\alpha} \overset{\leftrightarrow}{\partial_\mu}\, \xi_\alpha\right] = \frac{i}{2} \xi^\dagger \bar{\sigma}^\mu \overset{\leftrightarrow}{\partial_\mu}\, \xi \,. \tag{1.156}$$

Both kinetic terms can be added, if the two spinor fields are independent from each other, then the combined Lagrangian $\mathcal{L}^{\text{kin.}} = \mathcal{L}_L^{\text{kin.}} + \mathcal{L}_R^{\text{kin.}}$ obeys a symmetry of the form

$$\begin{array}{rclrcl} \xi'(x) & = & e^{i\theta_R}\, \xi(x) & \xi^{*\prime}(x) & = & e^{-i\theta_R}\, \xi^*(x) \\ \eta'(x) & = & e^{i\theta_L}\, \eta(x) & \eta^{*\prime}(x) & = & e^{-i\theta_L}\, \eta^*(x) \end{array} \,. \tag{1.157}$$

Such a symmetry is called a chiral symmetry, it is symbolized by $U(1)_R \times U(1)_L$, since it has two parameters, the two phase angles. This symmetry again leads to Noether currents and conserved charges[2]. Since again the action is dimensionless, the Lagrangian has mass dimension n, the number of dimensions. Therefore, each spinor field has mass dimension

$$\dim[\xi] = \dim[\eta] = \frac{n-1}{2} \xrightarrow{n\to 4} \frac{3}{2}\,. \tag{1.158}$$

As a next step, mass terms will be constructed. Obviously, the simplest Lorentz-scalars that can be made up for two spinors are

$$\eta^{\dot\alpha}\epsilon_{\dot\alpha\dot\beta}\eta^{\dot\beta} = \eta^T\epsilon\eta \quad\text{and}\quad \xi^{\alpha}\epsilon_{\alpha\beta}\eta^{\beta} = \xi^T\epsilon\xi\,. \tag{1.159}$$

The only possibility to keep such terms different from zero is to assume anti-commutating numbers - which, of course, is more than reasonable, since the spinors are meant to describe particles with half-integer spin, i.e. fermions obeying Fermi statistic. Only then the result of the terms above,

$$\eta_1\eta_2 - \eta_2\eta_1 \quad\text{and}\quad \xi_1\xi_2 - \xi_2\xi_1 \tag{1.160}$$

yield finite results. Therefore, introducing complex masses, suitable mass terms read

$$\begin{aligned}\mathcal{L}_L^{\mathrm{m}} &= -\frac{1}{2}\left(m_L\eta^T\epsilon\eta - m_L^*\eta^\dagger\epsilon\eta^*\right)\\ \mathcal{L}_R^{\mathrm{m}} &= -\frac{1}{2}\left(m_R^*\xi^\dagger\epsilon\xi^* - m_R\xi^T\epsilon\xi\right)\,,\end{aligned} \tag{1.161}$$

where

$$\epsilon^\dagger = \epsilon^T = -\epsilon \tag{1.162}$$

and

$$(\theta\xi)^* = \xi^*\theta^* \tag{1.163}$$

for two Grassmann-numbers θ and ξ has been used. A mass term of the kind above is called a Majorana mass term. It breaks the chiral symmetry. To have a mass term that respects a somewhat limited version of chiral symmetry, consider the Dirac mass term

$$\mathcal{L}_D^{\mathrm{m}} = -\left(m\xi^\dagger\eta + m^*\eta^\dagger\xi\right) \tag{1.164}$$

[2]It should be stressed here that kinetic terms with more factors of derivatives lead to unphysical tachyonic states

that connects the two kinds of spinors and thus establishes a connection between them. It obeys a $U(1)_V$ symmetry in which both angles of the transformation above, Eq. (1.157), are set equal,

$$\left(\begin{array}{c}\xi'(x)\\ \eta'(x)\end{array}\right) = e^{i\theta}\left(\begin{array}{c}\xi(x)\\ \eta(x)\end{array}\right) \quad \text{and} \quad \left(\begin{array}{c}\xi^{*\prime}(x)\\ \eta^{*\prime}(x)\end{array}\right) = e^{-i\theta}\left(\begin{array}{c}\xi^*(x)\\ \eta^*(x)\end{array}\right) . \tag{1.165}$$

This transformation leaves an $U(1)_V$ charge Q_V conserved, particles are thus carrying a charge $Q_V = +1$, anti-particles have a charge $Q_V = -1$. Note that so far three complex masses have been introduced, m_L, m_R, and m, By re-phasing of the fields two of them can be set real.

Dirac spinors

Apart from the Dirac mass term, the spinors so far have been completely independent from each other. Only the introduction of the Dirac mass term, maybe motivated by the wish to have the symmetry $U(1)_V$, couples the two kinds of spinors. In such a case the introduction of bi-spinors is of great help to analyze the structure of the theory.
In a first step a bi-spinor field

$$\psi(x) = \left(\begin{array}{c}\xi_\alpha(x)\\ \eta^{\dot\alpha}(x)\end{array}\right) \tag{1.166}$$

is introduced. The four-dimensional analogue to the two-dimensional Pauli matrices are the Dirac matrices γ^μ, which, in chiral representation, read

$$\begin{aligned}\gamma^\mu &= \left(\begin{array}{cc}0 & (\sigma^\mu)_{\alpha\dot\beta}\\ (\bar\sigma^\mu)^{\dot\alpha\beta} & 0\end{array}\right)\\ \gamma^0 &= \left(\begin{array}{cc}0 & 1\\ 1 & 0\end{array}\right)\,,\quad \vec\gamma = \left(\begin{array}{cc}0 & -\vec\sigma\\ \vec\sigma & 0\end{array}\right) .\end{aligned} \tag{1.167}$$

Similar to the Pauli matrices, Eq. (1.78), the Dirac matrices anti-commute,

$$\{\gamma^\mu,\,\gamma^\nu\} \equiv \gamma^\mu,\,\gamma^\nu + \gamma^\nu,\,\gamma^\mu = 2g^{\mu\nu} . \tag{1.168}$$

The free Lagrangian for massive Dirac spinors with real mass m is then

$$\mathcal{L}_D = \mathcal{L}_D^{\text{kin.}} + \mathcal{L}_D^{\text{m}} = \frac{i}{2}\bar\psi\gamma^\mu \overleftrightarrow{\partial}_\mu \psi - m\bar\psi\psi = \frac{i}{2}\bar\psi \overleftrightarrow{\not\partial}\, \psi - m\bar\psi\psi\,, \tag{1.169}$$

where the Dirac-conjugated (barred) bi-spinor is given by

$$\bar{\psi} = \psi^\dagger \gamma^0 \,, \tag{1.170}$$

and the Feynman dagger is defined as

$$\partial\!\!\!/ \equiv \partial^\mu \gamma_\mu = \partial_\mu \gamma^\mu \,. \tag{1.171}$$

Furthermore, a matrix γ_5 is defined, which in chiral representation is

$$\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \,. \tag{1.172}$$

Obviously, the two spinors are eigenvectors of the matrix γ_5 with eigenvalues ± 1. This finding can be used to decompose the Dirac spinor into two chiral spinors with chirality ± 1 through

$$\psi_R = P_R \psi \equiv \frac{1+\gamma_5}{2}\psi = \begin{pmatrix} \xi_\alpha \\ 0 \end{pmatrix} \quad \text{and} \quad \psi_L = P_L \psi \equiv \frac{1-\gamma_5}{2}\psi = \begin{pmatrix} 0 \\ \eta^{\dot{\alpha}} \end{pmatrix} \,. \tag{1.173}$$

To reconstruct the representation of the Lorentz transformation for Dirac spinors consider the tensor corresponding to Eq. (1.94). It reads

$$\sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^\mu ,\, \gamma^\nu\right] = \begin{pmatrix} (\sigma^{\mu\nu})_\alpha^{\ \beta} & 0 \\ 0 & (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}_{\ \dot{\beta}} \end{pmatrix} \,, \tag{1.174}$$

and, therefore, the Lorentz-transformed Dirac spinor is

$$\psi'(x') = \exp\left(-\frac{i}{4}\varepsilon^{\mu\nu}\sigma_{\mu\nu}\right)\psi(x) = A\psi(x) \,. \tag{1.175}$$

The transformation matrix A satisfies

$$\bar{A}\gamma^\mu A = \Lambda^\mu_{\ \nu}\gamma^\nu \,, \text{where} \quad \bar{A} = \gamma_0 A^\dagger \gamma_0 \,. \tag{1.176}$$

Obviously, under Lorentz transformations the terms

$$\bar{\psi}\psi \,, \quad \bar{\psi}\gamma_5\psi \,, \quad \bar{\psi}\gamma_\mu\psi \,, \quad \bar{\psi}\gamma_\mu\gamma_5\psi \,, \quad \bar{\psi}\sigma_{\mu\nu}\psi \tag{1.177}$$

transform like scalars, pseudo-scalars, vectors, axial vectors, and tensors, respectively. Consider now, once more, the chiral transformations belonging to the group $U(1)_R \times U(1)_L$ acting on the spinors, but with opposite phases, $\theta = \theta_R = -\theta_L$,

$$\begin{array}{rclrcl} \xi'(x) & = & e^{i\theta}\,\xi(x) & \xi^{*\prime}(x) & = & e^{-i\theta}\,\xi^*(x) \\ \eta'(x) & = & e^{-i\theta}\,\eta(x) & \eta^{*\prime}(x) & = & e^{i\theta}\,\eta^*(x) \end{array} .$$

Plugging this into the definition of the Dirac spinor, two transformations emerge, namely, cf. Eq. (1.165),

$$\psi'(x) = e^{i\theta}\psi(x)\,, \quad \psi^{*\prime}(x) = e^{-i\theta}\psi^*(x) \tag{1.178}$$

and

$$\psi'(x) = e^{i\gamma_5\theta}\psi(x)\,, \quad \psi^{*\prime}(x) = e^{-i\gamma_5\theta}\psi^*(x)\,. \tag{1.179}$$

The latter transformation is called the chiral transformation, both lead to Noether currents, namely

$$j^\mu = -\bar{\psi}\gamma^\mu\psi \quad \text{and} \quad j_5^\mu = -\bar{\psi}\gamma^\mu\gamma^5\psi\,. \tag{1.180}$$

However, as mentioned before, the transformations $U(1)_V$ lead to conserved charges $Q = \pm 1$ associated with the particles $\psi = (\xi,\, \eta)T$ and $\psi^* = (\xi^*,\, \eta^*)^T$, respectively. A transformation connecting these two particles types ψ and ψ^* is dubbed charge conjugation. But just applying complex conjugation on the spinors would not do the trick, because $(\xi_\alpha)^* = \xi^*_{\dot{\alpha}}$ connecting dotted and un-dotted spinors. This is bad news, since the resulting bi-spinor after charge conjugation should still have a proper behavior under Lorentz transformations. This requires to have upper components of the bi-spinor being an un-dotted spinor. Therefore, upper and lower components of the bi-spinors have to be interchanged, and in each component indices have to be raised or lowered. This is realized by a transformation

$$\psi = \begin{pmatrix} \xi_\alpha \\ \eta^{\dot{\beta}} \end{pmatrix} \xrightarrow{C} \psi^{\mathcal{C}} = \begin{pmatrix} \eta^{*\beta}\epsilon_{\beta\alpha} \\ \epsilon^{\dot{\alpha}\dot{\beta}}\xi^*_{\dot{\beta}} \end{pmatrix} = i\gamma^2\psi^* \equiv \mathcal{C}\bar{\psi}^T\,. \tag{1.181}$$

Therefore, the 4×4 matrix $\mathcal{C}$ reads

$$\mathcal{C} = i\gamma^2\gamma^0 = -\mathcal{C}^{-1} = -\mathcal{C}^\dagger\,. \tag{1.182}$$

It obeys

$$\mathcal{C}^{-1}\gamma^\mu\mathcal{C} = -\gamma^{\mu T}\,. \tag{1.183}$$

This immediately results in the fact that the free Lagrangian for a massive Dirac spinor is invariant under charge conjugation, leaving the question of which operator is constructor or destructor a matter of convention.

Majorana spinors

Assume a Lagrangian for only one $SL(2, C)$ spinor, say η, with real mass m

$$\mathcal{L} = \frac{i}{2}\eta^\dagger\sigma^\mu \overleftrightarrow{\partial_\mu} \eta - \frac{m}{2}\left(\eta^T\epsilon\eta - \eta^\dagger\epsilon\eta^*\right) . \tag{1.184}$$

This can be cast into a form with bi-spinors by defining the Majorana spinor

$$\psi_M \equiv \begin{pmatrix} \eta^{*\beta}\epsilon_{\alpha\beta} \\ \eta^{\dot{\alpha}} \end{pmatrix} = \begin{pmatrix} -i\sigma_2\eta^* \\ \eta \end{pmatrix} . \tag{1.185}$$

Obviously, this spinor is its own charge conjugate,

$$\psi_M^{\mathcal{C}} = \psi_M \tag{1.186}$$

and in this sense a Majorana spinor is a real Dirac spinor, which can be used to describe neutral particles. In terms of this spinor the Lagrangian reads

$$\mathcal{L}_M = \frac{i}{4}\bar{\psi}_M\gamma^\mu \overleftrightarrow{\partial_\mu} \psi_M - \frac{m}{2}\bar{\psi}_M\psi_M . \tag{1.187}$$

1.4 Gauge symmetries

1.4.1 $U(1)$ gauge fields and electromagnetism

Local gauge transformations

Consider, once again, the Lagrangians for free charged massive scalar or Dirac fields,

$$\begin{aligned} \mathcal{L}_{\text{scalar}} &= (\partial_\mu\phi^*)(\partial^\mu\phi) - \mu^2\phi^*\phi - \frac{\lambda}{2}(\phi^*\phi)^2 \\ \mathcal{L}_{\text{Dirac}} &= \bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi . \end{aligned} \tag{1.188}$$

As has been discussed, these Lagrangians are invariant under global $U(1)$ transformations acting on the fields like

$$\begin{aligned} \phi'(x) &= e^{i\theta}\phi(x) , & \phi^{*\prime}(x) &= e^{-i\theta}\phi^*(x) \\ \psi'(x) &= e^{i\theta}\psi(x) , & \bar{\psi}'(x) &= e^{-i\theta}\bar{\psi}(x) , \end{aligned} \tag{1.189}$$

where the phase parameter θ is obviously independent of space-time, i.e. global. But, in principle, at different points in space-time, different observers should be allowed to have their own, private phase convention, and this convention should not affect the measurement. In other words, it seems to be more physical to ask for the invariance of the Lagrangians above under local $U(1)$ phase transformations

$$\theta = \theta(x)\,. \tag{1.190}$$

Such local phase transformations are called gauge transformations, invariance under them defines a gauge theory. But obviously, the kinetic terms of the Lagrangians above are not invariant under gauge transformations. For instance,

$$\begin{aligned}\partial_\mu\phi(x) \longrightarrow \partial_\mu\phi'(x) = \partial_\mu\left[e^{i\theta(x)}\phi(x)\right] &= i\left[(\partial_\mu\theta(x))\,e^{i\theta(x)}\phi(x) + e^{i\theta(x)}\partial_\mu\phi(x)\right]\\ &\neq ie^{i\theta(x)}\partial_\mu\phi(x)\,,\end{aligned} \tag{1.191}$$

and terms proportional to the derivative of the phase emerge. To restore the invariance under such local transformations, such extra terms have to be compensated for. This can be done by modifying the derivatives accordingly, in the simplest case by "subtracting" the phase. Such a subtraction is achieved by defining the covariant derivative, which, acting on a scalar field, reads

$$D_\mu\phi(x) \equiv \left[\partial_\mu - ieA_\mu(x)\right]\phi(x)\,, \tag{1.192}$$

where the gauge field A_μ has been introduced and the particles have been given a charge e. Invariance under local phase transformations is then guaranteed by demanding that the gauge fields transforms like

$$A_\mu(x) \longrightarrow A'_\mu(x) = A_\mu(x) + \frac{1}{e}\partial_\mu\theta(x)\,. \tag{1.193}$$

Then Eq. (1.191) with the covariant derivative replacing the ordinary one reads

$$\begin{aligned}(D_\mu\phi(x))' &= \left[\partial_\mu - ieA'_\mu(x)\right]\phi'(x)\\ &= i\left[(\partial_\mu\theta(x))\,e^{i\theta(x)}\phi(x) + e^{i\theta(x)}\partial_\mu\phi(x)\right] - \qquad (1.194)\\ &\qquad i\left[eA_\mu(x) + \partial_\mu\theta(x)\right]e^{i\theta(x)}\phi(x)\\ &= e^{i\theta(x)}\left[\partial_\mu - ieA_\mu(x)\right]\phi(x) = D_\mu\phi(x)\,. \qquad (1.195)\end{aligned}$$

The construction principle of the covariant derivative acting on an arbitrary field Φ with charge q that transforms like

$$\Phi'(x) = e^{iq\theta(x)}\Phi(x) \tag{1.196}$$

is such that

$$(D_\mu \Phi(x))' = e^{iq\theta(x)} D_\mu \Phi(x)\,, \tag{1.197}$$

i.e.

$$D_\mu \Phi(x) = \left[\partial_\mu - iqeA_\mu(x)\right] \Phi(x)\,. \tag{1.198}$$

Also,

$$D_\mu \Phi^*(x) = \left[D_\mu \Phi(x)\right]^*\,. \tag{1.199}$$

This fixes the Lagrangians above and it introduces an interaction of the gauge field with the matter fields. Taken together, Eq. (1.188) now reads

$$\begin{aligned}
\mathcal{L}_{\text{scalar}} &= (D_\mu\phi)^*(D^\mu\phi) - \mu^2\phi^*\phi - \frac{\lambda}{2}(\phi^*\phi)^2 \\
\mathcal{L}_{\text{Dirac}} &= \bar{\psi}\,(i\gamma^\mu D_\mu - m)\,\psi\,.
\end{aligned} \tag{1.200}$$

From these equations, also the dimension of the gauge fields can be read off:

$$\dim\left[A_\mu\right] = \dim\left[\partial_\mu\right] = 1\,. \tag{1.201}$$

Kinetic terms for the gauge fields

But theories described by the Lagrangians of Eq. (1.200) are pretty boring, because the gauge fields are not dynamical. To cure this and to add some spice to life, one would like to add kinetic terms for the gauge fields as well. Again, the demand to be gauge invariant provides a guiding principle. The most naive idea for a Lorentz scalar with two gauge fields, mass dimension of four, would be a term like

$$(\partial_\mu A^\mu(x))^2 \tag{1.202}$$

but it is quite obvious that such a term violates gauge invariance and is therefore not acceptable. A quantity that consists of only one derivative and is gauge invariant is the field strength tensor

$$F^{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,. \tag{1.203}$$

With its help, a suitable kinetic term can be found as

$$\mathcal{L}^{\text{kin.}}_{\text{gauge}} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}\,. \tag{1.204}$$

This term also has a number of nice properties. Plugging in the form of the field strength tensor,

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix} \tag{1.205}$$

it can be immediately seen that this term is equal to

$$\mathcal{L}^{\text{kin.}}_{\text{gauge}} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} = \frac{1}{2}(\vec{E}^2 - \vec{B}^2)\,, \tag{1.206}$$

the Lagrangian for the electromagnetic field. Then, using the tensor

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} = \begin{pmatrix} 0 & -B^1 & -B^2 & -B^3 \\ B^1 & 0 & E^3 & -E^2 \\ B^2 & -E^3 & 0 & E^1 \\ B^3 & E^2 & -E^1 & 0 \end{pmatrix} \tag{1.207}$$

the Euler-Lagrange equations can be written in compact form as

$$\partial_\mu \tilde{F}^\mu \nu = 0 \quad \text{and} \quad \partial_\mu F^\mu \nu = j^\nu \tag{1.208}$$

and it turns out that they are identical to the homogeneous and inhomogeneous Maxwell equations. Taken together, Lagrangians for scalar electrodynamics or spinor electrodynamics are given by

$$\begin{aligned} \mathcal{L}_{\text{scalar}} &= -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + (D_\mu\phi)^*(D^\mu\phi) - \mu^2\phi^*\phi - \frac{\lambda}{2}\left(\phi^*\phi\right)^2 \\ \mathcal{L}_{\text{Dirac}} &= -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi\,. \end{aligned} \tag{1.209}$$

Alternative formalsim: Infinitesimal gauge transformations

Consider a set of fields Φ, which are not necessarily real or spin-less, whose dynamics is characterized by a Lagrangian $\mathcal{L}(\Phi,\, \partial_\mu\Phi)$. Let $\mathcal{L}$ be invariant under a single parameter group of transformations

$$\Phi \to \Phi' = e^{iQ\omega}\Phi\,, \tag{1.210}$$

where Q is a Hermitian matrix, called the charge matrix. Conventionally, the fields can be chosen to be complex in such a fashion that Q is diagonal.

However, for the purposes here it is simpler to choose a set of real fields such that iQ is a real antisymmetric matrix like any other generator of a group. The associated infinitesimal transformation of the fields reads

$$\delta\Phi = iQ\delta\omega\Phi\,. \tag{1.211}$$

Following the path towards a gauge theory, take the parameter ω to depend on space-time. Then, the theory above is not invariant under these transformation any longer, since

$$\delta(\partial_\mu\Phi) = iQ[\delta\omega(\partial_\mu\Phi) + \Phi(\partial_\mu\delta\omega)]\,. \tag{1.212}$$

Obviously, the second term spoils the invariance. This is taken care of by introducing a new field A_μ, which transforms as

$$\delta A_\mu = -\frac{1}{g}\partial_\mu(\delta\omega)\,, \tag{1.213}$$

where g is an essentially free parameter, the coupling constant. Defining a gauge-covariant derivative D by

$$D_\mu\Phi = \partial_\mu\Phi + igQA_\mu\Phi \tag{1.214}$$

yields

$$\delta(D_\mu\Phi) = iQ\Phi\delta\omega \tag{1.215}$$

and the modified Lagrangian $\mathcal{L}(\Phi,\, D_\mu\Phi)$ is gauge-invariant. Adding in the kinetic term for the gauge field A_μ,

$$\mathcal{L}_{\text{kin}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,, \tag{1.216}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = [D_\mu,\, D_\nu] \tag{1.217}$$

completes the Lagrangian. This is nothing but the usual Lagrangian of minimally coupled electrodynamics with the usual interpretation of charged fields Φ, massless photons A_μ etc., as long as the dynamics of the Φ fields does not suffer a spontaneous breakdown of symmetry. But what happens if symmetry is spontaneously broken?

1.4.2 Extension to $SU(N)$

Matter fields

To extent the findings of the previous section from the electromagnetic $U(1)$ symmetry to the more general case of an $SU(N)$ symmetry one first has to find an analogue to the charged scalar and the Dirac fields of Eq. (1.188). Concentrating in the following on Dirac fields (for scalars, the line of reasoning is identical), first of all, the bi-spinors ψ have to be replaced by an N-component vector of bi-spinors

$$\psi = (\psi_1,\, \psi_2,\, \ldots,\, psi_N)^T\,. \tag{1.218}$$

Invariance of the Lagrangian under $SU(N)$ gauge transformations implies that the Lagrangian remains invariant under

$$\psi'_i(x) = [U(x)]_i^{\,j}\,\psi_j(x) = \left[e^{ig\theta_a(x)T_a}\right]_i^{\,j}\,\psi_j(x)\,, \tag{1.219}$$

where the T_a are the generators of the group G and g is the corresponding coupling constant. Defining now the gauge fields (for mathematicians: connection fields)

$$[A_\mu(x)]_i^{\,j} = \sum_{a=1}^{\dim(G)} A_\mu^a(x)\,[T_a]_i^{\,j} \tag{1.220}$$

the covariant derivative can be defined as

$$[D_\mu]_i^{\,j} = \partial_\mu\delta_i^{\,j} - ig\,[A_\mu(x)]_i^{\,j}\,. \tag{1.221}$$

The field A_μ transforms under the local gauge transformation $U(x)$ as

$$A_\mu(x) \longrightarrow A'_\mu(x) = U(x)A_\mu(x)U^\dagger(x) + \frac{i}{g}\left(\partial_\mu U(x)\right)\,U^\dagger(x)\,, \tag{1.222}$$

and, correspondingly, the covariant derivative transforms like

$$D_\mu(x) \longrightarrow D'_\mu(x) = U(x)D_\mu(x)U^\dagger(x)\,. \tag{1.223}$$

Under these transformations the Lagrangian of Eq. (1.209) remains invariant.

Gauge fields

Again, the gauge fields defined above have to be made dynamic. To do so, an analogue to the field strength tensor has to be constructed. It can be proven that this analogue reads

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig\left[A_\mu,\, A_\nu\right] = \frac{i}{g}\left[D_\mu,\, D_\nu\right] . \tag{1.224}$$

The commutator does not vanish, because the fields A_μ are products of individual gauge (vector) fields with the non-commuting generators of the gauge group. Under gauge transformations, the field strength tensor behaves as

$$F_{\mu\nu} \longrightarrow F'_{\mu\nu} = UF_{\mu\nu}U^\dagger \,, \tag{1.225}$$

and only the term

$$\mathrm{Tr}\left[F_{\mu\nu}F^{\mu\nu}\right] = F^a_{\mu\nu}F^{a\mu\nu} \,, \tag{1.226}$$

where the trace is over the generator indices of the group, is gauge invariant. Taken together, the Lagrange density for an $SU(N)$ gauge group with spinors reads

$$\mathcal{L} = -\frac{1}{4}F^{a\mu\nu}F^a_{\mu\nu} + \bar{\psi}^i\left[i\gamma^\mu\left(D_\mu\right)_i^{\;j} - m\delta_i^{\;j}\right]\psi_j \,. \tag{1.227}$$

Transformation properties of a non-Abelian model

Now the findings of the previous discussion will be applied to the case of a larger internal symmetry group, a non-Abelian gauge group. Global transformations of the form

$$\delta\Phi = T^a\delta\omega^a\Phi \tag{1.228}$$

leave such a theory invariant. Here, again, the T^a are the N generators of the gauge group. Considering local transformation, it is simple to see that the second term of the corresponding infinitesimal transformations

$$\delta(\partial_\mu\Phi) = T^a\delta\omega^a(\partial_\mu\Phi) + T^a(\partial_\mu\delta\omega^a)\Phi \tag{1.229}$$

spoils this invariance. As usual, this can be cured by introducing gauge fields. This time, N gauge fields A^a_μ, one for each generator, need to be introduced. Then the gauge-covariant derivative

$$D_\mu\Phi = \partial_\mu\Phi + gT^aA^a_\mu\Phi \tag{1.230}$$

with the coupling constant g can be constructed. The transformation properties of the gauge fields are defined such that

$$\delta(D_\mu\Phi) = T^a\delta\omega^a(D_\mu\Phi)\,. \tag{1.231}$$

This implies that

$$\delta A_\mu^a = f^{abc}\delta\omega^b A_\mu^c - \frac{1}{g}\partial_\mu\delta\omega^a\,. \tag{1.232}$$

Again the fs are the structure constants of the group, defined through the commutator of the generators. In fact both terms of the equation above are easily understood: The second term is just the generalization of the corresponding term in Abelian theories, cf. Eq. (1.213), the first term is need to ensure gauge-invariance under global transformations. It is merely a statement that the gauge fields transform like the group generators. Anyways, taken everything together, the emerging Lagrangian $\mathcal{L}(\Phi,\, D_\mu\Phi)$ is invariant under local gauge transformations.

In order to construct a kinetic term for the gauge fields, the field-strength tensor $F_{\mu\nu}$ of electromagnetism needs to be generalized to the non-Abelian case. The trick is to observe that in electromagnetism

$$(D_\mu D_\nu - D_\nu D_\mu)\Phi = iQF_{\mu\nu}\Phi \tag{1.233}$$

implying that $F_{\mu\nu}$ is a gauge-invariant quantity. In non-Abelian theories this is neatly generalized to

$$(D_\mu D_\nu - D_\nu D_\mu)\Phi = iT^aF_{\mu\nu}^a\Phi\,, \tag{1.234}$$

where

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b A_\nu^c\,. \tag{1.235}$$

It is easy to see that this field-strength tensor is gauge-covariant, i.e.

$$\delta F_{\mu\nu}^a = f^{abc}\delta\omega^b F_{\mu\nu}^c\,. \tag{1.236}$$

However, the quadratic form ${F_{\mu\nu}^a}^2$ is gauge-invariant and therefore is a good candidate for the kinetic term in the Lagrangian. The total Lagrangian for non-Abelian (Yang-Mills) theories therefore reads

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^a F_a^{\mu\nu} + \mathcal{L}(\Phi,\, D_\mu\Phi)\,. \tag{1.237}$$

Chapter 2

The Standard Model

2.1 Spontaneous symmetry breaking

2.1.1 Hidden symmetry and the Goldstone bosons

Simple examples

So far, some of the symmetry properties of the Lagrangians encountered and their consequences in the quantum world have been discussed. The most notable symmetry there has been an internal symmetry - gauge symmetry. Quantizing a system of fields which enjoys this symmetry one is forced to introduce the concept of a gauge-covariant derivative and re-write the kinetic term of the Lagrangian accordingly. In so doing, gauge fields emerge naturally. Quantizing these fields as well, a gauge has to be fixed, leading to further problems. They can be solved with the Faddeev-Popov procedure, cf. Sec. **??**.
However, in general there is no real reason to assume that the ground state of a Hamiltonian enjoys the same symmetries as the fields in the Hamiltonian. To exemplify this, consider a Hamiltonian describing protons and neutrons, forming a nucleus in its ground state, plus, eventually its excitations. It is clear that such a Hamiltonian can be constructed in a rotationally invariant way - just assure that the kinetic terms and the interactions do not favor any direction. Independent of this the nucleus described through such a Hamiltonian does not necessarily be rotationally invariant, i.e. of spin zero. Any other spin of this nucleus translates into the ground state having a symmetry different from the Hamiltonian. This is a trivial example and in fact, many

nuclei exhibit this behavior. However trivial such a behavior is for nuclei, its consequences become highly non-trivial if it appears in spatially infinite systems. One of the prime examples here is the Heisenberg ferromagnet. It is a crystal of spin-1/2 dipoles with nearest neighbor interactions of the spins, which are such that the spins have a tendency to align. In three dimensions the spins may be labelled by their position labels i, j, and k. The Hamiltonian thus reads

$$H = \sum_{i,j,k=-\infty}^{\infty} \left[\vec{\sigma}_{ijk} \cdot \vec{\sigma}_{i\pm1jk} + \vec{\sigma}_{ijk} \cdot \vec{\sigma}_{ij\pm1k} + \vec{\sigma}_{ijk} \cdot \vec{\sigma}_{ijk\pm1} \right] . \tag{2.1}$$

This Hamiltonian clearly is rotationally invariant, since the scalar products of vectors are rotationally invariant. The ground state however is not rotationally invariant: since the spins preferably are aligned in the ground state there will be an arbitrary direction into which the spins point. This translates into the ground state to be infinitely degenerate. This is because there are infinitely many directions to point into, each of which is connected with another ground state. A little man living inside such a ferromagnet would have an extremely hard time to detect the full rotational symmetry and the corresponding invariance of the laws of nature and the system around him. Basically all experiments he could do would have to deal with the background magnetic field stemming from the aligned dipoles. Depending o the extent to which he was able to shield his experimental apparatus from this background field he would detect an approximate rotational invariance or no invariance altogether. In any case he would have no reason to believe that rotational invariance indeed is an exact symmetry of the Hamiltonian describing his world. In addition his chances to realize that his world is just in one of the states of an infinite multiplet of equivalent ground states are null. Since he is of finite extent only (this is the real meaning of “little”) he would be able to alter the direction of a finite number of dipoles only - clearly not enough to move his world into another ground state.

In order to generalize the above discussion to the case of quantum field theory, the spins have to be replaced by fields, the rotational invariance by an internal symmetry such as gauge symmetry, the ground state becomes the vacuum state and the little man is us. This means that here the following conjecture is made: There are laws of nature with symmetries that are hard to detect for us, because the vacuum manifestly does not respect them. This situation usually is called “spontaneous symmetry breaking” - a deceptive terminology since the symmetry in fact is not broken, it is merely hidden.

To exemplify this consider the case of n real scalar fields ϕ_i organized in a vector Φ with Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi)(\partial^\mu \Phi) - \mathcal{U}(\Phi)\,, \tag{2.2}$$

where the potential $\mathcal{U}$ is a function of Φ and not of its derivatives. The energy density is given by the Hamiltonian

$$\mathcal{H} = \frac{1}{2}\left[(\partial_0 \Phi)^2 + (\vec{\nabla}\Phi)^2\right] + \mathcal{U}(\Phi)\,, \tag{2.3}$$

and thus the lowest energy state (the vacuum) is a state for which Φ is a constant, denoted by $\langle \Phi \rangle$. This value is called the vacuum expectation value and depends on the detailed dynamics of the theory, or, more specifically, by the location of the minima of the potential $\mathcal{U}$. Within this class of theories there are theories with spontaneously broken and those with manifest symmetries. The simplest theory is that of a single field ϕ (i.e. $n = 1$) with potential

$$\mathcal{U} = \frac{\lambda}{4!}\phi^4 + \frac{\mu^2}{2}\phi^2\,, \tag{2.4}$$

where the coupling constant λ is a positive real number and the mass term μ^2 may, despite its name, be either a positive or negative number. This theory obviously is invariant under reflections, i.e. under

$$\phi \to -\phi\,. \tag{2.5}$$

Depending on the sign of μ^2 the shape of the potential is quite different. If $\mu^2 > 0$, the minimum is at $\phi = 0$, and hence in this case

$$\langle \phi \rangle = 0 \quad \text{for} \quad \mu^2 > 0\,. \tag{2.6}$$

If $\mu^2 < 0$ the minimum is not at $\phi = 0$ any longer but rather there are two minima, given by ϕ_0 with

$$\phi_0^2 = -\frac{6\mu^2}{\lambda}\,. \tag{2.7}$$

Up to an irrelevant constant the potential can then be written as

$$\mathcal{U} = \frac{\lambda}{4!}\left(\phi^2 - \phi_0^2\right)^2\,. \tag{2.8}$$

Obviously the minima are located at $\phi = \pm\phi_0$, and due to the reflection symmetry it is irrelevant which one is chosen as the vacuum (or ground state, i.e. the state of least energy)[1]. However, independently of which one is the actual vacuum, the reflection symmetry is broken. To continue, chose $\langle\phi\rangle = \phi_0$ and define a new field

$$\phi' = \phi - \langle\phi\rangle := \phi - v \, . \tag{2.9}$$

In terms of this field, the potential reads

$$\mathcal{U} = \frac{\lambda}{4!}\left(\phi'^2 + 2v\phi'\right)^2 = \frac{\lambda}{4!}\phi'^4 + \frac{\lambda v}{6}\phi'^3 + \frac{\lambda v^2}{6}\phi'^2 \, . \tag{2.10}$$

This implies that the true mass of the visible particle connected to the field ϕ' is $\lambda\langle\phi\rangle^2/3$ rather than μ^2. Note also that due to the shift the potential exhibits a cubic term which would render detection of the true symmetry an even more complicated task.

Goldstone bosons in Abelian theories

A new phenomenon appears when considering the spontaneous breakdown of continuous symmetries. To highlight this consider a theory with two real scalar fields ϕ_1 and ϕ_2 with a potential given by

$$\begin{aligned} \mathcal{U} &= \frac{\lambda}{4!}\left[\phi_1^2 + \phi_2^2 - v^2\right]^2 \\ &= \frac{\lambda}{4!}\left[\phi_1^4 + \phi_2^4 + 2\phi_1^2\phi_2^2\right] - \frac{1}{2}\frac{\lambda v^2}{6}\left[\phi_1^2 + \phi_2^2\right] \, . \end{aligned} \tag{2.11}$$

This theory admits a two-dimensional rotation symmetry, $SO(2)$. The corresponding transformations of the fields are characterized by a rotation angle ω and read

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \longrightarrow \begin{pmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} . \tag{2.12}$$

[1] It should be noted here that, in principle, the ground state could also be given by the linear combination of the two minima. This is similar to the ground state of a quantum mechanical system with a double-well potential, where the two minima are connected through tunneling. The same reasoning would also apply here, and the system could eventually tunnel between the two minima. However, the further discussion will focus on systems with continuous symmetries and, hence, infinitely many minima. For such systems the tunneling probability from one ground state to another one is basically zero - just because there are infinitely many equivalent minima to chose from.

The minima of the potential are characterized by

$$\phi_1^2 + \phi_2^2 = v^2 \,, \tag{2.13}$$

and, just as before, one is free to chose one of the infinitely many minima. Also, just as before, although it is irrelevant which vacuum is chosen, the $SO(2)$ symmetry is broken. The choice made here is

$$\langle \phi_1 \rangle = v \,, \quad \langle \phi_2 \rangle = 0 \,. \tag{2.14}$$

Shifting the fields again,

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \longrightarrow \begin{pmatrix} \phi_1' \\ \phi_2' \end{pmatrix} = \begin{pmatrix} \phi_1' - \langle \phi_1 \rangle \\ \phi_2' - \langle \phi_2 \rangle \end{pmatrix} \,, \tag{2.15}$$

the new potential, again up to inconsequential constants, reads

$$\begin{aligned} \mathcal{U} &= \frac{\lambda}{4!} \left[\phi'^2_1 + \phi'^2_2 + 2v\phi'_1 \right]^2 \\ &= \frac{\lambda}{4!} \left[\phi'^4_1 + \phi'^4_2 + 2\phi'^2_1\phi'^2_2 \right] + \frac{\lambda v}{6} \left[\phi'^3_1 + \phi'^2_2\phi'_1 \right] + \frac{1}{2}\frac{\lambda v^2}{3}\phi'^2_1 \,. \end{aligned} \tag{2.16}$$

It becomes apparent immediately that the ϕ_1 field still has the same mass as before, but the other field ϕ_2 is now massless. Such a massless real scalar field is called a Goldstone boson. For the theories under consideration, the emergence of Goldstone bosons does not depend on the form of the potential $\mathcal{U}$. Rather, it is connected with the breaking of a symmetry. In particular, as it will turn out, there will be one Goldstone boson connected to each generator of the continuous symmetry group that is broken. To understand this better, it is useful to introduce new fields ρ and θ through

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \rho \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \,. \tag{2.17}$$

In terms of these fields the symmetry transformations of Eq. (2.12) become

$$\begin{aligned} \rho &\longrightarrow \rho \\ \theta &\longrightarrow \theta + \omega \,. \end{aligned} \tag{2.18}$$

Written in these variables, the $SO(2)$ invariance is just a more formal statement of the fact that the potential $\mathcal{U}$ does not depend on the "angular field"

θ. The transformation of these angular fields is, of course, ill-defined at the origin, and this is reflected in the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \rho)(\partial^\mu \rho) + \frac{1}{2}\rho^2(\partial_\mu \theta)(\partial^\mu \theta) - \mathcal{U}(\rho)\,. \tag{2.19}$$

However, this obstacle is not relevant here, since the perturbative expansion won't be around the origin but rather around a minimum characterized by

$$\langle \rho \rangle = v\,, \quad \langle \theta \rangle = 0\,. \tag{2.20}$$

Introducing, as before, shifted fields

$$\rho' = \rho - v\,, \quad \theta' = \theta \tag{2.21}$$

the Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \rho')(\partial^\mu \rho') + \frac{1}{2}(\rho' + v)^2(\partial_\mu \theta')(\partial^\mu \theta') - \mathcal{U}(\rho' + v)\,. \tag{2.22}$$

It is obvious from this expression that the rotational mode θ' is massless, because it enters only through its derivative. This has a simple geometrical interpretation: Since the vacuum states are those with a constant $\rho = v$, they differ only by different values of θ. In this respect the field θ connects the vacua. Choosing one of the vacua and expanding around it, no terms should appear that allow the connection to another vacuum.

The Goldstone theorem in a nutshell

The reasoning above can easily be extended to the case of non-Abelian theories. As an example consider n real scalar fields ϕ_i, organized in an n-dimensional vector Φ. Assume that the Lagrangian is invariant under a group of transformations

$$\Phi \to \Phi' = \exp\left[T^a \omega^a\right]\, \Phi\,, \tag{2.23}$$

where the ω^a are arbitrary real parameters and the T^a are the generators of the symmetry group. In the case considered here, these generators are real anti-symmetric $n \times n$ matrices, which satisfy the commutation relation

$$\left[T^a,\, T^b\right] = f^{abc} T^c \tag{2.24}$$

with the structure constants f^{abc}. Choosing the Ts to be orthonormal, these structure constants are antisymmetric. The corresponding infinitesimal transformations read

$$\delta\Phi = T^a \delta\omega^a \Phi \,. \tag{2.25}$$

Invariance of the Lagrangian implies that the transformed potential is

$$\mathcal{U}(\Phi) = \mathcal{U}\left(\exp\left[T^a\omega^a\right]\, \Phi\right) \,. \tag{2.26}$$

Consider now the subgroup that leaves the minima of $\mathcal{U}$ invariant. Depending on the structure of the potential and the symmetry group, this group may be anything between the trivial identity subgroup (i.e. the full group), translating into all generators being connected to a breakdown of symmetry, or the full group, i.e. no symmetry broken. In any case, the generators can always be chosen such that the first m generators ($n \geq m \geq 0$) leave the minima invariant. Formally, this can be written as

$$T^{a\in[1,\ldots m]}\langle\Phi\rangle = 0 \,. \tag{2.27}$$

By construction the remaining $n-m$ generators do not leave $\langle\Phi\rangle$ invariant. Thus we have, passing through $\langle\Phi\rangle$ an $n-m$-dimensional surface of constant $\mathcal{U}$. Thus, by the same arguments as before, there must be $n-m$ spin-less fields, one for each broken generator[2]. These fields are called Goldstone-bosons. The discussion above is a special case of the Goldstone theorem which can be formulated and proven in much larger generality: Given a field theory obeying the usual axioms (Lorentz-invariance, locality, properly defined Hilbert space with positive-definite inner product, ...), if there is a local conserved current such that the space integral of its time component (the associated charge) does not annihilate the vacuum, then the theory must contain a massless spin-less boson with the same internal symmetry and parity properties as the time component of the current.

2.1.2 The Higgs mechanism

Abelian model

At first sight this seems to be a killer to the idea that spontaneous breakdown of symmetries may be at work in the real world, since there is not a glimpse

[2]Note that here spin-less is the correct wording, not scalar or pseudo-scalar. This depends on whether parity is broken as well; in extreme cases the bosons may even be of mixed behavior under parity transformation, i.e. neither scalar nor pseudo-scalar.

of a hint that there is a massless spin-less particle around. However, there is a loophole. There are in fact perfectly respectable theories which do allow for an extra trick of nature. These are the gauge theories, of which QED is the most simple one. In fact, these theories do not fully satisfy the usual axioms named above, and this is related to gauge invariance and the enforced choice of gauge. If such theories are quantized in a covariant gauge, the norm is no longer positive-definite (remember the Gupta-Bleuler metric), if such a theory is quantized in another gauge, Lorentz-covariance is not manifestly satisfied.

To be definite, consider a Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(D_\mu\Phi)(D^\mu\Phi) - \frac{\lambda}{4!}\left(\Phi^2 - \Phi_0^2\right)^2 , \tag{2.28}$$

where $\Phi = (\phi_1, \phi_2)^T$. To investigate the effect of symmetry breaking, note that

$$D_\mu\Phi = \partial_\mu\Phi + gA_\mu\frac{\delta\Phi}{\delta\omega} . \tag{2.29}$$

Writing the fields again in the "angular" form, this can be cast into

$$\begin{aligned} D_\mu\rho' &= \partial_\mu\rho' \\ D_\mu\theta' &= \partial_\mu\theta' + gA_\mu . \end{aligned} \tag{2.30}$$

Inserting this into the Lagrangian yields

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_\mu\rho')(\partial^\mu\rho') \\ &\quad + \frac{1}{2}(\rho'+v)^2\,(\partial_\mu\theta' + gA_\mu)(\partial^\mu\theta' + gA^\mu) + \mathcal{U}(\rho'+v) . \end{aligned} \tag{2.31}$$

It is hard to read off the physical spectrum and properties of this Lagrangian for small oscillations around the vacuum because of terms proportional to $A_\mu\partial^\mu\theta$. These terms, however, can be eliminated by introducing

$$c_\mu = A_\mu + \frac{1}{g}\partial_\mu\theta . \tag{2.32}$$

Then, the Lagrangian reads

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4}(\partial_\mu c_\nu - \partial_\nu c_\mu)(\partial^\mu c^\nu - \partial^\nu c^\mu) + \frac{1}{2}(\partial_\mu\rho')(\partial^\mu\rho') \\ &\quad + \frac{g}{2}(\rho'+v)^2\,c_\mu c^\mu + \mathcal{U}(\rho'+v) , \end{aligned} \tag{2.33}$$

and is wonderfully quadratic in the dynamic fields ρ' and c_μ. But what is this? The rotational mode θ has been absorbed by the field A_μ, which, like any other gauge field, was massless. After "eating" the Goldstone boson, however, the photon becomes a massive spin-1 particle with mass

$$m_c^2 = g^2 v^2 \,. \tag{2.34}$$

This looks a bit awkward, since it seems as if some fields have been lost. Counting degrees of freedom however, it becomes apparent that their number remains the same. To understand this, just note that massive spin-1 bosons have three degrees of freedom (or, stated differently three polarizations) rather than two degrees of freedom associated with a massless spin-1 boson. The additional polarization mode is a longitudinal one, which adds in to the two transversal ones. This magic trick of the Goldstone boson being eaten by a massless gauge field, which in turn grows heavy, was discovered independently by a number of people, among them Peter Higgs and is therefore called the Higgs-phenomenon (to give reference to all others, in fact, this effect should be known as the Brout-Englert-Guralnik-Hagen-Higgs-Kibble phenomenon).

Further insight into it can be gained by remembering the motivation for the minimal coupling prescription: gauge invariance. It tells us that the theory should be invariant under local transformations of the form

$$\theta \to \theta + \omega \tag{2.35}$$

with ω an arbitrary function of space and time. In particular, ω may be picked to be $-\theta$ at every point in space-time, such that the net field θ is zero everywhere and vanishes. The reason that the Goldstone boson disappears in gauge theory is thus pretty simple: It was never really there as dynamic degree of freedom. In fact, the degree of freedom associated with it is a mere gauge phantom - an object that may be gauged away like, e.g., a longitudinal photon.

Particles			Charge(s)
ν_e	ν_μ	ν_τ	0
< 0.8 eV	< 0.8 keV	< 17 MeV	
e^-	μ^-	τ^-	-1
511 keV	105.7 MeV	1.777 GeV	
u	c	t	$\frac{2}{3}$
$3 - 7$ MeV	$1.3 - 1.7$ GeV	175 ± 5 GeV	
d	s	b	$-\frac{1}{3}$
$3 - 7$ MeV	$100 - 200$ MeV	4.8 ± 0.3 GeV	

Table 2.1: Matter fields (fermions) in the Standard Model and some of their properties. For the quarks, which occur in bound states only, their current mass has been given. This is not their constituent mass, for the u and d quarks around 300 MeV. The different fermions, and in particular the quarks, are the different "flavors".

2.2 Constructing the Standard Model

2.2.1 Inputs

Matter sector

In the Standard Model, the fundamental matter particles are spin-1/2 fermions, quarks and leptons, where the former engage in strong interactions and the latter don't. The Standard Model does not attempt to explain the pattern of their interactions or their masses: These particles are just taken as truly elementary point particles. This property has been tested to some extent, up to now, no deviation from this assumption of point-like electrons etc. has been found. In other words there is no evidence hinting at quark or lepton compositeness, such as excited states, form factors, etc..
The quarks and leptons come in three families - there's compelling theoretical evidence that these families must be complete in order to cancel an unwanted feature called anomaly (see below), and there's a compelling measurement by LEP that there are only three neutrino species: ν_e, ν_μ, ν_τ with mass below roughly 45 GeV. The fermions and some of their properties are listed in Table 2.1. It should be noted here that the quarks come in three colors.

Chiral fermions

Ignoring for the time being recent findings indicating small mass differences of neutrinos (and, hence, massive neutrinos) the Standard Model assumes massless neutrinos, which couple only weakly. Analysis of the spin structure of weak interactions, there's evidence that only left-handed fermions and in particular neutrinos take part in the weak interactions. This immediately implies that left-handed neutrinos only are interesting in this respect: the right-handed ones, if existing at all, do not interact and can thus be ignored. However, this implies that some of the interactions of the Standard Model distinguish between left- and right-handed fermions. Therefore it makes sense to decompose the fermion fields into left- and right-handed components through suitable projectors P_L and P_R,

$$\psi = (P_L + P_R)\psi = \left(\frac{1-\gamma_5}{2} + \frac{1+\gamma_5}{2}\right)\psi = \psi_L + \psi_R\,. \tag{2.36}$$

It is simple to incorporate this into the Lagrangian for massless fermions. From

$$\begin{aligned} \mathcal{L} &= i\bar{\psi}\partial\!\!\!/\psi = i(\bar{\psi}_L + \bar{\psi}_R)\partial\!\!\!/(\psi_L + \psi_R) \\ &= i\bar{\psi}_L\partial\!\!\!/\psi_L + i\bar{\psi}_R\partial\!\!\!/\psi_R = \mathcal{L}_L + \mathcal{L}_R \end{aligned} \tag{2.37}$$

it is apparent that the kinetic terms for the spinor fields decompose into two sectors, a left- and a right-handed one. This would allow to define weak interactions as a local gauge theory, acting on the left-handed fermions only and ignoring the right-handed ones. Mass-terms, however, look like

$$\mathcal{L}_m = im\bar{\psi}\psi = im(\bar{\psi}_L + \bar{\psi}_R)(\psi_L + \psi_R) = im(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L) \tag{2.38}$$

and thus mix left- and right-handed components. This clearly spoils a gauge invariant formulation for the weak interactions, since there is no way to compensate for shifts on the left-handed fermions induced by local gauge transformations.

Gauge sector

The gauge group of the Standard Model is $SU(3) \times SU(2) \times U(1)$. The corresponding gauge bosons (eight for $SU(3)$, three for $SU(2)$ and one for $U(1)$) are massless due to gauge invariance - this is in striking contrast to

experimental findings and necessitates the inclusion of spontaneous symmetry breaking and the Higgs mechanism. Anyways, before this is discussed in more detail, the gauge sector and the interactions will be discussed.
As discussed before, interactions emerge through minimal coupling and the gauge principle, which boils down to replacing the partial derivative in the kinetic terms by a gauge-covariant derivative. To do so, consider first the strong interactions, which act on the quark fields only. The group related to strong interactions is $SU(3)$, with the 3×3 Gell-Mann matrices λ^a as generators. They satisfy

$$\left[\lambda^a,\, \lambda^b\right] = 2if^{abc}\lambda^c\,. \tag{2.39}$$

Therefore, the quark fields, which reside in the fundamental representation of this group come with three indices. Since the charge of the strong interactions is called "color" these indices are known as color indices. Thus, the kinetic terms including the strong interactions, read

$$\begin{aligned}\mathcal{L}_{\text{quarks}} & \\ = & \sum_{q\in[u,\,d,\,c,\,s,\,t,\,b]} \left[\bar{q}^i_L\left(i\partial\!\!\!/\delta_{ij} - g_3 G\!\!\!/^a\frac{\lambda^a_{ij}}{2}\right)q^j_L + \bar{q}^i_R\left(i\partial\!\!\!/\delta_{ij} - g_3 G\!\!\!/^a\frac{\lambda^a_{ij}}{2}\right)q^j_R\right]\,, \\ = & \sum_{q\in[u,\,d,\,c,\,s,\,t,\,b]} \left[\bar{q}^i_L iD\!\!\!/_{ij}q^j_L + \bar{q}^i_R iD\!\!\!/_{ij}q^j_R\right]\,,\end{aligned} \tag{2.40}$$

where the spinors are denoted by q, the G^a_μ are the gluon fields related to the generators λ^a of $SU(3)$ with $a \in [1,\, \ldots,\, 8]$ and i and j are the color indices. The strong coupling constant is denoted by g_3, hindsighting the gauge group. From this the gauge-covariant derivative can be read off, namely

$$D^\mu_{ij}q^j_{L,\,R} = \left(\partial^\mu\delta_{ij} + ig_3G^{\mu;a}\frac{\lambda^a_{ij}}{2}\right)q^j_{L,\,R}\,. \tag{2.41}$$

Of course, the gluon fields need a kinetic term as well. It is given by

$$\mathcal{L}_{\text{gluons}} = -\frac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a\,, \tag{2.42}$$

where

$$G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu - g_3 f^{abc}G^b_\mu G^c_\nu\,. \tag{2.43}$$

	T_W	T_{W3}	Y_W	Q		T_W	T_{W3}	Y_W	Q
$L_L := \begin{pmatrix} \nu_e \\ e_L \end{pmatrix}$	$\frac{1}{2}$	$+\frac{1}{2}$ $-\frac{1}{2}$	-1	0 -1	e_R	0	0	-2	-1
$Q_L := \begin{pmatrix} u_L \\ d_L \end{pmatrix}$	$\frac{1}{2}$	$+\frac{1}{2}$ $-\frac{1}{2}$	$\frac{1}{3}$	$+\frac{2}{3}$ $-\frac{1}{3}$	u_R d_R	0	0	$+\frac{4}{3}$ $-\frac{2}{3}$	$+\frac{2}{3}$ $-\frac{1}{3}$

Table 2.2: Charge assignments for the left- and right-handed fields.

For the interactions related to the group $SU(2)\times U(1)$ life is not that simple, since some care has to be spent on the assignment of charges and quantum numbers before electroweak symmetry breaking (EWSB) in order that in he end contact to phenomenological findings can be made. The groups $SU(2)$ and $U(1)$ before EWSB are related to the weak isospin T_W and the weak hyper-charge Y_W, and only after EWSB the usual electromagnetic charges Q emerge. It will be sufficient to check assignments for one family only, the others just emerge as copies of this template. Due to the structure of weak interactions as acting on left-handed fermions only and connecting the electron with its neutrino and the up with the down quark, the structure of multiplets reads

$$\begin{array}{lll} \text{leptons:} \quad L_L := \begin{pmatrix} \nu_e \\ e_L \end{pmatrix} & & e_R \\ \text{quarks:} \quad Q_L := \begin{pmatrix} u_L \\ d_L \end{pmatrix} & & u_R,\, d_R\,. \end{array}$$

In other words, the matter fields come in left-handed iso-doublets - related to the weak isospin - and in right-handed iso-singlets, where right-handed neutrinos are ignored. The charge assignments are connected to the fermions ability to take part in weak interactions, they are listed in Tale 2.2. A priori there are five different assignments for the weak hyper-charge Y_W, but as will be seen quickly, they are connected with the electric charges through

$$Y_W = 2(Q - T_{W3}) \quad \text{or} \quad Q = T_{W3} + \frac{1}{2}Y_W\,. \tag{2.44}$$

Having assigned quantum numbers, the interactions can be added. The weak-isospin ($SU(2)$) interactions are transmitted through the fields

$$\vec{W}_\mu = (W^1_\mu,\, W^2_\mu,\, W^3_\mu)^T\,, \tag{2.45}$$

whereas the gauge field related to the weak hyper-charge is denoted by B_μ. The kinetic terms of the gauge fields read

$$\mathcal{L}_{\rm gauge} = -\frac{1}{4} W^i_{\mu\nu} W_i^{\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} \,, \tag{2.46}$$

where

$$\begin{aligned} W^i_{\mu\nu} &= \partial_\mu W^i_\nu - \partial_\nu W^i_\mu - g_2 \epsilon^{ijk} W^j_\mu W^k_\nu \\ B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu \,, \end{aligned} \tag{2.47}$$

where the indices i, j, and k run from one to three and ϵ^{ijk} is the antisymmetric tensor of rank three, the structure constants of $SU(2)$.
The fermion pieces can easily be added. For each family I they are given by

$$\mathcal{L}_{\rm fermions} = i \sum_{I=1}^{3} \left[\bar{L}^I_L \not{D} L^I_L + \bar{Q}^I_L \not{D} Q^I_L + \bar{e}^I_R \not{D} e^I_R + \bar{u}^I_R \not{D} u^I_R + \bar{d}^I_R \not{D} d^I_R \right] \,, \tag{2.48}$$

where it should be noted that the left-handed fields L^I_L and Q^I_L are iso-doublets whereas the right-handed ones are iso-singlets. This means that with respect to the $SU(2) \times U(1)$ gauge piece of the Standard Model, the L^I_L are composed as a vector of two Dirac spinors, whereas the right-handed fields are just single Dirac spinors. The covariant derivatives read

$$\begin{aligned} D_\mu \psi_R &= \left(\partial_\mu + i g_1 \frac{Y_W}{2} B_\mu \right) \psi_R \\ D_\mu \psi_L &= \left[\mathbf{I} \left(\partial_\mu + i g_1 \frac{Y_W}{2} B_\mu \right) + i g_2 \frac{\tau^{\mathbf{i}}}{2} W^i_\mu \right] \psi_L \,, \end{aligned} \tag{2.49}$$

where, in order to underline the iso-doublet character, a 2×2 unit-matrix $\mathbf{I}$ as well as the generators τ^i of $SU(2)$ have been made explicit.

Adding in the Higgs fields

The reasoning so far describes a perfectly consistent gauge theory of strong interactions, weak isospin and weak hyper-charge. However, phenomenologically it is not acceptable, since it is a theory of massless quanta only, in striking contrast to experimental findings, indicting the existence of massive fermions and three massive gauge bosons of weak interactions. In order to overcome this, a scalar sector - also dubbed Higgs sector - must be added in order to allow for EWSB. Since the four generators I and τ^i are to be broken down to one generator related to electrical charges and since there are three gauge bosons that need to acquire a mass there are at least three Goldstone which need to be introduced. In the Standard Model, he Higgs sector consists of a doublet of complex scalars,

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} , \tag{2.50}$$

which is coupled to the $SU(2) \times U(1)$ gauge fields through

$$D_\mu \Phi = \left[I \left(\partial_\mu + i\frac{g_1}{2} B_\mu \right) + i g_2 \frac{\tau^i}{2} W^i_\mu \right] \Phi \, . \tag{2.51}$$

The Higgs sector then reads

$$\mathcal{L}_{\text{Higgs}} = (D^\mu \Phi)^* (D_\mu \Phi) + \mu^2 \Phi^\dagger \Phi - \lambda (\Phi^\dagger \Phi)^2 \, . \tag{2.52}$$

In addition, the Higgs fields are allowed to couple to the fermions through

$$\mathcal{L}_{\text{HF}} = -f_u^{IJ} \bar{Q}^I_L \tilde{\Phi} u^J_R - f_d^{IJ} \bar{Q}^I_L \Phi d^J_R - f_e^{IJ} \bar{L}^I_L \Phi e^J_R + \text{h.c.} \, , \tag{2.53}$$

where $\tilde{\Phi}$ denotes the charge conjugate of Φ,

$$\tilde{\Phi} = i \tau_2 \Phi^* \, . \tag{2.54}$$

Summary: Lagrangian before EWSB

So far, all fermions and gauge bosons in this theory are massless:

$$\mathcal{L} = \mathcal{L}_{\text{G}} + \mathcal{L}_{\text{F}} + \mathcal{L}_{\text{H}} + \mathcal{L}_{\text{HF}} \, , \tag{2.55}$$

where

$$\begin{aligned}
\mathcal{L}_{\text{G}} &= -\frac{1}{4} G^a_{\mu\nu} G^{a,\mu\nu} - \frac{1}{4} W^i_{\mu\nu} W^{i,\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} \\
\mathcal{L}_{\text{F}} &= \sum_{\psi_L} i \bar{\psi}_L \not{D} \psi_L + \sum_{\psi_R} i \bar{\psi}_R \not{D} \psi_R \\
\mathcal{L}_{\text{H}} &= (D^\mu \Phi)^* (D_\mu \Phi) - (-\mu^2 \Phi^\dagger \Phi + \lambda (\Phi \Phi^\dagger)^2) \\
\mathcal{L}_{\text{HF}} &= -\sum_{I,J=1}^{3} f_u^{IJ} \bar{Q}^I_L \tau_2 \Phi^* u^J_R - \sum_{I,J=1}^{3} f_d^{IJ} \bar{Q}^I_L \Phi d^J_R - \sum_{I,J=1}^{3} f_e^{IJ} \bar{L}^I_L \Phi e^J_R
\end{aligned} \tag{2.56}$$

Obviously, the above Lagrangian does not have any mass terms.
There are eight gluon fields, labelled by $a \in [1, \ldots, 8]$, three gauge fields related to weak isospin, labelled by $i \in [1, \ldots, 3]$, and one gauge field connected with the weak hyper-charge. Their kinetic terms are given through the field-strength tensors,

$$\begin{aligned} G^a_{\mu\nu} &= \partial_\mu G^a_\nu - \partial_\nu G^a_\mu - g_3 f^{abc} G^b_\mu G^c_\nu \\ W^i_{\mu\nu} &= \partial_\mu W^i_\nu - \partial_\nu W^i_\mu - g_2 \epsilon^{ijk} W^j_\mu W^k_\nu \\ B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu \,. \end{aligned} \tag{2.57}$$

Taken together the gauge sector has twelve fields with two polarisations each, in sum: 24 degrees of freedom.
The fermions are organized in three families with family index I. Each family consists of left-handed doublets of right handed-singlets of leptons and quarks,

$$Q^I_L = (u^I_L, \, d^I_L)^T \,, \quad L^I_L = (\nu^I_L, \, e^I_L)^T \,, \quad u^I_R \,, \quad d^I_R \,, \quad e^I_R \,. \tag{2.58}$$

The u- and d-quarks of the families are called up- and down-type quarks, it should be noted here that all quark fields have color indices, which are suppressed here. These indices, however, translate into multiplets with three copies, labelled through color indices, of each quark field inside the doublets or singlets. Since $SU(3)$ is not in the focus of this section, this will not be further discussed.
Also, ignoring evidences for neutrino masses for the moment, there are no right-handed neutrinos. This leaves the Standard Model with seven chirality states per family, in sum there are 21 degrees of freedom.
These fermions couple to the gauge bosons above through covariant deriva-

tives, where the I and τ-matrices act on the fields inside the doublets,

$$\begin{aligned}
D_\mu Q_L^I &= \left[I\left(\partial_\mu + i\frac{g_1 Y_W}{2}B_\mu + i\frac{g_3\lambda_a}{2}G_\mu^a\right) + i\frac{g_2\tau_i}{2}W_\mu^i\right]Q_L^I \\
D_\mu L_L^I &= \left[I\left(\partial_\mu + i\frac{g_1 Y_W}{2}B_\mu\right) + i\frac{g_2\tau_i}{2}W_\mu^i\right]L_L^I \\
D_\mu u_R^I &= \left(\partial_\mu + i\frac{g_1 Y_W}{2}B_\mu\right)u_R^I \\
D_\mu d_R^I &= \left(\partial_\mu + i\frac{g_1 Y_W}{2}B_\mu\right)d_R^I \\
D_\mu e_R^I &= \left(\partial_\mu + i\frac{g_1 Y_W}{2}B_\mu\right)e_R^I\,. \qquad (2.59)
\end{aligned}$$

The Higgs fields in the minimal Standard Model consist of four scalar fields, organized in a doublet of complex scalars Φ,

$$\Phi = (\phi^+,\, \phi^0)^T\,, \qquad (2.60)$$

altogether 4 degrees of freedom.
Similar to the fermions, the Higgs doublet couples to the gauge fields through a covariant derivative,

$$D_\mu\Phi = \left[I\left(\partial_\mu + i\frac{g_1}{2}B_\mu\right) + i\frac{g_2\tau_i}{2}W_\mu^i\right]\Phi\,, \qquad (2.61)$$

it obviously has weak hyper-charge equal to one. Anyways, the reason why the Higgs-doublet couples to the weak bosons W_μ^i and B_μ is quite simple: It must couple in order to break the $SU(2)\times U(1)$ symmetry. Finally, the Higgs-doublet also couples to the fermions through the terms in $\mathcal{L}_{\mathrm{HF}}$, the reason for this is as simple as the one before: So far the fermions do not have any mass, as soon as the Higgs field acquires a vacuum expectation value (vev), this problem will also be cured.

2.2.2 Electroweak symmetry breaking

EWSB for non-Abelian theories

Starting from the Lagrangian before EWSB discussed above, in a first step the true ground state of the scalars is determined by minimizing the potential. This leads to

$$\Phi(-\mu^2 + 2\lambda\Phi\Phi^\dagger) = 0\,. \qquad (2.62)$$

Choosing the parameters μ^2 and λ in the potential and the equation above to be real positive numbers, the ground states are given by

$$\langle \Phi\Phi^\dagger \rangle_0 = \frac{\mu^2}{2\lambda} = \frac{v^2}{2}\,. \tag{2.63}$$

Picking, as in the Abelian toy model discussed in Sec.2.1.2, a ground state (here

$$\langle \Phi \rangle_0 = \left(\begin{array}{c} 0 \\ v/\sqrt{2} \end{array} \right) \tag{2.64}$$

is chosen) and expanding around this minimum will actually break the $SU(2)\times U(1)$. Before continuing, however, it will be useful to define charged fields $W_\mu^\pm$ through

$$W_\mu^\pm = \frac{1}{\sqrt{2}} \left(W_\mu^1 \mp iW_\mu^2 \right)\,, \tag{2.65}$$

which correspondingly can be put into the Lagrangian through the operators

$$\tau_+ = \sqrt{2} \left(\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right)\,,\quad \tau_- = \sqrt{2} \left(\begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array} \right)\,. \tag{2.66}$$

Then, the covariant derivative reads

$$D^\mu \Phi = \left[I\left(\partial_\mu + i\frac{g_1}{2} B_\mu \right) + i \sum_{i=\pm,\,3} \frac{g_2 \tau_i}{2} W_\mu^i \right] \Phi\,. \tag{2.67}$$

In full analogy to the Abelian case, in $\mathcal{L}_{\rm H}$ and $\mathcal{L}_{\rm HF}$, the Higgs fields in the doublet Φ will be replaced by

$$\Phi' = \Phi - \langle \Phi \rangle_0 = \left(\begin{array}{c} \phi^+ \\ \phi^0 - v/\sqrt{2} \end{array} \right)\,. \tag{2.68}$$

However, rather than using the field ϕ^+ and ϕ^0[3], it will turn out that it is

[3]Written in the complex $\phi^{+,0}$ components therefore

$$\begin{aligned}
(D^\mu \Phi) &= \left(\begin{array}{c} \partial^\mu \phi^+ + \frac{ig_1}{2} B^\mu \phi^+ + \frac{ig_2}{2} W^{3,\mu} \phi^+ + \frac{ig_2}{\sqrt{2}} W^{+,\mu} (\phi^0 - v/\sqrt{2}) \\ \partial^\mu \phi^0 + \frac{ig_2}{\sqrt{2}} W^{-,\mu} \phi^+ + \left[\frac{ig_1}{2} B^\mu - \frac{ig_2}{2} W^{3,\mu} \right] (\phi^0 - v/\sqrt{2}) \end{array} \right) \\
(D_\mu \Phi)^* &= \left(\begin{array}{c} \partial_\mu \phi^- - \frac{ig_1}{2} B_\mu \phi^- - \frac{ig_2}{2} W^{3,\mu} \phi^- - \frac{ig_2}{\sqrt{2}} W_\mu^- (\phi^{0*} - v/\sqrt{2}) \\ \partial_\mu \phi^{0*} - \frac{ig_2}{\sqrt{2}} W_\mu^+ \phi^- - \left[\frac{ig_1}{2} B_\mu - \frac{ig_2}{2} W_\mu^3 \right] (\phi^{0*} - v/\sqrt{2}) \end{array} \right)\,.
\end{aligned}$$

more convenient to write Φ in polar coordinates as

$$\Phi(x) = \exp\left[-\frac{i}{v}\sum_{i=\pm,0}\xi^i(x)\tau^i\right]\begin{pmatrix} 0 \\ \frac{v+\eta(x)}{\sqrt{2}} \end{pmatrix} = U^{-1}(\xi)\frac{v+\eta(x)}{\sqrt{2}}\chi\,, \quad (2.69)$$

where

$$\chi = (0,\,1)^T \quad (2.70)$$

and the new fields ξ^i and η have zero vev,

$$\langle\xi^i\rangle_0 = \langle\eta\rangle_0 = 0\,. \quad (2.71)$$

In order to display the particle spectrum, a gauge has to be fixed. Here, the unitary gauge is chosen. It is characterized, first of all, by transforming the Higgs doublet

$$\Phi' = U(\xi)\Phi = \frac{v+\eta(x)}{\sqrt{2}}\chi\,. \quad (2.72)$$

This shift has to be compensated for. In the Higgs-fermion piece of the Lagrangian the structure is such that

$$\mathcal{L}_{\rm HF} = \bar{Q}_L\Phi d_R + \ldots \quad (2.73)$$

and similar, thus the left-handed fermion fields in the Lagrangian must experience a suitable transformation such that the transformed Lagrangian remain invariant. This is achieved through

$$\begin{aligned} u'_R &= u_R\,, \quad d'_R = d_R\,, \quad e'_R = e_R\,, \quad e'_R = e_R\,, \\ Q'_L &= U(\xi)Q_L\,, \quad L'_L = U(\xi)L_L\,, \end{aligned} \quad (2.74)$$

leading to

$$\mathcal{L}'_{\rm HF} = \frac{v+\eta}{\sqrt{2}}\left[f_u^{IJ}\bar{u}'^I_L u'^J_R + f_d^{IJ}\bar{d}'^I_L d'^J_R + f_e^{IJ}\bar{e}'^I_L e'^J_R\right]\,, \quad (2.75)$$

and it is simple to see that mass terms for the fermions emerged.
In addition, the gauge fields must transform like

$$\begin{aligned} \frac{1}{2}\sum_{i=1}^{3}\tau^i W'^i_\mu &= U(\xi)\left(\frac{1}{2}\sum_{i=1}^{3}\tau^i W^i_\mu\right)U^{-1}(\xi) - \frac{i}{g_2}\left(\partial_\mu U(\xi)\right)U^{-1}(\xi) \\ B'_\mu &= B_\mu\,. \end{aligned} \quad (2.76)$$

These transformations actually lead to an absorption of the three Goldstone bosons ξ^i into the gauge fields. Their corresponding masses are contained in the square of the covariant derivative,

$$D_\mu \Phi' = \left(\partial_\mu + \frac{ig_1}{2} B'_\mu + \frac{ig_2}{2} \sum_{i=1}^{3} \tau^i {W'}^i_\mu \right) \left(\frac{v + \eta}{\sqrt{2}} \right) \chi \tag{2.77}$$

Also, in the transformed fields, the Higgs potential reads

$$\mathcal{U}(\Phi) = \lambda v^2 \eta^2 + \lambda v \eta^3 + \frac{\lambda}{4} \eta^4 \, . \tag{2.78}$$

Hence, there is a physical mass term for the Higgs-boson, where the mass is given by

$$m_\eta = \sqrt{\lambda v^2} \, . \tag{2.79}$$

Massive gauge bosons

The mass terms for the gauge bosons can be read off from $(D_\mu \Phi)^* (D^\mu \Phi')$, namely

$$\begin{aligned} \mathcal{L}_{\mathrm{M}} &= \frac{v^2}{2} \chi^\dagger \left(\frac{g_2}{2} \vec{\tau} \cdot \vec{W'}_\mu + \frac{g_1}{2} B'_\mu \right) \left(\frac{g_2}{2} \vec{\tau} \cdot \vec{W'}^\mu + \frac{g_1}{2} B'^\mu \right) \chi \\ &= \frac{v^2}{8} \left\{ g_2^2 \left[\left({W'}^1_\mu \right)^2 + \left({W'}^2_\mu \right)^2 \right] + \left[g_2 {W'}^3_\mu - g_1 B'_\mu \right]^2 \right\} \, . \end{aligned} \tag{2.80}$$

Forgetting about the primes (they occur everywhere), and transforming on the charged $W^\pm$ fields this can be written as

$$\mathcal{L}_{\mathrm{M}} = \frac{v^2 g_2^2}{4} W^-_\mu W^{+,\mu} + \frac{v^2}{8} \begin{pmatrix} B_\mu \\ W^3_\mu \end{pmatrix}^T \begin{pmatrix} g_1^2 & -g_1 g_2 \\ -g_1 g_2 & g_2^2 \end{pmatrix} \begin{pmatrix} B^\mu \\ W^{3,\mu} \end{pmatrix} , \tag{2.81}$$

leading to

$$M_W = \frac{v g_2}{2} \, . \tag{2.82}$$

Also, the second term in $\mathcal{L}_{\mathrm{M}}$ implies that the two neutral fields B_μ and W^3_μ mix. What does this mean? The answer is clear: Although the fields are responsible for each interaction (weak isospin and weak hyper-charge)

separately, the mass eigenstates do not respect this separation. This, in fact, is a common feature of quantum field theories: Whenever particles have the same quantum numbers, they are allowed to mix, and in most cases they will do so. This will be seen again in the fermion sector of the theory.
However, physically meaningful fields are the mass eigenstates, hence the mass matrix in the B_μ-W^3_μ basis must be diagonalized, leading to the eigenvalues

$$m_1 = M_\gamma = 0 \quad \text{and} \quad m_2 = M_Z = \frac{v}{2}\sqrt{g_1^2 + g_2^2}\,. \tag{2.83}$$

This is good news! The mechanism presented here leads to a suitable result of one massive and one massless neutral gauge boson, plus two massive charged gauge bosons. The Z boson and the photon are thus realized as linear combinations of the B_μ and the W^3_μ,

$$\begin{aligned} Z_\mu &= \cos\theta_W W^3_\mu - \sin\theta_W B_\mu \\ A_\mu &= \sin\theta_W W^3_\mu + \cos\theta_W B_\mu\,. \end{aligned} \tag{2.84}$$

Simple calculation shows that the Weinberg angle θ_W is given by

$$\tan\theta_W = \frac{g_1}{g_2}\,. \tag{2.85}$$

The model also predicts that the mass ratio of the massive gauge bosons is given by the coupling constants of the (unbroken) $SU(2)$ and $U(1)$ only,

$$\frac{M_W}{M_Z} = \frac{g_2}{\sqrt{g_1^2+g_2^2}} = \cos\theta_W\,. \tag{2.86}$$

Anyway, coming back to the equations above, Eqs. (**??**) - (**??**), it is now clear that in them, linear combinations of B_μ and W^3_μ should be replaced by A_μ and Z_μ. To do so, consider

$$\begin{aligned} W^3_\mu &= \cos\theta_W Z_\mu + \sin\theta_W A_\mu \\ B_\mu &= -\sin\theta_W Z_\mu + \cos\theta_W A_\mu\,. \end{aligned} \tag{2.87}$$

The fermion sector

For the fermion sector, there are two relevant pieces in the Lagrangian, namely

$$\mathcal{L}_{\rm HF} = \frac{v+\eta}{\sqrt{2}}\left[f_u^{IJ}\bar{u}'^I_L u'^J_R + f_d^{IJ}\bar{d}'^I_L d'^J_R + f_e^{IJ}\bar{e}'^I_L e'^J_R\right] \tag{2.88}$$

leading to both mass terms for the fermions and their couplings to the Higgs bosons. Assuming the coupling matrices to be diagonal, simple inspection shows that the masses for the fermions are given by

$$m_{u^I} = \frac{f_u^I v}{\sqrt{2}}\,, \quad m_{d^I} = \frac{f_d^I v}{\sqrt{2}}\,, \quad \text{and} \quad m_{e^I} = \frac{f_e^I v}{\sqrt{2}},. \tag{2.89}$$

Hence, rather than having nine fundamental masses, there are nine fundamental coupling constants of the fermions to the Higgs-doublet. These coupling both drive the mass terms for the fermions after EWSB and the interaction strength with the Higgs boson. Thus, the ratio of interaction strengths of the Higgs boson to different fermions equals the ratio of their masses.

In addition there are the gauge interactions with the (gauge-fixed) $SU(2) \times U(1)$ fields, omitting the primes these interactions read

$$\begin{aligned} \mathcal{L}_{\rm F} &= +\bar{Q}_L\left(\frac{ig_2}{2}\vec{\tau}\vec{\not{W}}+\frac{ig_1}{2}\frac{1}{3}\not{B}\right)Q_L+\bar{L}_L\left(\frac{ig_2}{2}\vec{\tau}\vec{\not{W}}-\frac{ig_1}{2}\not{B}\right)L_L \\ &\quad +\bar{u}_R\frac{4}{3}\frac{ig_1}{2}\not{B}u_R-\bar{d}_R\frac{2}{3}\frac{ig_1}{2}\not{B}d_R-\bar{e}_R 2\frac{ig_1}{2}\not{B}e_R\,. \end{aligned} \tag{2.90}$$

The following discussion can be greatly simplified when writing this as interactions of the gauge bosons with fermion currents.

For the W^1 and W^2 bosons, the transformation on charged bosons $W^\pm$, cf. Eq. (2.65), implies that there are charged current interactions, i.e.

$$\begin{aligned} \mathcal{L}_{\rm CC} &= ig_2\left(J^1_\mu W^{1,\mu}+J^2_\mu W^{2,\mu}\right) \\ &= \frac{ig_2}{\sqrt{2}}\left(J^+_\mu W^{+,\mu}+J^-_\mu W^{-,\mu}\right)\,, \end{aligned} \tag{2.91}$$

where the currents are given by

$$J^+_\mu = J^1_\mu + iJ^2_\mu = \bar{\nu}^I_L\gamma_\mu e^I_L + \bar{u}^I_L\gamma_\mu d^I_L = \bar{\nu}^I\gamma_\mu\frac{1-\gamma_5}{2}e^I + {}^-u^I\gamma_\mu\frac{1-\gamma_5}{2}d^I \tag{2.92}$$

and its Hermitian conjugate. For low energies, well below the $W^\pm$ mass it seems to be natural that the effect of momenta in a propagating W can be neglected. Formally speaking, it can be integrated out [4]. Then, an effec-

[4] To see this, it suffices to check the form of a massive W-propagator, namely

$$\frac{-g_\mu\nu+\frac{q_\mu q_\nu}{M_W^2}}{q^2-M_W^2}\ \stackrel{|q|\ll M_W}{\longrightarrow}\ \frac{-g_{\mu\nu}}{M_W^2}\,.$$

tive Lagrangian emerges, which is described by a current-current interaction, namely

$$\mathcal{L}_{\rm CC}^{(\rm eff.)} = -\frac{g_2^2}{2M_W^2} J_\mu^+ J^{-,\mu} = -\frac{4G_F}{\sqrt{2}} J_\mu^+ J^{-,\mu}\,, \tag{2.93}$$

which is just the Fermi theory. The Fermi constant is given by

$$G_F = \frac{g_2^2}{4\sqrt{2}M_W^2} = \frac{g_2^2}{\sqrt{2}v^2} \tag{2.94}$$

which ultimately yields the value of the vacuum expectation value

$$v \approx 250 \text{ GeV}\,. \tag{2.95}$$

For the neutral bosons, life is not that simple; the neutral interactions, written in currents, read

$$\begin{aligned}\mathcal{L}_{\rm NC} &= ig_2\left(J_\mu^3 W^{3,\mu} + \frac{ig_1}{2} J_\mu^Y B^\mu\right)\\ &= +ig_2 J_\mu^3 \left(\cos\theta_W Z^\mu + \sin\theta_W A^\mu\right)\\ &\quad +ig_2 \tan\theta_W \left(\cos\theta_W A^\mu - \sin\theta_W Z^\mu\right)\left(J_\mu^{\rm em.} - J_\mu^3\right)\,,\end{aligned} \tag{2.96}$$

where Eqs. (2.84) and (2.44) have been used. Therefore,

$$\mathcal{L}_{\rm NC} = ieJ_\mu^{\rm em.} A^\mu + \frac{g_2}{\cos\theta_W} J_\mu^0 Z^\mu\,, \tag{2.97}$$

where

$$\begin{aligned} e &= g_2 \sin\theta_W\\ J_\mu^0 &= J_\mu^3 - \sin^2\theta_W J_\mu^{\rm em.}\end{aligned} \tag{2.98}$$

and the currents are given by

$$\begin{aligned} J_\mu^{\rm em.} &= \sum_{f\in\text{fermions}} e_f \bar\psi_f \gamma_\mu \psi_f\\ J_\mu^0 &= \sum_{f\in\text{fermions}} \left[g_L^f \bar\psi_f \gamma_\mu \frac{1-\gamma_5}{2}\psi_f + g_R^f \bar\psi_f \gamma_\mu \frac{1+\gamma_5}{2}\psi_f\right]\,.\end{aligned} \tag{2.99}$$

The former is the well-known electromagnetic current, known from QED. The latter is a bit more complicated, since here different charge and hypercharge assignments play a role. This is cast into the coupling constant, which are essentially given by

$$g_{L,R}^f = T_{W3}(f_L, R) - e_f \sin^2\theta_W\,. \tag{2.100}$$

2.2.3 Quark mixing: The CKM-picture

Mass vs. gauge eigenstates

There are two contributions in the Lagrangian, where fermions show up, namely, first of all the mass- and Higgs-terms, and, secondly, in the kinetic terms, which include the gauge interactions of the fermions. Omitting the primes of the previous discussion, these terms read

$$\mathcal{L}_{\rm HF} \;=\; \frac{v+\eta}{\sqrt{2}}\left[f_u^{IJ}\bar{u}_L^I u_R^J + f_d^{IJ}\bar{d}_L^I d_R^J + f_e^{IJ}\bar{e}_L^I e_R^J\right] \tag{2.101}$$

and

$$\begin{aligned}\mathcal{L}_{\rm F} \;=\; & +\bar{Q}_L^I\left(\frac{ig_2}{2}\vec{\tau}\vec{W\!\!\!/} + \frac{ig_1}{2}\frac{1}{3}B\!\!\!/\right)Q_L^I + \bar{L}_L^I\left(\frac{ig_2}{2}\vec{\tau}\vec{W\!\!\!/} - \frac{ig_1}{2}B\!\!\!/\right)L_L^I \\ & +\bar{u}_R^I\frac{4}{3}\frac{ig_1}{2}B\!\!\!/ u_R^I - \bar{d}_R^I\frac{2}{3}\frac{ig_1}{2}B\!\!\!/ d_R^I - \bar{e}_R^I 2\frac{ig_1}{2}B\!\!\!/ e_R^I\,,\end{aligned} \tag{2.102}$$

respectively. The fermion fields in both terms are the gauge eigenstates. This is the representation (linear combination) of fermions, in which the gauge interactions are diagonal in the flavor indices, i.e. have the structure $\psi^I D\!\!\!\!/\,\psi^J\delta_{IJ}$. These states, however, are not necessarily the - physically better defined - mass eigenstates. This is due to the fact that the mass matrices $f^{IJ}_{u,d,e}$ in $\mathcal{L}_{\rm HF}$ can very well be non-diagonal. Diagonalization of these matrices would lead to mass eigenstates of the quarks and leptons, that are linear combinations of the gauge eigenstates.

Diagonalization of the mass matrix

In order to analyze this situation, it must be investigated, how such matrices, which do not even need to be Hermitian or symmetric, can be diagonalized. In general, this is possible through bi-unitary transformations. Given a matrix M, it can be shown that there exist unitary matrices S and T such that

$$S^\dagger M T = M_{\rm diag.}\,, \tag{2.103}$$

where $M_{\rm diag.}$ is a diagonal matrix with positive eigenvalues. Noting that every matrix M can be written as the product of a Hermitian matrix H and a unitary matrix U,

$$M = HU \tag{2.104}$$

the proof for the above statement bases on the fact that $M^\dagger M$ is Hermitian and positive. Therefore, it can be diagonalized by

$$S^\dagger(M^\dagger M)S = (M^2)_{\text{diag.}} = \begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix}, \tag{2.105}$$

where S is unique up to a diagonal phase matrix, i.e. up to

$$S^\dagger F^\dagger(M^\dagger M)FS = (M^2)_{\text{diag.}} \tag{2.106}$$

with

$$F = \begin{pmatrix} e^{i\phi_1} & 0 & 0 \\ 0 & e^{i\phi_2} & 0 \\ 0 & 0 & e^{i\phi_3} \end{pmatrix}. \tag{2.107}$$

These phases will be studied later in the context of CP-violation, here it should suffice to state that they ensure them_i^2 to be positive. Defining now a hermitian matrix H through

$$H = SM_{\text{diag}}S^\dagger \tag{2.108}$$

it can be shown that U is given by

$$U = H^{-1}M \quad \text{and} \quad U^\dagger = M^\dagger H^{-1}. \tag{2.109}$$

It is a unitary matrix, since

$$\begin{aligned} UU^\dagger &= H^{-1}MM^\dagger H^{-1} = H^{-1}S(M^2)_{\text{diag}}S^\dagger H^{-1} \\ &= H^{-1}SM_{\text{diag}}S^\dagger SM_{\text{diag}}S^\dagger H^{-1} = H^{-1}HHH^{-1} = 1. \end{aligned} \tag{2.110}$$

Thus,

$$S^\dagger HS = S^\dagger MU^\dagger S = S^\dagger MT = M_{\text{diag}}, \tag{2.111}$$

equivalent to

$$T = U^\dagger S. \tag{2.112}$$

Connecting the eigenstates

From the reasoning above, it is clear that the connection of gauge and mass eigenstates follows from

$$\bar{\psi}_L M \psi_R = (\bar{\psi}_L S)(S^\dagger M T)(T^\dagger \psi_R) = \bar{\psi}'_L M_{\text{diag}} \psi'_R \,, \tag{2.113}$$

where the unprimed spinors denote the gauge eigenstates and the primed ones are the mass eigenstates, given by

$$\psi'_L = S^\dagger \psi_L \quad \text{and} \quad \psi'_R = T^\dagger \psi_R \,. \tag{2.114}$$

Expressing the charged weak quark currents in mass rather than in gauge eigenstates,

$$\begin{aligned} J_\mu^+ &= \bar{Q}_L^I \gamma_\mu \tau^+ Q_L^I = \bar{u}_L^I \gamma_\mu d_L^I \\ &= \bar{u}'^J_L \gamma_\mu \left[(S^\dagger_{(u)})^{JI} (S_{(d)})^{IK} \right] d'^K_L \,, \end{aligned} \tag{2.115}$$

it becomes obvious that the mass eigenstates of the quarks of different generations now mix in the weak interactions. The mixing matrix is known as the Cabibbo-Kobayashi-Maskawa matrix, V_{CKM} and is given by

$$V = S^\dagger_{(u)} S_{(d)} \,. \tag{2.116}$$

It is a unitary matrix (since it emerges as the product of two unitary matrices). A careful analysis of degrees of freedom shows that a complex $n \times n$ matrix has $2n^2$ real parameters. Unitary imposed reduces this number to n^2; further $(2n-1)$ phases can be removed y redefining quark states, i.e. absorbing the phases in the fields. Keeping in mind that there are $(n-1)(n-2)/2$ angles in an $n \times n$ orthogonal matrix, it is clear that there are

$$2n^2 - n^2 - (2n-1) - \frac{(n-1)(n-2)}{2} = \frac{n(n-1)}{2} \tag{2.117}$$

independent parameters. Hence, it is clear that for two generations, there is one real angle, mixing the generations, known as the Cabibbo angle; for the case of three generations there are three real angles plus a complex phase needed in order to fully describe V_{CKM}. This mixing, however, does not affect the neutral currents, since there the same matrices act on each other.
In addition, for leptons, there is no mixing, since in the framework of the Standard Model, there are no neutrino masses at all. This additional degeneracy makes the phases obsolete.

Chapter 3

Renormalization of gauge theories

3.1 QED at one-loop

The QED Lagrangian in Feynman gauge ($\alpha = 1$ in front of the gauge fixing term) is given by

$$\mathcal{L}^{\text{cl.}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\alpha}(\partial_\mu A^\mu)^2 + i\bar{\psi}\partial\!\!\!/\psi - m\bar{\psi}\psi - e\bar{\psi}A\!\!\!/\psi\,, \tag{3.1}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,. \tag{3.2}$$

3.1.1 Regularization

Catalogue of diagrams

Before starting out with the program of Regularization, the superficial degree of divergence for each graph in QED should be counted. This is to list all relevant diagrams that have to be regularized. Defining, as before L as the number of loops, V as the number of vertices and E_ψ, I_ψ and E_A, I_A as the numbers of external (E) or internal (I) fermion (ψ) or photon fields (A), the superficial degree of divergence for a QED graph is

$$D = 4L - 2I_A - I_\psi\,. \tag{3.3}$$

The goal now is to cast this into a form such that only the external legs show up, allowing to classify the generic types of diagrams or n-point functions that need to be cured. Since there is only one kind of vertex in QED with two fermion and one photon line, it is easy to convince oneself that

$$V = I_\psi + \frac{1}{2} E_\psi \tag{3.4}$$

and that

$$V = 2I_A + E_A \,. \tag{3.5}$$

Also, the number of independent momenta is equal to L, which in turn equals the number of internal lines minus the number of vertices (momentum conservation at each vertex) plus one (total four momentum conservation),

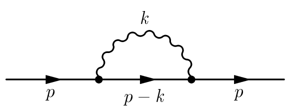
$$L = I_\psi + I_A - V + 1 \,. \tag{3.6}$$

Putting everything together, one finds

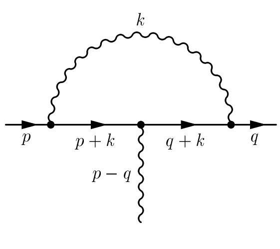
$$D = 4 - \frac{3}{2} E_\psi - E_A \,. \tag{3.7}$$

This is quite fortunate, since once again (after the example of ϕ^4 theory) it shows that the total divergence of any Feynman diagram, no matter how complicated, just depends on the number and type of external particles. In fact, the only superficially divergent graphs are

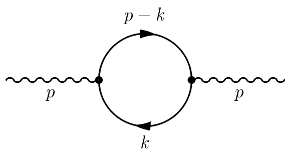
1. the electron self-energy ($D = 1$),



2. the electron-photon vertex ($D = 0$),



3. the photon vacuum polarization (nominally $D = 2$, but gauge invariance reduces the degree of divergence by 2),



4. the three-photon graphs ($D = 1$); they vanish because there are two graphs, one with the electron clockwise and one with the electron counterclockwise leading to exactly the same results with opposite sign - this is a manifestation of Furry's theorem, and, finally,

5. the light-by-light scattering ($D = 0$). This graph becomes convergent by employing gauge invariance.

Self-energy of the electron

Working in the Feynman gauge and suppressing the spinors the one-loop correction on the fermion line, cf. Fig.3.1, also known as self-energy, reads

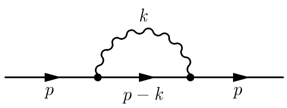


Figure 3.1: Electron self energy graph at order e^2.

$$
\begin{aligned}
\Sigma(p) &= \\
&= i(-i)(ie)^2\mu^{4-D}\int \frac{\mathrm{d}^D k}{(2\pi)^D}\frac{\gamma^\mu\left(\not{p}-\not{k}+m\right)\gamma^\nu\left(g_{\mu\nu}\right)}{\left[(p-k)^2-m^2\right]k^2} \\
&= -e^2\mu^{4-D}\int \frac{\mathrm{d}^D k}{(2\pi)^D}\frac{(2-D)\left(\not{p}-\not{k}\right)+Dm}{\left[(p-k)^2-m^2\right]k^2} \\
&= -e^2\mu^{4-D}\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^D k}{(2\pi)^D}\frac{(2-D)\left(\not{p}-\not{k}\right)+Dm}{\left[k^2-2xp\cdot k+xp^2-xm^2\right]^2} \\
&= -e^2\mu^{4-D}\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^D K}{(2\pi)^D}\frac{(2-D)\left((1-x)\not{p}-\not{K}\right)+Dm}{\left[K^2+x(1-x)p^2-xm^2\right]^2} \\
&= -\frac{ie^2\mu^{4-D}\Gamma\left(2-\frac{D}{2}\right)}{(4\pi)^{D/2}\Gamma\left(2\right)}\int_0^1 \mathrm{d}x\frac{(2-D)(1-x)\not{p}+Dm}{\left[x(x-1)p^2+xm^2\right]^{2-D/2}}\,,
\end{aligned}
\tag{3.8}
$$

where, in intermediate steps, a Feynman trick of Eq. (**??**) and the substitution $k^\mu \to K^\mu = k^\mu - xp^\mu$ has been applied. Also, Eq. (**??**) has been used for the evaluation of the D-dimensional integral. To continue, one uses $D = 4 - 2\eta$ and expands around small η. Using

$$
\hat{\mu}^2 = 4\pi e^{-\gamma_E}\mu^2 \quad \text{and} \quad M^2(x) = (x^2-x)p^2 + xm^2 \tag{3.9}
$$

this yields

$$
\begin{aligned}
\Sigma(p) &= -\frac{ie^2}{(4\pi)^2}\frac{1}{\eta}\left[1+\eta\log\left(\hat{\mu}^2\right)\right] \\
&\quad \int_0^1 \mathrm{d}x\left[(2\eta-2)(1-x)\not{p}+(4-2\eta)m\right]\left[1-\eta\log M^2(x)\right] \\
&= -\frac{ie^2}{(4\pi)^2}\left\{\frac{-\not{p}+4m}{\eta}+\not{p}-2m+\right. \\
&\qquad \left.\int_0^1 \mathrm{d}x\left[2(1-x)\not{p}+4m\right]\log\frac{M^2(x)}{\hat{\mu}^2}\right\}.
\end{aligned}
\tag{3.10}
$$

The finite integral can be done analytically.

Vacuum polarization

The one-loop correction to the photon line is mediated through a fermion loop, also known as vacuum polarization, cf. Fig.3.2. It reads

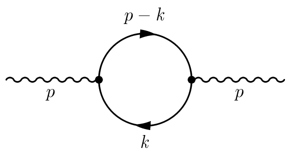


Figure 3.2: Vacuum polarization graph at order e^2.

$$
\begin{aligned}
&\Pi_{\mu\nu}(p) = \\
&= -i^2(ie)^2\mu^{4-D}\int\frac{\mathrm{d}^4k}{(2\pi)^D}\mathrm{Tr}\left[\frac{\gamma_\mu(\not{p}+\not{k}+m)\gamma_\nu(\not{k}+m)}{[(p+k)^2-m^2][k^2-m^2]}\right] \\
&= -4e^2\mu^{4-D}\int\frac{\mathrm{d}^4k}{(2\pi)^D}\frac{(p+k)_\mu k_\nu+(p+k)_\nu k_\mu-[(p+k)\cdot k-m^2]g_{\mu\nu}}{[(p+k)^2-m^2][k^2-m^2]} \\
&= -4e^2\mu^{4-D}\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^4k}{(2\pi)^D} \\
&\qquad\times\frac{(p+k)_\mu k_\nu+(p+k)_\nu k_\mu-[(p+k)\cdot k-m^2]g_{\mu\nu}}{[k^2+2xp\cdot k+x(p^2-m^2)]^2} \\
&= -4e^2\mu^{4-D}\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^4K}{(2\pi)^D} \\
&\qquad\times\frac{2K_\mu K_\nu-2x(1-x)p_\mu p_\nu-[K^2-x(1-x)p^2-m^2]g_{\mu\nu}}{[K^2+x(1-x)p^2-m^2]^2} \\
&= -4e^2\mu^{4-D}\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^4K}{(2\pi)^D}\left\{\frac{2g_{\mu\nu}}{D}\frac{K^2}{[K^2+x(1-x)p^2-m^2]^2}\right. \\
&\qquad\left.-\frac{g_{\mu\nu}}{K^2+x(1-x)p^2-m^2}-\frac{2x(1-x)(p_\mu p_\nu-p^2g_{\mu\nu})}{[K^2+x(1-x)p^2-m^2]^2}\right\} \\
&= -\frac{4ie^2\mu^{4-D}}{(4\pi)^D/2}\int_0^1\mathrm{d}x\left\{-\frac{2g_{\mu\nu}}{D}\frac{\Gamma\left(1+\frac{D}{2}\right)\Gamma\left(1-\frac{D}{2}\right)}{\Gamma\left(\frac{D}{2}\right)\Gamma\left(2\right)}\frac{1}{[M^2(x)]^{1-D/2}}\right. \\
&\qquad\left.+g_{\mu\nu}\frac{\Gamma\left(1-\frac{D}{2}\right)}{[M^2(x)]^{1-D/2}}-\frac{2x(1-x)(p_\mu p_\nu-p^2g_{\mu\nu})}{[M^2(x)]^{2-D/2}}\frac{\Gamma\left(2-\frac{D}{2}\right)}{\Gamma\left(2\right)}\right\} \\
&= \frac{4ie^2\mu^{4-D}\Gamma\left(2-\frac{D}{2}\right)}{(4\pi)^D/2}\left(p_\mu p_\nu-p^2g_{\mu\nu}\right)\int_0^1\mathrm{d}x\frac{2x(1-x)}{[M^2(x)]^{2-D/2}}\,,
\end{aligned}
\tag{3.11}
$$

where the minus sign and the trace stem from the fermion loop. In the computation above the substitution $k^\mu \to K^\mu = k^\mu + xp^\mu$ and the usual integral identities already employed for the evaluation of the self-energy diagram have been applied. Also, the function $M^2(x)$ equals the one introduced before. Setting $D = 4 - 2\eta$ the result above is expanded around small η

yielding

$$
\begin{aligned}
\Pi_{\mu\nu}(p) &= \frac{4ie^2\left(p_\mu p_\nu - p^2 g_{\mu\nu}\right)}{(4\pi)^2}\frac{1}{\eta}\left[1+\eta\log\left(\hat{\mu}^2\right)\right] \\
&\qquad \int\limits_0^1 \mathrm{d}x 2x(1-x)\left[1-\eta\log M^2(x)\right] \\
&= \frac{4ie^2\left(p_\mu p_\nu - p^2 g_{\mu\nu}\right)}{(4\pi)^2}\left[\frac{2}{3\eta}+\int\limits_0^1 \mathrm{d}x 2x(1-x)\log\frac{M^2(x)}{\hat{\mu}^2}\right].
\end{aligned} \tag{3.12}
$$

Again, the remaining integral is finite and can be computed analytically.

Vertex correction

The vertex correction at one-loop, cf. Fig.3.3 is given by

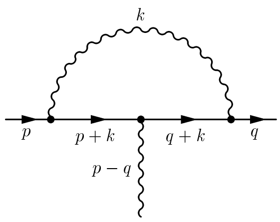


Figure 3.3: Electron-photon vertex correction graph at order e^3.

$$
\begin{aligned}
&\Gamma_\rho(p,\,q) = \\
&= \; i^2(-i)(ie)^3\mu^{4-D}\int\frac{\mathrm{d}^4k}{(2\pi)^D}\frac{g^{\mu\nu}\gamma_\mu(\not{p}+\not{k}+m)\gamma_\rho(\not{q}+\not{k}+m)\gamma_\nu}{[(p+k)^2-m^2][(q+k)^2-m^2]k^2} \\
&= \; 2e^3\mu^{4-D}\int\limits_0^1\mathrm{d}x\int\limits_0^{1-x}\mathrm{d}y\int\frac{\mathrm{d}^4k}{(2\pi)^D} \\
&\qquad\frac{\gamma_\mu(\not{p}+\not{k}+m)\gamma_\rho(\not{q}+\not{k}+m)\gamma^\mu}{[k^2+(2xp+2yq)\cdot k+xp^2+yq^2-(x+y)m^2]^3} \\
&= \; 2e^3\mu^{4-D}\int\limits_0^1\mathrm{d}x\int\limits_0^{1-x}\mathrm{d}y\int\frac{\mathrm{d}^4K}{(2\pi)^D} \\
&\qquad\frac{\gamma_\mu(\not{K}-y\not{q}+(1-x)\not{p}+m)\gamma_\rho(\not{K}+(1-y)\not{q}-x\not{p}+m)\gamma^\mu}{[K^2+x(1-x)p^2+y(1-y)q^2-2xyp\cdot q-(x+y)m^2]^3}
\end{aligned}
\tag{3.13}
$$

To continue, the numerator has to be inspected in some detail. To separate out those pieces leading to ultraviolet divergences, one has to separate the terms quadratic in K from the others. Then, with

$$
-V^2(x,\,y) = x(1-x)p^2+y(1-y)q^2-2xyp\cdot q-(x+y)m^2 \tag{3.14}
$$

the divergent part of the vertex correction reads

$$
\begin{aligned}
&\Gamma_\rho^{\mathrm{U.V.}}(p,\,q) = -2(D-2)e^3\mu^{4-D}\int\limits_0^1\mathrm{d}x\int\limits_0^{1-x}\mathrm{d}y\int\frac{\mathrm{d}^4K}{(2\pi)^D}\frac{\not{K}\gamma_\rho\not{K}}{[K^2-V^2(x,\,y)]^3} \\
&= \; \frac{2(2-D)^2\gamma_\rho}{D}e^3\mu^{4-D}\int\limits_0^1\mathrm{d}x\int\limits_0^{1-x}\mathrm{d}y\int\frac{\mathrm{d}^4K}{(2\pi)^D}\frac{K^2}{[K^2-V^2(x,\,y)]^3} \\
&= \; \frac{2ie^3\mu^{4-D}}{(4\pi)^{D/2}}\gamma_\rho\frac{(2-D)^2\Gamma\left(1+\frac{D}{2}\right)\Gamma\left(2-\frac{D}{2}\right)}{\Gamma\left(\frac{D}{2}\right)\Gamma\left(3\right)D}\int\limits_0^1\mathrm{d}x\int\limits_0^{1-x}\mathrm{d}y\frac{1}{[V^2(x,\,y)]^{2-D/2}} \\
&= \; \frac{2ie^3}{(4\pi)^2}\gamma_\rho\frac{1-2\eta}{\eta}\left[1+\eta\log\hat{\mu}^2\right]\int\limits_0^1\mathrm{d}x\int\limits_0^{1-x}\mathrm{d}y\left[1-\eta\log V^2(x,\,y)\right] \\
&= \; \frac{ie^3}{(4\pi)^2}\gamma_\rho\left[\frac{1}{\eta}-2-2\int\limits_0^1\mathrm{d}x\int\limits_0^{1-x}\mathrm{d}y\log\frac{V^2(x,\,y)}{\hat{\mu}^2}\right]\,.
\end{aligned}
\tag{3.15}
$$

Again the last integral yields a final result, as well as those parts that have been neglected before.

3.1.2 Renormalization

Counter-terms

Starting from the QED Lagrangian, Eq. (3.1), one is tempted to try a counter-term Lagrangian of the form

$$\mathcal{L}^{\text{c.t.}} = -\frac{K_3}{4}F_{\mu\nu}F^{\mu\nu} - \frac{K_\alpha}{2\alpha}(\partial_\mu A^\mu)^2 + iK_2\bar{\psi}\partial\!\!\!/\psi - K_m m\bar{\psi}\psi - K_1 e\bar{\psi}A\!\!\!/\psi\,. \tag{3.16}$$

Then the renormalized Lagrangian

$$\mathcal{L}^{\text{ren.}} = \mathcal{L}^{\text{cl.}} + \mathcal{L}^{\text{c.t.}} \tag{3.17}$$

can be rewritten in terms of bare quantities

$$\begin{aligned}
\psi_0 &= \sqrt{1+K_2}\,\psi := \sqrt{Z_2}\,\psi \\
A_0^\mu &= \sqrt{1+K_3}\,A^\mu := \sqrt{Z_3}\,A^\mu \\
e_0 &= \frac{1+K_1}{(1+K_2)\sqrt{1+K_3}}\,e := \frac{Z_1}{Z_2\sqrt{Z_3}}\,e \\
m_0 &= \frac{1+K_m}{1+K_2}\,m := \frac{Z_m}{Z_2}\,m \\
\alpha_0^{-1} &= \frac{1+K_\alpha}{1+K_3}\alpha^{-1} := \frac{Z_\alpha}{Z_3}\alpha^{-1}
\end{aligned} \tag{3.18}$$

and reads

$$\begin{aligned}
\mathcal{L}^{\text{ren.}} &= \frac{1}{4}F_{\mu\nu 0}F_0^{\mu\nu} + \frac{1}{2\alpha_0}(\partial_\mu A_0^\mu)^2 + i\bar{\psi}_0\partial\!\!\!/\psi_0 + im_0\bar{\psi}_0\psi_0 + ie\bar{\psi}_0A\!\!\!/_0\psi_0 \\
&= \frac{Z_3}{4}F_{\mu\nu}F^{\mu\nu} + \frac{Z_\alpha}{2\alpha}(\partial_\mu A^\mu)^2 + iZ_2\bar{\psi}\partial\!\!\!/\psi + iZ_m m\bar{\psi}\psi + eZ_1\bar{\psi}A\!\!\!/\psi\,.
\end{aligned} \tag{3.19}$$

Written in this form it is rather straightforward to see that gauge invariance will be preserved under loop corrections if $Z_1 = Z_2$. In fact this is exactly the case as will be discussed in the following. However, this is not so obvious,

since by putting in a gauge fixing term the Z_i, or, basically, the counter terms, became gauge dependent.
Now, from the various contributions calculated in the previous section, the counter terms can be read off. From the self energy

$$\Sigma(p) = \frac{ie^2}{(4\pi)^2}\frac{\not{p} - 4m}{\eta} + \text{finite terms} \tag{3.20}$$

one finds

$$\begin{aligned} K_2 &= \frac{e^2}{(4\pi)^2}\left[\frac{1}{\eta} + F_2\left(\eta,\, \frac{m}{\hat{\mu}}\right)\right] \\ K_m &= -\frac{4e^2}{(4\pi)^2}\left[\frac{1}{\eta} + F_m\left(\eta,\, \frac{m}{\hat{\mu}}\right)\right] , \end{aligned} \tag{3.21}$$

where the finite pieces are encapsulated in the finite functions F, which are analytic for $\eta \to 0$. The vertex correction, given by

$$\Pi_{\mu\nu}(p) \;=\; \frac{ie^2\left(p_\mu p_\nu - p^2 g_{\mu\nu}\right)}{(4\pi)^2}\left[\frac{4}{3\eta} + \text{finite terms}\right] \tag{3.22}$$

yields

$$\begin{aligned} K_3 &= -\frac{e^2}{(4\pi)^2}\left[\frac{4}{3\eta} + F_3\left(\eta,\, \frac{m}{\hat{\mu}}\right)\right] \\ K_\alpha &= \frac{e^2}{(4\pi)^2}\left[\frac{4}{3\eta} + F_\alpha\left(\eta,\, \frac{m}{\hat{\mu}}\right)\right] , \end{aligned} \tag{3.23}$$

where the F have the same properties as the ones above. The renormalization of α corresponds to the correction of the longitudinal part of the photon propagator, as given by the term $\sim p_\mu p_\nu$. Lastly, the vertex corrections

$$\Gamma_\rho^{\rm U.V.}(p,\, q) = \frac{ie^3}{(4\pi)^2}\gamma_\rho\left[\frac{1}{\eta} + \text{finite terms}\right] \tag{3.24}$$

give

$$K_1 \;=\; -\frac{e^2}{(4\pi)^2}\left[\frac{1}{\eta} + F_1\left(\eta,\, \frac{m}{\hat{\mu}}\right)\right] , \tag{3.25}$$

where again F_1 is the finite piece of the counter term.

To summarize

$$\begin{aligned} Z_1 &= 1 - \frac{e^2}{(4\pi)^2}\left(\frac{1}{\eta} + F_1\right) + \mathcal{O}(e^4) \\ Z_2 &= 1 - \frac{e^2}{(4\pi)^2}\left(\frac{1}{\eta} + F_2\right) + \mathcal{O}(e^4) \\ Z_3 &= 1 - \frac{e^2}{(4\pi)^2}\left(\frac{4}{3\eta} + F_3\right) + \mathcal{O}(e^4) \\ Z_m &= 1 - \frac{e^2}{(4\pi)^2}\left(\frac{4}{\eta} + F_m\right) + \mathcal{O}(e^4) \\ Z_\alpha &= 1 - \frac{e^2}{(4\pi)^2}\left(\frac{4}{3\eta} + F_\alpha\right) + \mathcal{O}(e^4)\,. \end{aligned} \tag{3.26}$$

In fact, to this order in perturbation theory, the divergent pieces of $Z_{1,2}$ satisfy the relation $Z_1 = Z_2$. The bare charge therefore can be expressed as

$$e_0 = \mu^\eta \left[1 + \frac{e^2}{(4\pi)^2}\frac{2}{3\eta} + \text{finite terms} + \mathcal{O}(e^4)\right]\,. \tag{3.27}$$

Thus, by using a mass independent renormalization prescription, one can read off the scale variation of the coupling constant,

$$\hat{\mu}\frac{\partial e}{\partial \hat{\mu}} = \frac{e^3}{12\pi^2}\,, \tag{3.28}$$

in fact the same sign as for the scalar ϕ^4 theory. The solution to the equation above yields a running coupling constant given by

$$e^2(\mu) = \frac{e^2(\mu_0)}{1 - \frac{e^2(\mu_0)}{6\pi^2}\log\frac{\mu}{\mu_0}}\,, \tag{3.29}$$

leading to the Landau singularity at

$$\mu_L = \mu_0 \exp\left[\frac{6\pi^2}{e^2(\mu_0)}\right]\,. \tag{3.30}$$

The fact that the coupling constant decreases for smaller scales i.e. longer distances is good news: it proves that QED as a theory is perfectly suited to define asymptotic states and according scattering cross sections.

Ward identities: simple version

When moving to an all-orders discussion of QED and its renormalization, the work will be vastly simplified by a set of identities known as Ward-Takahashi identities. Their benefit is that they reduce the number of independent renormalization constants Z_i, and they are intimately related to the gauge structure of the theory. One such identity is already present in the one-loop renormalization constants listed in Eq. (3.26), namely $Z_1 = Z_2$. This identity could have been seen already before explicit calculation, to see how this works recall the expression for the self energy correction,

$$\Sigma(p) = -e^2\mu^{4-D}\int\frac{\mathrm{d}^Dk}{(2\pi)^D}\frac{\gamma^\mu\,(\not{p}-\not{k}+m)\,\gamma_\mu}{[(p-k)^2-m^2]\,k^2} \tag{3.31}$$

and compute its derivative

$$\begin{aligned}
-\frac{\partial\Sigma(p)}{\partial p^\rho} &= -e^2\mu^{4-D}\int\frac{\mathrm{d}^Dk}{(2\pi)^D}\frac{1}{k^2}\gamma^\mu\left[-\frac{\partial}{\partial p^\rho}\frac{1}{\not{p}-\not{k}-m}\right]\gamma_\mu\\
&= e^2\mu^{4-D}\int\frac{\mathrm{d}^Dk}{(2\pi)^D}\frac{1}{k^2}\gamma^\mu\left[\frac{1}{\not{p}-\not{k}-m}\gamma_\rho\frac{1}{\not{p}-\not{k}-m}\right]\gamma_\mu\\
&= e^2\mu^{4-D}\int\frac{\mathrm{d}^Dk}{(2\pi)^D}\frac{\gamma^\mu(\not{p}-\not{k}+m)\gamma_\rho(\not{p}-\not{k}+m)\gamma_\mu}{k^2[(p-k)^2-m^2]^2}\,,
\end{aligned} \tag{3.32}$$

which is, up to a constant factor (the coupling constant e) identical with the vertex correction, if the two fermion momenta are identical,

$$\begin{aligned}
\Gamma_\rho(p,\,p) &= \lim_{q\to p}\Gamma_\rho(p,\,q)\\
&= \lim_{q\to p}e^3\mu^{4-D}\int\frac{\mathrm{d}^4k}{(2\pi)^D}\frac{\gamma_\mu(\not{p}-\not{k}+m)\gamma_\rho(\not{q}-\not{k}+m)\gamma_\mu}{[(p-k)^2-m^2][(q-k)^2-m^2]k^2}\\
&= e^3\mu^{4-D}\int\frac{\mathrm{d}^4k}{(2\pi)^D}\frac{\gamma_\mu(\not{p}-\not{k}+m)\gamma_\rho(\not{p}-\not{k}+m)\gamma_\mu}{[(p-k)^2-m^2]^2k^2}\,.
\end{aligned} \tag{3.33}$$

This explains why the two ultraviolet structures of the respective diagrams can be related. To generalize the procedure above to obtain a more useful form, consider

$$\frac{1}{\not{p}-m}-\frac{1}{\not{p}+\not{q}-m} = \frac{\not{p}+\not{q}-m-\not{p}+m}{[\not{p}-m][\not{p}+\not{q}-m]} = \frac{\not{q}}{[\not{p}-m][\not{p}+\not{q}-m]}\,, \tag{3.34}$$

which allows to identify

$$(p-q)^\rho\,\Gamma_\rho(p,\,q) \;=\; -\left[\Sigma(p)-\Sigma(q)\right]\,. \tag{3.35}$$

To show that this relationship holds true to all orders, consider a specific member contributing to the electron self-energy at arbitrary order. Then, schematically, one has, for an external electron with momentum p_0

$$\Sigma(p_0) \;\sim\; \sum_{\text{N=numberofloops}} \prod_{\text{N}} \int \left(\prod_{i=0}^{N} \not{a}_i \frac{1}{\not{p}_i - m}\right) \not{a}_n\,, \tag{3.36}$$

where the $p_i = p_0 + q_i$ with the q some loop momenta and $p_n = p_0$. The a_i denote the momentum-dependent photon lines that are connected to other fermion loops, which are not shown explicitly. As an example consider a graph, where the photon of the self-energy correction is itself corrected by some vacuum polarization, cf. Fig.3.4. Then, after performing the traces of the vacuum polarization piece, the γ matrices of the photon vertices on the electron line are contracted with loop momenta from the vacuum bubble. is

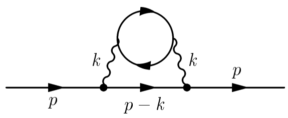


Figure 3.4: Electron self energy graph at higher order.

given by In the scheme above, these guys are denoted by the a_i. From the considerations just made it is clear that the total momenta inside the a_i must add up to zero - after all there is only one incoming and one outgoing particle, both with momentum p_0. Similarly, a corresponding diagram contributing to the vertex correction can be represented as

$$\Gamma_\rho(p_0, p_N) \;\sim\; \sum_{\text{N=numberofloops}} \prod_{\text{N}} \int \sum_{r=1}^{N-1} \left(\prod_{i=0}^{r} \not{a}_i \frac{1}{\not{p}_i - m}\right) \gamma_\rho \frac{1}{\not{p}_r + \not{q} - m} \left(\prod_{i=r+1}^{N} \not{a}_i \frac{1}{\not{p}_i + \not{q} - m}\right) \not{a}_n\,. \tag{3.37}$$

Here, q denotes the momentum of the external photon line and the external fermion momenta are p_0 and p_N. Obviously, the photon is inserted at point

r, and the sum is over all positions r. The essential point of the proof is that there is a similarity of the self-energy and the vertex graphs. Graphically speaking, taking any self-energy graph and making all possible insertions of an extra photon along all fermion propagators leads to the sum over all vertex correction diagrams. To show this, contract $\Gamma_\rho(p_0, p_N)$ with q^ρ and use the previous identity. Then each term of the sum over r splits into two pieces with a relative minus sign. Performing the sum it becomes apparent that only the first and the last term survive, leading to

$$\begin{aligned} q^\rho \Gamma_\rho(p_0, p_N) \quad \sim \quad & \sum_{\mathrm{N=number of loops}} \prod_{\mathrm{N}} \int \sum_{r=1}^{N-1} \left(\prod_{i=0}^{r-1} \not{a}_i \frac{1}{\not{p}_i - m} \right) \not{a}_r \\ & \left(\frac{1}{\not{p}_r - m} - \frac{1}{\not{p}_r + \not{q} - m} \right) \left(\prod_{i=r+1}^{N} \not{a}_i \frac{1}{\not{p}_i + \not{q} - m} \right) \not{a}_n \\ = \quad & \Sigma(p_0) - \Sigma(p_N = p_0 + q) \,. \end{aligned} \tag{3.38}$$

The proof is complete by realizing that attaching the photon along closed fermion lines (i.e. vacuum polarization insertions) leads to exact pairwise cancellations yielding a contribution of exactly zero from such closed loops. These considerations prove the Ward-Takahashi identity to all orders.

BRS transformations

The Ward-Takahashi identities discussed above play a crucial role for the proof of renormalizability of QED. To cast them into a more general form, however, it is useful to make a little detour. The main point of this detour is to define a symmetry valid for gauge theories which covers up for the gauge-invariance breaking effects of the gauge-fixing term.
Because of local gauge invariance not all of the connected Greens functions generated by

$$\begin{aligned} Z[J_\mu,\, \bar{\chi},\, \chi] &= Z[0,\, 0,\, 0] \exp\{i\mathcal{W}[J]\} \\ &= \mathcal{N} \int \mathcal{D}A_\mu \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left\{ i\mathcal{S} + \int \mathrm{d}^4x \left[J_\mu A^\mu + i\bar{\chi}\psi + i\bar{\psi}\chi \right] \right\} \end{aligned} \tag{3.39}$$

where

$$\mathcal{S} = \int \mathrm{d}^4x \left[-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\alpha}(\partial_\mu A^\mu)^2 + i\bar{\psi}\not{\partial}\psi - m\bar{\psi}\psi - e\bar{\psi}\not{A}\psi \right] \tag{3.40}$$

are independent. In fact, due to the gauge fixing term and the various sources, the generating functional above is not invariant under the local gauge transformations

$$\begin{aligned}
\delta A_\mu(x) &= \frac{1}{e}\partial_\mu \Lambda(x) \\
\delta\psi(x) &= = -i\Lambda(x)\psi(x) \\
\delta\bar{\psi}(x) &= = i\bar{\psi}(x)\Lambda(x)\,.
\end{aligned} \tag{3.41}$$

In this and the next section a set of functional constraints on $\mathcal{W}$ will be derived which will result in relations between various Greens functions known as Ward identities. To derive these constraints, the approach due to Becchi, Rouet, and Stora will be used: it generalizes neatly to the case of Yang-Mills theories.

The first step of their approach consists in restoring a weaker form of invariance even in the presence of the gauge fixing term. In doing so, the sources will be neglected for a moment. This restoration is achieved by switching to a ghost representation of the constant determinant stemming from the gauge fixing term, resulting also in a different normalization factor. Then the new action, including the ghost fields η, is

$$\mathcal{S}' = \int \mathrm{d}^4x \left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\alpha}(\partial_\mu A^\mu)^2 + \bar{\psi}\left(i\partial\!\!\!/ - m - eA\!\!\!/\right)\bar{\psi} + i\partial_\mu\eta^*\partial^\mu\eta \right] \tag{3.42}$$

This new action is invariant under the special gauge transformations

$$\begin{aligned}
\delta A_\mu(x) &= \frac{1}{e}\partial_\mu\left[\xi^*\eta(x) + \xi\eta^*(x)\right] \\
\delta\psi(x) &= = -i\left[\xi^*\eta(x) + \xi\eta^*(x)\right]\psi(x) \\
\delta\bar{\psi}(x) &= = i\bar{\psi}(x)\left[\xi^*\eta(x) + \xi\eta^*(x)\right] \\
\delta\eta(x) &= -\frac{i}{\alpha e}(\partial\cdot A)\xi \\
\delta\eta^*(x) &= \frac{i}{\alpha e}(\partial\cdot A)\xi^*\,,
\end{aligned} \tag{3.43}$$

where ξ are arbitrary complex Grassmann numbers independent of x. The transformations of Eq. (3.43) are also known also as BRS-transformations. Applying them, it is quite simple to check that the kinetic term of the gauge fields and the fermion piece of the action are invariant; hence the action is shifted by

$$\delta\mathcal{S}' = \frac{1}{e}\int \mathrm{d}^4x \left\{\frac{1}{\alpha}(\partial\cdot A)\partial^2\left[\xi^*\eta(x)+\xi\eta^*(x)\right] -\frac{1}{\alpha}(\partial\cdot A)\partial^2\left[\xi^*\eta(x)\right] - \frac{1}{\alpha}(\partial\cdot A)\partial^2\left[\xi\eta^*(x)\right]\right\} = 0 \tag{3.44}$$

as advertised. To obtain this result the variation of the ghost term of the new action has been integrated by parts and the identity

$$(\omega\chi)^* = \chi^*\omega^* \tag{3.45}$$

for Grassmann numbers has been used. Including sources one has to add source terms σ for the ghost fields; again, they are scalars obeying Fermi statistics. The generating functional then reads

$$\exp\{i\mathcal{W}[J]\} = \mathcal{N}'\int \mathcal{D}A_\mu\mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\eta^*\mathcal{D}\eta \exp\left\{i\mathcal{S}' + \int \mathrm{d}^4x\left[J_\mu A^\mu + i\bar{\chi}\psi + i\bar{\psi}\chi + i\eta^*\sigma + i\eta\sigma^*\right]\right\} \tag{3.46}$$

Applying now a BRS transformation, two things can be noted: first, the Jacobian is unity; second, since the action is invariant under the transformations only the source terms are affected. Then, the generating functional, after having applied the BRS transformation, yields

$$\exp\{i\mathcal{W}[J]\} = \mathcal{N}'\int \mathcal{D}A_\mu\mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\eta^*\mathcal{D}\eta \exp\left\{i\mathcal{S}' + \int \mathrm{d}^4x\left[J_\mu A^\mu + i\bar{\chi}\psi + i\bar{\psi}\chi + i\eta^*\sigma + i\eta\sigma^*\right]\right\} \exp\left\{\int \mathrm{d}^4x\left[J_\mu\delta A^\mu + i\bar{\chi}\delta\psi + i\delta\bar{\psi}\chi + i\delta\eta^*\sigma + i\delta\eta\sigma^*\right]\right\}. \tag{3.47}$$

Specializing to those BRS transformations for which ξ is real, i.e. $\xi^* = \xi$, and demanding that the Greens functions remain invariant means that the difference of both corresponding generating functionals vanishes. By expanding the exponential in ξ and noting that the second term $\sim \xi^2$ is zero because of $\xi^2 = 0$, one has

$$
\begin{aligned}
0 = \int & \mathcal{D}A_\mu \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\eta^*\mathcal{D}\eta \\
& \exp\left\{ i\mathcal{S}' + \int \mathrm{d}^4x \left[J_\mu A^\mu + i\bar{\chi}\psi + i\bar{\psi}\chi + i\eta^*\sigma + i\eta\sigma^* \right] \right\} \\
& \left\{ \int \mathrm{d}^4x \left[J_\mu \partial^\mu \frac{\eta + \eta^*}{e} - i\bar{\chi}(\eta + \eta^*)\psi + i\bar{\psi}(\eta + \eta^*)\chi \right.\right. \\
& \left.\left. \qquad + \frac{i}{\alpha e}(\partial \cdot A)(\sigma - \sigma^*) \right] \right\} .
\end{aligned}
\tag{3.48}
$$

Effective action and Ward-Takahashi identities

To continue, one may introduce the effective action through a Legendre transformation from the path integral.

$$
\Gamma[\phi_{\text{cl.}}] = \mathcal{W}[J] - \int \mathrm{d}^4x \left(J \cdot A_{\text{cl.}} + i\bar{\chi}\psi_{\text{cl.}} + i\bar{\psi}_{\text{cl.}}\chi + \eta^*_{\text{cl.}}\sigma + \eta_{\text{cl.}}\sigma^* \right) . \tag{3.49}
$$

this effective action serves as the generating functional of all n point functions, i.e. proper vertices, and it is formulated in terms of the classical fields. There, the J_μ, etc., are the classical sources, defined through equation of the form

$$
J_\mu = -\frac{\delta\Gamma}{\delta A^\mu_{\text{cl.}}} . \tag{3.50}
$$

In terms of Γ the equation above, Eq.3.48, can be written in manageable form as

$$
\begin{aligned}
0 \;=\; \int \mathrm{d}^4x & \left[-\frac{1}{e}\frac{\partial\Gamma}{\partial A^\mu_{\text{cl.}}}\partial^\mu(\eta_{\text{cl.}} + \eta^*_{\text{cl.}}) \right. \\
& + i\frac{\partial\Gamma}{\partial\psi_{\text{cl.}}}(\eta_{\text{cl.}} + \eta^*_{\text{cl.}})\psi_{\text{cl.}} - i\bar{\psi}_{\text{cl.}}(\eta_{\text{cl.}} + \eta^*_{\text{cl.}})\frac{\partial\Gamma}{\partial\bar{\psi}_{\text{cl.}}} \\
& \left. + \frac{i}{\alpha e}\partial \cdot A_{\text{cl.}}\frac{\partial\Gamma}{\partial\eta_{\text{cl.}}} - \frac{i}{\alpha e}\partial \cdot A_{\text{cl.}}\frac{\partial\Gamma}{\partial\eta^*_{\text{cl.}}} \right] .
\end{aligned}
\tag{3.51}
$$

This is the most transparent form of the Ward-Takahashi identities for QED. In the following it will be applied to some of the simplest cases.

1. The dependence of Γ on the ghost fields is simple, since they do not interact. Therefore one can write

$$\Gamma = i \int \mathrm{d}^4x\mathrm{d}^4y\eta^*_{\mathrm{cl.}}(x)\Delta^{-1}(x-y)\eta_{\mathrm{cl.}}(y) + \Gamma'[A^\mu_{\mathrm{cl}},\, \psi_{\mathrm{cl}},\, \bar{\psi}_{\mathrm{cl}}] \tag{3.52}$$

with the free propagator

$$\Delta(x-y) = -\partial^2\delta^4(x-y)\,. \tag{3.53}$$

In contrast the expression for Γ' is not so simple. It starts out with

$$\begin{aligned}\Gamma' \;&=\; \int \mathrm{d}^4x\mathrm{d}^4y\left[\bar{\psi}_{\mathrm{cl.}}(x)S_F^{-1}(x-y)\psi_{\mathrm{cl.}}(y) + \frac{1}{2}A^\mu_{\mathrm{cl}}(x)\Delta^{-1}_{\mu\nu}(x-y)A^\nu_{\mathrm{cl}}(y)\right]\\ &\quad+\int \mathrm{d}^4x\mathrm{d}^4y\mathrm{d}^4z\left[\bar{\psi}_{\mathrm{cl.}}(x)A^\mu_{\mathrm{cl}}(y)\Gamma_\mu(x,\,y,\,z)\psi_{\mathrm{cl.}}(z) + \ldots\right]\,,\end{aligned} \tag{3.54}$$

where Γ_μ is the three point function and S_F and $\Delta_{\mu\nu}$ are the free field propagators for the fermion and photon field, respectively. Of course, Γ' contains many more interaction terms corresponding to various n point functions.

Applying now Eq. (3.51) to Eqs. (3.52) and (3.54) and keeping only those terms that contain the photon or the ghost fields, i.e. A^μ or $\eta + \eta^*$, one obtains in momentum space

$$k^\mu\Delta^{-1}_{\mu\nu}(k) + \frac{1}{\alpha}k_\nu k^2 = 0\,. \tag{3.55}$$

Writing the inverse propagator in a general form as

$$\Delta^{-1}_{\mu\nu}(k) = A(k^2)g_{\mu\nu} + B(k^2)k_\mu k_\nu \tag{3.56}$$

and plugging it into the equation above one finds the Ward identity

$$A(k^2) + B(k^2)k^2 + \frac{1}{\alpha}k^2 = 0 \tag{3.57}$$

and in Feynman gauge ($\alpha = 1$) the general form of the inverse propagator is

$$\Delta^{-1}_{\mu\nu}(k) = -g_{\mu\nu}k^2 + \left(g_{\mu\nu}k^2 - k_\mu k_\nu\right)F(k^2)\,. \tag{3.58}$$

Here $F(k^2)$ is at least of order e^2.

2. As another application, consider terms which contain ψ, $\bar{\psi}$ and the η's. One finds

$$\frac{\partial}{\partial y_\mu}\Gamma_\mu(x,y,z) - iS_F^{-1}(x-z)\delta(z-y) + iS_F^{-1}(x-z)\delta(x-y) = 0 \quad (3.59)$$

In momentum space this yields

$$(p-p')^\mu \Gamma_\mu(p, p-p', p') = S_F^{-1}(p) - S_F^{-1}(p')\,, \quad (3.60)$$

an identity that has been encountered and discussed before.

In the respective perturbative expansion of both terms this is easy to test at leading order.

$$\begin{aligned} \Gamma_\mu &= i\gamma_\mu + \dots \\ S_F^{-1}(p) &= i(\not{p} - m) + \dots \end{aligned} \quad (3.61)$$

and since S_F^{-1} is multiplicatively renormalized by Z_2 and Γ_μ by Z_1 the identity above enforces

$$Z_1 = Z_2\,. \quad (3.62)$$

Moreover, it is clear that it would be not a smart move to adopt any subtraction procedure (i.e. any prescription for the construction of counter-terms) that violates the Ward identities because then the BSR symmetry would be violated.

Renormalization prescriptions

Recalling the result of the vacuum polarization diagram and the corresponding counter-term, cf. Eqs. (3.12) and (3.23), yields for the photon propagator up to order e^4

$$\begin{aligned} \Delta_{\mu\nu}(p) &= \frac{ig_{\mu\nu}}{p^2}\left[1 + \frac{e^2}{2\pi^2}\left(-\int\limits_0^1 \mathrm{d}x x(1-x)\log\frac{m^2 + p^2x(1-x)}{\hat{\mu}^2} - \frac{1}{6}F_3\right)\right] \\ &\quad - \frac{ip_\mu p_\nu}{p^4}\left[\frac{e^2}{2\pi^2}\left(-\int\limits_0^1 \mathrm{d}x x(1-x)\log\frac{m^2 + p^2x(1-x)}{\hat{\mu}^2} - \frac{1}{6}F_\alpha\right)\right]\,, \end{aligned} \quad (3.63)$$

where the $F_{3,\alpha}$ were a priori unknown terms in the counter terms. Requiring that for $p^2 \to 0$ the photon propagator remains in the same form fixes these terms. In other words from

$$\left\{ \Delta_{\mu\nu}(p) = \frac{i g_{\mu\nu}}{p^2} \right\} \Bigg|_{p^2=0} \tag{3.64}$$

it follows that

$$F_3 = F_\alpha = -\log \frac{m^2}{\hat{\mu}^2} \,. \tag{3.65}$$

Appendix A

Brief review of group theory

Algebra, and especially group theory, provide the mathematical framework in which many aspects of symmetry can be formulated in an elegant and compact fashion. Modern day particle physics is - to a large extent - based on external or internal symmetries such as Lorentz or gauge invariance. It is fair to say that these foundations form the most important cornerstone in the formulation of, e.g., the Standard Model and its supersymmetric extensions. To gain insight into this structure and into the beauty of theoretical particle physics some terms of group theory are necessary. Therefore, in this first chapter of the lecture, basics of group theory and Lie algebras will be recalled. Starting out with finite groups, the concept of the representation of groups will be briefly discussed, before the focus shifts onto Lie groups and algebras that are used to formalize continuous symmetries like the ones mentioned above. Finally, the concepts introduced so far will be exemplified by analyzing the structure of $SU(2)$.

A.1 Finite Groups and their Representations

A.1.1 Finite Groups

Definition of a group

A group consists of a set $\mathcal{G}$, on which an operation "$\cdot$" ("multiplication") is defined. This operation exhibits the following properties

1. Closure : If g_i and g_j are elements of $\mathcal{G}$, then also $g_k = g_i \cdot g_j$ is a member of $\mathcal{G}$.

2. Identity element (existence of a "1") : There is exactly one identity element g_1 in $\mathcal{G}$, that maps all elements g_i exactly on themselves, $g_i = g_i \cdot g_1 = g_1 \cdot g_i$.

3. Inverse element : For every element g_i of $\mathcal{G}$ there is exactly one inverse element g_i^{-1} with the property $g_i^{-1} \cdot g_i = g_i \cdot g_i^{-1} = g_1$.

4. Associativity : $g_i \cdot (g_j \cdot g_k) = (g_i \cdot g_j) \cdot g_k$ is fulfilled.

The property of commutativity, i.e. the validity of $g_i \cdot g_j \stackrel{?}{=} g_j \cdot g_i$, decides, whether a group as defined above is Abelian or non–Abelian.
Some examples of groups include :

- $\{1, -1, i, -i\}$ together with the ordinary multiplication (note here that the subset $\{-1, 1\}$ with the multiplication forms a subgroup);

- the integer numbers with the addition as operation;

- the set $\{1, 2, \dots, p-1\}$ forms a group under multiplication modulo p, if p is a prime number;

- the set of all permutations of three objects a, b, and c, sitting on positions 1, 2, and 3. The group of permutations contains the six operations

( )	do nothing, $(a, b, c) \to (a, b, c)$
(12)	exchange objects on positions 1 and 2, $(a, b, c) \to (b, a, c)$
(13)	exchange objects on positions 1 and 3, $(a, b, c) \to (c, b, a)$
(23)	exchange objects on positions 2 and 3, $(a, b, c) \to (a, c, b)$
(123)	cyclic permutation, and $(a, b, c) \to (c, a, b)$
(321)	anti-cyclic permutation $(a, b, c) \to (b, c, a)$;

- the group D_4 of symmetry transformations on a square with corners labeled clockwise from 1 to 4. This group consists of the following elements:

E	Identity element.
$\mathcal{R}_{90,180,270}$	Clockwise rotations around the center of the square, with angle as indicated.
$\mathcal{M}_{x,y}$	Reflections around the horizontal (x) and vertical (y) axis.
$\mathcal{M}_{13,24}$	Reflections around the axis (13) or (24), respectively.

The “multiplication” of two such operations is defined as their sequential application. Here and in the following it is understood that in the product $A \cdot B$ first B and then A is executed, i.e. the order of operation is from the right to the left. It cannot be over-stressed that the sequence of the operations matters - as will be seen later, group elements can be expressed as matrices, where also the sequence of multiplication matters.

Generators of finite groups

In general it is possible to generate all the elements of a finite group by starting from a certain limited set of elements subject to some relations. Consider now the smallest set of elements generating all other elements by repeated application of the operation belonging to the group. The elements of this minimal set are called the generators of the group. Note that this set is not necessarily unique. Coming back to the example of the permutation group over three elements, it is quite easy to check that for instance any two out of the three (ij) generate the whole group.

Direct Products

The direct product of two groups $\mathcal{H}_1$ and $\mathcal{H}_2$ of order h_1 and h_2, respectively, is defined as

$$\mathcal{G} = \mathcal{H}_1 \otimes \mathcal{H}_2 = \{h_1^i \cdot h_2^j\} \, . \tag{A.1}$$

This definition holds true only, if both groups have only the identity element in common and if the elements of both groups commute mutually. As an example, consider

$$\{E\, ,\ \mathcal{M}_x\} \otimes \{E\, ,\ \mathcal{M}_y\} = \{E\, ,\ \mathcal{M}_x\, ,\ \mathcal{M}_y\, ,\ \mathcal{R}_{180}\}\, . \tag{A.2}$$

A.1.2 Representations of finite groups

Definitions

Any mapping $\mathcal{D}$ of the group $\mathcal{G}$ with the property

$$\mathcal{D}(g_i) \cdot \mathcal{D}(g_j) = \mathcal{D}(g_i \cdot g_j) \tag{A.3}$$

for all elements g_i and g_j of $\mathcal{G}$ is called a representation of this group. More particularly, the name representation is reserved for a collection of square

matrices T with the property above. The dimension of the representation is then identified with the rank of the matrices.
So, returning once more to one of the favorite examples, a representation of the permutation group of three elements in terms of square 3×3 matrices is given by,

$$\begin{aligned}
\mathcal{D}(\ ) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \mathcal{D}(12) &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
\mathcal{D}(13) &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} & \mathcal{D}(23) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
\mathcal{D}(321) &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} & \mathcal{D}(123) &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} .
\end{aligned}$$

In this particular representation, the operation "$\cdot$" can be identified with the multiplication of matrices, for instance $(12) \cdot (23) = (123)$ or

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} . \tag{A.4}$$

Since the matrices T do not need to be distinct, the set of the T does not necessarily form a group under matrix multiplication.

Equivalent representations

Two representations T_1 and T_2 of a group are called equivalent, if there is a nonsingular matrix S with

$$T_1 = S^{-1} T_2 S \, . \tag{A.5}$$

In other words if the two representations can be mapped onto each other by a similarity transformation within the same vector space, in which both representations (matrices) are defined and act, then the two representations are equivalent. A very useful theorem on representations is that each representation T of a finite group is equivalent to a representation by unitary matrices.

This can be used to cast any representation T into a block-diagonal form,

$$\tilde{T} = \begin{pmatrix} \tilde{t}^{(1)} & & & \\ & \tilde{t}^{(2)} & & \\ & & \ddots & \\ & & & \tilde{t}^{(n)} \end{pmatrix} . \tag{A.6}$$

The block-diagonal elements $\tilde{t}^{(i)}$ are called irreducible representations of the group.

A.2 Lie groups and Lie algebras

A.2.1 Lie groups, the basics

Definition

Stated in an intuitive manner (suitable for physicists), a Lie group is a continuous group, i.e. the elements are labeled by some continuous real parameter or parameters, where the multiplication law depends smoothly on these parameters. In other words, the multiplication law can be rewritten as a function of these parameters, the corresponding inverse elements exist and can also be labeled as smooth functions of the parameters. This translates into the possibility to generate group elements from each other that differ only by an infinitesimal amount of one of the real parameters:

$$x(0,0,\ldots,\epsilon_j,\ldots,0) = x(0,0,\ldots,0) + i\epsilon_j \mathbf{I}_j(0,0,\ldots,0) + \mathcal{O}(\epsilon^2) \tag{A.7}$$

For obvious reasons, the r operators $\mathbf{I}_j$ are called the generators of the Lie group, since their multiple application spans the group. Therefore, as indicated above, any group element, which can be derived from the identity element by continuous changes in the parameters can be re-expressed as

$$\exp(i\alpha_l \mathbf{I}_l) \ , \tag{A.8}$$

where the summation over l is understood. The $\mathbf{I}_l$ are the linearly independent generators defined by

$$\mathbf{I}_l = \lim_{\epsilon_l \to 0} \left[\frac{x(0,\ldots,\epsilon_l,\ldots,0) - x(0,0,\ldots,0)}{i\epsilon_l} \right] \tag{A.9}$$

and the α_l are real parameters.

Representations of a Lie group

This leads immediately to the issue of how to represent a Lie group. The answer to this is pretty straightforward and already indicated above : Lie groups are represented by finite-dimensional matrices. Extending the previous results on finite groups, representations of Lie groups have the following properties :

1. For any representation of a Lie group an equivalent representation with unitary matrices can be found.
2. Any unitary representation can be cast into block diagonal form as displayed in Eq. (A.6).
3. Any irreducible representation is finite dimensional.

Generator algebra

The more interesting thing about the generators $\mathbf{I}_l$ is that to a large extend they define the structure of the group. The basic point is that they satisfy simple commutation relations

$$[\mathbf{I}_a,\, \mathbf{I}_b] = \mathbf{I}_a\mathbf{I}_b - \mathbf{I}_b\mathbf{I}_a = if_{abc}\mathbf{I}_c\,, \tag{A.10}$$

and

$$[\mathbf{I}_a,\, [\mathbf{I}_b,\, \mathbf{I}_c]] + \text{ cycl.} = 0\,. \tag{A.11}$$

The commutator relation above, Eq. (A.10), determines completely the structure of the Lie group. Thus, the constants f_{abc} are called structure constants of the group.

A.2.2 Lie algebras

Definition

The r generators of a Lie group form the basis vectors of an r–dimensional vector space. The set of all real linear combinations of the generators is called a Lie algebra. Such an algebra L has the following properties:

1. If $\mathbf{x},\, \mathbf{y} \in L$, then $[\mathbf{x},\, \mathbf{y}] \in L$.

2. $[\mathbf{x}, \mathbf{y}] = -[\mathbf{y}, \mathbf{x}]$.
3. The Jacobi identity holds, $[\mathbf{x}, [\mathbf{y}, \mathbf{z}]]$+cycl.$= 0$.

Since the generators of a Lie group together with the commutation relation form a Lie algebra, it is easy to see that the structure constants fulfil

$$f_{abc} = -f_{bac} \,. \tag{A.12}$$

Even more, Eq. (A.11) translates into

$$f_{bcd}f_{ade} + f_{abd}f_{cde} + f_{cad}f_{bde} = 0 \,. \tag{A.13}$$

Defining a set of new matrices $\mathbf{J}$ through

$$(\mathbf{J}_a)_{bc} = -if_{abc} \tag{A.14}$$

this then implies

$$[\mathbf{J}_a, \mathbf{J}_b] = if_{abc}\mathbf{J}_c \,. \tag{A.15}$$

Fundamental and adjoint representation

Stated in less formal words, the structure constants f_{abc} themselves generate a representation of the algebra, which is called the adjoint representation. In contrast, the representation generated by the I is called fundamental representation. The dimension of this adjoint representation equals the number of generators, since this is the space, on which it acts. This is exactly the number of real parameters needed to describe every group element.

Construction of representations

As mentioned before, the structure constants can not be uniquely defined. Instead, they depend on the choice of the basis for the vector space of the generators. A suitable choice can be obtained via the adjoint representation, defined in Eq. (A.14).
To see how this works, consider the trace

$$\mathrm{Tr}(\mathbf{J}_a\mathbf{J}_b) = \sum_{d,e} f_{ade}f_{edb} \,. \tag{A.16}$$

Here, it is easy to convince oneself that $\mathbf{J}_a\mathbf{J}_b$ is a real, symmetric matrix. Therefore, it can be diagonalized with real eigenvalues by appropriate choices of the generators $\mathbf{I}$, i.e. by replacing the original $\mathbf{I}$ with suitable linear combinations, and, correspondingly, by linear combinations of the generators of the adjoint representation $\mathbf{J}$. Thus,

$$\mathrm{Tr}(\mathbf{J}_a\mathbf{J}_b) = k_a\delta_{ab}\,, \tag{A.17}$$

and the freedom of rescaling the generators can be used to choose all k_a in a way such that $|k_a| = 1$. In the framework of this lecture, only compact, semi–simple Lie–groups will be encountered. They are characterized by positive k_a that can be rescaled to some common κ. In this particular basis, the f_{abc} are completely antisymmetric, because

$$f_{abc} = -\frac{i}{\kappa}\,\mathrm{Tr}([\mathbf{J}_a,\,\mathbf{J}_b]\,\mathbf{J}_c) \tag{A.18}$$

and by using the cyclic property of the trace. In this particular basis, all $\mathbf{J}_a$ are Hermitian matrices.

Casimir operators

Finally, note that in many cases further operators can be constructed that commute with all generators of a Lie group. Such operators are dubbed Casimir operators, and the number of independent Casimir operators defines the rank of the group. For $SU(2)$, for instance, it can be shown that the operator

$$\mathbf{J}^2 = \vec{\mathbf{J}}^2 = \mathbf{J}_1^2 + \mathbf{J}_2^2 + \mathbf{J}_3^2 \tag{A.19}$$

commutes with all the $\mathbf{J}_a$. For $SO(3)$ the only Casimir operator looks similar. Due to their commutation properties,the Casimir operators can be diagonalized simultaneously with the generators, and therefore their eigenvalues may be used to label the irreducible representations of the group.

A.2.3 Fun with $SO(2)$ and $SU(2)$

$SO(2)$

The group $SO(2)$ describes the set of all rotations of a circle around the axis perpendicular to its plane through its center. Each element can be characterized by one angle $\theta \in [0, 2\pi[$. Integer multiples of 2π added to the interval can obviously be ignored, since they do not add any new physical information. The law of composition for group elements $T(\theta)$ is

$$T(\theta) \cdot T(\phi) = T(\phi) \cdot T(\theta) = \begin{cases} T(\theta + \phi) & \text{if } \theta + \phi < 2\pi \\ T(\theta + \phi - 2\pi) & \text{if } \theta + \phi \geq 2\pi \end{cases} \tag{A.20}$$

The identity element is $T(0)$, inverse elements are

$$T^{-1}(\phi) = T(2\pi - \phi) \,. \tag{A.21}$$

A representation of this group can be constructed by studying the effect of the group elements on some point (x, y) in the plane. Then,

$$T(\phi) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} , \tag{A.22}$$

which is clearly an orthogonal 2×2–matrix with determinant 1. Coming back to Eq. (A.20), it is obvious that this group is Abelian, and following previous remarks, some alternative representation through $U(1)$ factors, namely $\exp(ic\phi)$ is recovered.
Due to its property as one–parameter group, $SO(2)$ has one generator, namely either

$$\mathbf{I} = \lim_{\phi\to 0} \frac{1}{i\phi} \left[\exp(im\phi) - 1\right] = m \tag{A.23}$$

or

$$\mathbf{I} = \lim_{\phi\to 0} \frac{1}{i\phi} \left[\begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \tag{A.24}$$

depending on the representation chosen for the group elements. Having identified the result for the generator in the second choice with the Pauli–matrix σ_y, the generator reads

$$T(\phi) = \exp(i\phi\sigma_y) \,. \tag{A.25}$$

Yet another generator can be found that has some more physical meaning by considering the effect of the $T(\phi)$ on some function $f = f(x, y)$ defined on the plane, the group acts on. Then,

$$T(\phi) f(x, y) = f(x\cos\phi + y\sin\phi, -x\sin\phi + y\cos\phi) \tag{A.26}$$

and the generator can be found through

$$
\begin{aligned}
\mathbf{I} \cdot f(x,y) &= \lim_{\phi\to 0} \frac{1}{i\phi} \left[f(x\cos\phi + y\sin\phi, -x\sin\phi + y\cos\phi) - f(x,y) \right] \\
&= -i \left(\frac{y\partial}{\partial x} - \frac{x\partial}{\partial y} \right) f(x,y)\,. \qquad \text{(A.27)}
\end{aligned}
$$

Identifying the component of the angular momentum normal to the (x, y)–plane $\mathbf{L}_z$ with

$$
\mathbf{L}_z = i\hbar \left(\frac{y\partial}{\partial x} - \frac{x\partial}{\partial y} \right) = -i\hbar \frac{\partial}{\partial \phi} \qquad \text{(A.28)}
$$

the generator finally reads

$$
T(\phi) = \exp\left(-\frac{i\phi \mathbf{L}_z}{\hbar} \right)\,. \qquad \text{(A.29)}
$$

SU(2)

The simplest non–Abelian Lie–algebra consists of three generators $\mathbf{J}_a$ with structure constants given by the completely antisymmetric tensor, $f_{abc} = \epsilon_{abc}$. Then the commutation relations read

$$
[\mathbf{J}_a,\, \mathbf{J}_b] = i\epsilon_{abc}\mathbf{J}_c\,, \qquad \text{(A.30)}
$$

the angular momentum algebra, known from the quantum mechanical treatment of spin and/or angular momentum. In principle, representations of the algebra of Eq. (A.30) could be constructed employing the Casimir operator $\vec{\mathbf{L}}^2$ in angular momentum or, analogously, of $\mathbf{J}^2$ for $SU(2)$. This is the textbook way of dealing with spin and angular momentum in Quantum Mechanics.

But rather than that, an approach employing ladder operators will be employed here, paving the way for a more general discussion in the next section. To this end, assume an N–dimensional irreducible representation of $SU(2)$, which can be diagonalized by one of the Hermitian operators, say $\mathbf{J}_3$. Then α eigenstates of $\mathbf{J}_3$ can be constructed $|j,\alpha\rangle$ corresponding to the maximal eigenvalue j. Hence,

$$
\mathbf{J}_3|j,\alpha\rangle = j|j,\alpha\rangle\,. \qquad \text{(A.31)}
$$

They can now be reshuffled in such a way that they are orthogonal,

$$\langle j, \beta | j, \alpha \rangle = \delta_{\alpha\beta} \,. \tag{A.32}$$

Non–Hermitian ladder operators

$$\mathbf{J}_\pm = \frac{1}{\sqrt{2}}(\mathbf{J}_1 \pm i\mathbf{J}_2) \tag{A.33}$$

can be constructed, which enjoy the following commutation relation with $\mathbf{J}_3$:

$$[\mathbf{J}_3, \mathbf{J}_\pm] = \pm\mathbf{J}_\pm \,, \quad [\mathbf{J}_+, \mathbf{J}_-] = \mathbf{J}_3 \,. \tag{A.34}$$

By acting on eigenstates $|m\rangle$ of $\mathbf{J}_3$, these ladder operators raise or lower the eigenvalue m of them by one unit, giving credit to their name,

$$\mathbf{J}_3\mathbf{J}_\pm|m\rangle = \mathbf{J}_\pm\mathbf{J}_3|m\rangle \pm \mathbf{J}_\pm|m\rangle = (m \pm 1)\mathbf{J}_\pm|m\rangle \,. \tag{A.35}$$

Consequently, when acting on the eigenstate with the highest eigenvalue, $\mathbf{J}_+$ yields

$$\mathbf{J}_+|j, \alpha\rangle = 0 \,. \tag{A.36}$$

Conversely, with some appropriate normalization $N_j(\alpha)$ the states

$$\mathbf{J}_-|j, \alpha\rangle = N_j(\alpha)|j-1, \alpha\rangle \tag{A.37}$$

are orthogonal,

$$\begin{aligned} N_j^*(\beta)N_j(\alpha)\langle j-1, \beta|j-1, \alpha\rangle &= \langle j, \beta|\mathbf{J}_+\mathbf{J}_-|j, \alpha\rangle \\ &= \langle j, \beta|[\mathbf{J}_+, \mathbf{J}_-]|j, \alpha\rangle \\ &= \langle j, \beta|\mathbf{J}_3|j, \alpha\rangle = j\delta_{\alpha\beta} \,. \end{aligned} \tag{A.38}$$

When replacing the product with the commutator in the equation above, the fact has been used that $\mathbf{J}_+$ annihilates the state $|j, \alpha\rangle$. Thus,

$$|N_j(\alpha)|^2 = j \tag{A.39}$$

and the phases of the $N_j(\alpha)$ can be chosen such that they are all real and positive numbers,

$$N_j(\alpha) = \sqrt{j} \,. \tag{A.40}$$

Therefore,

$$\begin{aligned}
\mathbf{J}_+|j-1,\alpha\rangle &= \frac{1}{N_j}\mathbf{J}_+\mathbf{J}_-|j,\alpha\rangle \\
&= \frac{1}{N_j}[\mathbf{J}_+,\mathbf{J}_-]|j,\alpha\rangle = N_j|j,\alpha\rangle \qquad \text{(A.41)}
\end{aligned}$$

This procedure can be repeated, and step by step orthonormal states $|j-k,\alpha\rangle$ are found which satisfy

$$\begin{aligned}
\mathbf{J}_-|j-k,\alpha\rangle &= N_{j-k}|j-k-1,\alpha\rangle \\
\mathbf{J}_+|j-k-1,\alpha\rangle &= N_{j-k}|j-k,\alpha\rangle
\end{aligned} \qquad \text{(A.42)}$$

with a recursion relation among the N_{j-k},

$$\begin{aligned}
N^2_{j-k} &= \langle j-k,\alpha|\mathbf{J}_+\mathbf{J}_-|j-k,\alpha\rangle \\
&= \langle j-k,\alpha|[\mathbf{J}_+,\mathbf{J}_-]|j-k,\alpha\rangle + \langle j-k,\alpha|\mathbf{J}_-\mathbf{J}_+|j-k,\alpha\rangle \\
&= j-k+N^2_{j-k+1}\,. \qquad \text{(A.43)}
\end{aligned}$$

Adding up these normalization factors from N_j^2 down to N^2_{j-k},

$$N^2_{j-k} = \frac{(k+1)(2j-k)}{2} \qquad \text{(A.44)}$$

or, introducing $m = j-k$

$$N_m = \sqrt{\frac{(j-m+1)(j+m)}{2}}\,. \qquad \text{(A.45)}$$

Climbing down the ladder, sooner or later some state $|j-l\rangle$ must be reached that gets annihilated by $\mathbf{J}_-$, $\mathbf{J}_-|j-l\rangle = 0$. But this requires

$$N_{j-l} = 0 \implies l = 2j\,. \qquad \text{(A.46)}$$

Since l is an integer number, the j are half–integers, and the space breaks up into α subspaces of dimension $2j+1$ plus possibly a subspace which might not have been found yet. This is exactly, what would have been found taking into account the knowledge about the Casimir operator J^2 and its eigenvalues. On the subspaces labeled by α, however, the J's act in a block-diagonal

form, since the orthogonality in the indices α and β, Eq. (A.38), remains conserved throughout this ladder procedure. But it was assumed that this procedure takes place in some irreducible representation, therefore there is only one α. This can be used to break down any representation into its irreducible components. Starting with any representation the block-diagonal form is reconstructed, where the highest j–representation is explicitly singled out. This procedure can be repeated with the remaining block until the representation has been decomposed completely.
The eigenvalues of $\mathbf{J}_3$ are called weights, and j is the highest weight. The irreducible representations are characterized by the highest weight j and they are called spin–j representations. In fact, they are associated with a particle at rest and spin angular momentum j, given by $J^2 = \hbar^2 j(j+1)$. The existence of Hermitian angular momentum operators in a problem allows to decompose the Hilbert space of the full problem into states $|j, m, \alpha\rangle$ in the spin–j representation, where m is the eigenvalue of $\mathbf{J}_3$, j is related to the eigenvalues of J^2 and α stands for all other labels, which have to be introduced, in order to characterize the states. States can be constructed which satisfy

$$\langle m, j, \alpha | m', j' \alpha' \rangle = \delta_{mm'} \delta_{jj'} \delta_{\alpha\alpha'} \,. \tag{A.47}$$

The δ–function in the α is in analogy with what has been discussed above, the δ–function in m and m' results from the fact that they are eigenvalues of an Hermitian operator (which has different eigenvectors and eigenvalues).
Even without employing the fact that the operator J^2 is Hermitian, too, it can be shown that the δ–function in j and j' holds. Suppose that $j' > j$ and consider

$$\langle j, j, \alpha | j, j', \beta \rangle = \sqrt{\frac{2}{(j'+j)(j'-j+1)}} \langle j, j, \alpha | \mathbf{J}_- | j+1, j', \beta \rangle = 0 \tag{A.48}$$

To round the discussion off, some irreducible representations of $SU(2)$ will explicitly be constructed:

- <u>$j = 1/2$:</u>
  The simplest non–trivial representation of this group has dimension 2, i.e. $j = 1/2$. The vectors, in which $\mathbf{J}_3$ is already diagonal are $|1/2\rangle = (1,0)^T$ and $|-1/2\rangle = (0,1)^T$. Hence, the ladder operators have the

form

$$\mathbf{J}_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \;,\; \mathbf{J}_- = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} . \qquad \text{(A.49)}$$

Thus,

$$\mathbf{J}_1 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \;,\; \mathbf{J}_2 = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad \text{(A.50)}$$

and by use of the commutator relations,

$$\mathbf{J}_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \qquad \text{(A.51)}$$

Not surprisingly, the common Pauli–matrices σ_i can be recovered. So, the simplest, non–trivial representation of $SU(2)$ is generated by $\sigma_i/2$.

- $\underline{j = 1 :}$
  The spin–1 representation is spanned by the following basis vectors : $|1\rangle = (1,0,0)^T$, $|0\rangle = (0,1,0)^T$ and $|-1\rangle = (0,0,1)^T$. The ladder operators are

$$\mathbf{J}_+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \;,\; \mathbf{J}_- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} . \qquad \text{(A.52)}$$

and

$$\begin{aligned} \mathbf{J}_1 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \;,\; \mathbf{J}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} , \\ \mathbf{J}_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} . \end{aligned} \qquad \text{(A.53)}$$

A.2.4 Roots and weights

Weights and the Cartan subalgebra

The next issue on the agenda is a generalization of the highest weight construction outlined above for the (almost trivial) case of $SU(2)$ to an arbitrary simple Lie–algebra, i.e. to a Lie algebra without non–Abelian subalgebras.

The basic idea for this is quite simple : Just divide the generators of the algebra into two sets. One set with Hermitian operators to be diagonalized, like $\mathbf{J}_3$ in the case of $SU(2)$, the other set consisting of non–Hermitian ladder operators, like the $\mathbf{J}_\pm$. In other words, suppose a simple Lie algebra with generators $\mathbf{X}_a$ in the irreducible representation D is to be dealt with. Then, a maximal set of M Hermitian operators $\mathbf{H}_{1,\ldots,M}$ is to be constructed, which satisfy

- the $\mathbf{H}_i = c_{ia}\mathbf{X}_a$ are linear combinations of the $\mathbf{X}_a$ with *real* coefficients c_{ia} (corresponding to a suitable choice of basis),
- they are Hermitian operators,
- the $\mathbf{H}_i$ all mutually commute,

$$[\mathbf{H}_i,\, \mathbf{H}_j] = 0\,, \tag{A.54}$$

- and they are normalized according to

$$\mathrm{Tr}(\mathbf{H}_i\mathbf{H}_j) = k_D\delta_{ij}\,. \tag{A.55}$$

This establishes that the $\mathbf{H}_i$ are in perfect analogy with the single $\mathbf{J}_3$ of the $SU(2)$ case. Obviously, their eigenvalues can then be used to label any state of the representation D in an unambiguous way. This again is in analogy with $SU(2)$, where the spin j is used to denote states. Consequently, M parameters are needed to label the states completely, and M is therefore dubbed the *rank* of the algebra.
Since the $\mathbf{H}_i$ are Hermitian and commute, they can simultaneously be diagonalized, with eigenvectors $|\mu, D\rangle$ which satisfy

$$\mathbf{H}_i|\mu, D\rangle = \mu_i|\mu, D\rangle\,. \tag{A.56}$$

These eigenvalues are called *weights*, the vectors with components μ_i are called *weight vectors* and the set of the $\mathbf{H}_i$ form the *Cartan subalgebra*. In other words, a Cartan subalgebra is the maximal Abelian subalgebra of a Lie–algebra.

Roots

For every representation of the group in question, weights can be defined, using the same generators H_i. To exemplify this, consider the adjoint representation defined by Eq. (A.14). Beyond the formal definition given there, one may think of the adjoint representation as a realization of the group in a basis, where the generators themselves become states, i.e. where $\mathbf{X}_a \to |X_a\rangle$. Scalar products of such states are given by

$$\langle \mathbf{X}_a | X_b \rangle = -\frac{1}{\lambda}\mathrm{Tr}(\mathbf{X}_a^\dagger \mathbf{X}_b) \tag{A.57}$$

with normalization λ and, furthermore,

$$\mathbf{X}_a | X_b \rangle = |[\mathbf{X}_a,\, \mathbf{X}_b]\rangle \tag{A.58}$$

for the action of the generators on the states. As a next step, a convenient basis spanning this space needs to be found. This sounds complicated, but it is not. There already is a very suitable set of operators, namely the $\mathbf{H}_i$. Using them yields

$$\mathbf{H}_i | H_j \rangle = |[\mathbf{H}_i,\, \mathbf{H}_j]\rangle = 0\,, \tag{A.59}$$

see Eq. (A.54). This allows to straightforwardly diagonalize the rest of the space. This leads to states $|E_\alpha\rangle$, which satisfy

$$\mathbf{H}_i | E_\alpha \rangle = \alpha_i | E_\alpha \rangle\,. \tag{A.60}$$

These states correspond to generators $\mathbf{E}_\alpha$ through

$$[\mathbf{H}_i,\, \mathbf{E}_\alpha] = \alpha_i \mathbf{E}_\alpha\,. \tag{A.61}$$

At this stage, it is important to keep in mind that the $\mathbf{H}_i$ and the $\mathbf{E}_\alpha$ are linear combinations of the generators $\mathbf{X}_a$ of the fundamental representation, and thus states in the adjoint representation. Therefore, the $\alpha_{i=1,\ldots,M}$ are components of the weight vectors α in the adjoint representation. Note also that the generators $\mathbf{E}_\alpha$ are not Hermitian,

$$[\mathbf{H}_i,\, \mathbf{E}_\alpha^\dagger] = -\alpha_i \mathbf{E}_\alpha^\dagger \implies \mathbf{E}_\alpha^\dagger = \mathbf{E}_{-\alpha}\,. \tag{A.62}$$

This property can be seen by remembering that the $\mathbf{H}_i$ are Hermitian and by taking the Hermitian conjugate of Eq. (A.61). The weights of the adjoint representation, the α_i are called *roots*, forming root vectors. Finally, normalize

$$\begin{aligned} \langle \mathbf{E}_\alpha | E_\beta \rangle &= -\frac{1}{\lambda} \mathrm{Tr}(\mathbf{E}_\alpha^\dagger \mathbf{E}_\beta) = \delta_{\alpha\beta} \\ \langle \mathbf{H}_i | H_j \rangle &= -\frac{1}{\lambda} \mathrm{Tr}(\mathbf{H}_i \mathbf{H}_j) = \delta_{ij} \,. \end{aligned} \tag{A.63}$$

Comparing the second of these equations with Eq. (A.55) $k_D = \lambda$ can be identified, if D is the adjoint representation. In fact, once the normalization λ is fixed, all other k_D are fixed as well.

The E_α as ladder operators

From

$$\mathbf{H}_i \mathbf{E}_\alpha |\mu, D\rangle = [\mathbf{H}_i,\, \mathbf{E}_\alpha] |\mu, D\rangle + \mathbf{E}_\alpha \mathbf{H}_i |\mu, D\rangle = (\alpha_i + \mu_i)\, \mathbf{E}_\alpha |\mu, D\rangle \tag{A.64}$$

the E_α are readily identified as raising and lowering operators for the weights. This is in complete analogy with the $\mathbf{J}_\pm$ of $SU(2)$, where their particular root vector α in fact is just a number, namely $\alpha = 1$.
Exploiting this correspondence allows to derive properties of the roots and weights, which are in close analogy to the statement that in $SU(2)$ the eigenvalues of $\mathbf{J}_3$, i.e. the weights, are half–integers. Consider any state $|\mu, D\rangle$. If there is more than one state with weight μ, the starting state $|\mu, D\rangle$ is chosen to be an eigenstate of $\mathbf{E}_{-\alpha}\mathbf{E}_\alpha$ for some fixed α. Imposing this condition on the state ensures that the state is not a linear combination of states with the same μ but from different representations. Then the state may be normalized through a constant $N_{\pm\alpha,\mu}$ namely

$$\mathbf{E}_{\pm\alpha} |\mu, D\rangle = N_{\pm\alpha,\mu} |\mu \pm \alpha, D\rangle \,. \tag{A.65}$$

Turning back to the adjoint representation shows that $\mathbf{E}_\alpha |E_{-\alpha}\rangle$ has weight zero, since

$$\begin{aligned} \mathbf{H}_i \left[\mathbf{E}_\alpha | E_{-\alpha} \rangle \right] &= \mathbf{H}_i | [\mathbf{E}_\alpha,\, \mathbf{E}_{-\alpha}] \rangle \\ &= | \left[\mathbf{H}_i,\, [\mathbf{E}_\alpha,\, \mathbf{E}_{-\alpha}] \right] \rangle = 0 \end{aligned} \tag{A.66}$$

by employing the Jacobi–identity and Eq. (A.61). For the state $|E_{-\alpha}\rangle$ the following ansatz can be made

$$\mathbf{E}_\alpha|E_{-\alpha}\rangle = \beta_i|\mathbf{H}_i\rangle\,, \tag{A.67}$$

which is motivated by the fact that $[\mathbf{H}_i,\,\mathbf{H}_j] = 0$ and by identifying $[\mathbf{E}_\alpha,\,\mathbf{E}_{-\alpha}] \to \mathbf{H}_j$. Multiplying from the left with $\langle H_j|$ and with the normalization condition for the products $\langle H_j|H_i\rangle$ the β_i can be fixed as

$$\beta_i \;=\; \langle H_i|\mathbf{E}_\alpha|E_{-\alpha}\rangle = \frac{\mathrm{Tr}(\mathbf{H}_i[\mathbf{E}_\alpha,\,\mathbf{E}_{-\alpha}])}{\lambda} = \frac{\mathrm{Tr}(\mathbf{E}_{-\alpha}[\mathbf{H}_i,\,\mathbf{E}_\alpha])}{\lambda} = \alpha_i\,. \tag{A.68}$$

Thus,

$$[\mathbf{E}_\alpha,\,\mathbf{E}_{-\alpha}] = \alpha_i\mathbf{H}_i\,. \tag{A.69}$$

To further analyze the commutator $[\mathbf{E}_\alpha,\,\mathbf{E}_\beta]$ an equation similar to Eq. (A.64) is employed

$$\begin{aligned}&\mathbf{H}_i[\mathbf{E}_\alpha,\,\mathbf{E}_\beta]|\mu,D\rangle\\ &\quad= \{[\mathbf{H}_i,\,\mathbf{E}_\alpha] + \mathbf{E}_\alpha\mathbf{H}_i\}\,\mathbf{E}_\beta|\mu,D\rangle - \{[\mathbf{H}_i,\,\mathbf{E}_\beta] + \mathbf{E}_\beta\mathbf{H}_i\}\,\mathbf{E}_\alpha|\mu,D\rangle\\ &\quad= (\alpha_i+\beta_i+\mu_i)\,[\mathbf{E}_\alpha,\,\mathbf{E}_\beta]|\mu,D\rangle\,. \end{aligned}\tag{A.70}$$

From this

$$[\mathbf{E}_\alpha,\,\mathbf{E}_\beta] \sim \mathbf{E}_{\alpha+\beta}\,. \tag{A.71}$$

Climbing ladders

As a first step, the normalization constants are calculated, in order to construct recursion relations similar to Eq. (A.43) in the case of $SU(2)$.

$$\langle\mu,D|\alpha_i\mathbf{H}_i|\mu,D\rangle \;=\; \langle\mu,D|[\mathbf{E}_\alpha,\mathbf{E}_{-\alpha}]|\mu,D\rangle \tag{A.72}$$

leading to

$$\begin{aligned}\alpha\cdot\mu \;&=\; \langle\mu,D|\mathbf{E}_\alpha\mathbf{E}_{-\alpha}|\mu,D\rangle - \langle\mu,D|\mathbf{E}_{-\alpha}\mathbf{E}_\alpha|\mu,D\rangle\\ &=\; |N_{-\alpha,\mu}|^2 - |N_{\alpha,\mu}|^2\,.\end{aligned}\tag{A.73}$$

In addition, the normalization constants also satisfy

$$\begin{aligned} N_{-\alpha,\mu} &= \langle \mu-\alpha, D|\mathbf{E}_{-\alpha}|\mu, D\rangle = \langle \mu-\alpha, D|E_\alpha^\dagger|\mu, D\rangle \\ &= \langle \mu, D|\mathbf{E}_\alpha|\mu-\alpha, D\rangle^* = N^*_{\alpha,\mu-\alpha} \end{aligned} \tag{A.74}$$

resulting in

$$|N_{\alpha,\mu-\alpha}|^2 - |N_{\alpha,\mu}|^2 = \alpha\cdot\mu\,. \tag{A.75}$$

Since, by assumption, the Lie-algebra under consideration is finite-dimensional, eventually zero must emerge when applying $\mathbf{E}_{\pm\alpha}$ repeatedly on some state $|\mu, D\rangle$. Suppose, this is the case for

$$\mathbf{E}_\alpha|\mu+p\alpha, D\rangle = 0 \text{ and } \mathbf{E}_{-\alpha}|\mu-q\alpha, D\rangle = 0 \tag{A.76}$$

with positive integers p and q. Then climbing down the full ladder,

$$\begin{array}{lcllcl} |N_{\alpha,\mu+(p-1)\alpha}|^2 & - & 0 & = & \alpha\cdot[\mu+p\alpha] \\ |N_{\alpha,\mu+(p-2)\alpha}|^2 & - & |N_{\alpha,\mu+(p-1)\alpha}|^2 & = & \alpha\cdot[\mu+(p-1)\alpha] \\ & & \cdots & & \\ |N_{\alpha,\mu}|^2 & - & |N_{\alpha,\mu+\alpha}|^2 & = & \alpha\cdot[\mu+\alpha] \\ |N_{\alpha,\mu-\alpha}|^2 & - & |N_{\alpha,\mu}|^2 & = & \alpha\cdot\mu \\ & & \cdots & & \\ |N_{\alpha,\mu-q\alpha}|^2 & - & |N_{\alpha,\mu-(q-1)\alpha}|^2 & = & \alpha\cdot[\mu-(q-1)\alpha] \\ 0 & - & |N_{\alpha,\mu-q\alpha}|^2 & = & \alpha\cdot[\mu-q\alpha] \end{array} \tag{A.77}$$

and summing the differences on the left and the scalar products on the right yields

$$\begin{aligned} 0 &= (p+q+1)\alpha\cdot\mu + \left(\frac{p(p+1)}{2} - \frac{q(q+1)}{2}\right)\alpha^2 \\ &= (p+q+1)\left(\alpha\cdot\mu + \alpha^2\frac{p-q}{2}\right)\,. \end{aligned} \tag{A.78}$$

But this immediately implies that

$$\frac{\alpha\cdot\mu}{\alpha^2} = -\frac{p-q}{2} \tag{A.79}$$

is a half–integer. Correspondingly, all $|N|^2$ can now be determined in terms of μ, α, p, and q. In fact, this holds true for any representation, but it has particularly simple and strong consequences for the adjoint representation. In the adjoint representation, μ is a root, say β with

$$\frac{\alpha \cdot \beta}{\alpha^2} = -\frac{p-q}{2} = \frac{m}{2} \tag{A.80}$$

and integer m. Applying $\mathbf{E}_{\pm\beta}$ on $|\mathbf{E}_\alpha\rangle$ results in

$$\frac{\beta \cdot \alpha}{\beta^2} = -\frac{p'-q'}{2} = \frac{m'}{2} \tag{A.81}$$

and integer m'. Multiplying Eqs. (A.80) and (A.81) implies that

$$\frac{mm'}{4} = -\frac{(\alpha \cdot \beta)^2}{\alpha^2 \beta^2} \equiv \cos^2\theta\,, \tag{A.82}$$

where θ is the angle between the root vectors. As a consequence, only a limited set of angles between root vectors is allowed at all, namely

$$\begin{array}{ll} mm' & \theta \\ 0 & \pi/2 \\ 1 & \pi/3\,,\ 2\pi/3 \\ 2 & \pi/4\,,\ 3\pi/4 \\ 3 & \pi/6\,,\ 5\pi/6 \\ 4 & 0\,,\ \pi\,. \end{array} \tag{A.83}$$

So far the implicit assumption was that for every non–zero root vector α, there is some unique generator $\mathbf{E}_\alpha$. Now this can be proven. Suppose there are two generators $\mathbf{E}_\alpha$ and $\mathbf{E}'_\alpha$ associated with the same root, which, without loss of generality can be chosen to be orthogonal,

$$\langle \mathbf{E}_\alpha | E'_\alpha \rangle \sim \mathrm{Tr}(\mathbf{E}_\alpha \mathbf{E}'_\alpha) = 0\,. \tag{A.84}$$

Applying now $\mathbf{E}_{\pm\alpha}$ on the state $|E'_\alpha\rangle$ gives

$$\frac{\alpha \cdot \alpha}{\alpha^2} = -\frac{p-q}{2}\,. \tag{A.85}$$

But since also

$$\begin{aligned}
\mathbf{E}_{-\alpha}|E'_\alpha\rangle &= \beta_i|H_i\rangle \\
\text{with } \beta_i &= \langle H_i|\mathbf{E}_{-\alpha}|E'_\alpha\rangle \sim \mathrm{Tr}(\mathbf{H}_i[\mathbf{E}_{-\alpha}, E'_\alpha]) \\
&= \mathrm{Tr}(\mathbf{E}'_\alpha[\mathbf{H}_i, \mathbf{E}_{-\alpha}]) = -\alpha_i \mathrm{Tr}(\mathbf{E}_{-\alpha}\mathbf{E}'_\alpha) = 0
\end{aligned} \tag{A.86}$$

$$\mathbf{E}_{-\alpha}|E'_\alpha\rangle = 0 \tag{A.87}$$

and therefore $q = 0$. Hence,

$$\frac{\alpha \cdot \alpha}{\alpha^2} = -\frac{p}{2} < 0\,. \tag{A.88}$$

But this is impossible, and therefore $\mathbf{E}_\alpha$ is unique.

Connection to $SU(2)$

The steps leading to Eq. (A.79) are essentially the same as the ones performed in Sec. A.2.3 to determine the representation of $SU(2)$. This connection might become even clearer in the following. To start with, define rescaled operators

$$\begin{aligned}
\mathbf{E}_\pm &= \frac{1}{|\alpha|}\mathbf{E}_{\pm\alpha} \\
\mathbf{E}_3 &= \frac{1}{|\alpha|^2}\alpha_i \mathbf{H}_i
\end{aligned} \tag{A.89}$$

with $|\alpha|^2 = \alpha \cdot \alpha$. Using Eqs. (A.61), (A.62), and (A.69), it is easy to show that these operators satisfy the $SU(2)$ algebra,

$$[\mathbf{E}_3,\, \mathbf{E}_\pm] = \pm\mathbf{E}_\pm\,, \quad [\mathbf{E}_+,\, \mathbf{E}_-] = \mathbf{E}_3\,. \tag{A.90}$$

Thus, each $\mathbf{E}_\alpha$ is associated with an $SU(2)$ subalgebra, comprising $\mathbf{E}_\alpha$ and its adjoint $\mathbf{E}_{-\alpha}$ and a linear combination of Cartan operators $\mathbf{H}_i$. If $|\mu, D\rangle$ is an eigenstate of $\mathbf{E}_{-\alpha}\mathbf{E}_\alpha$, then the states $|\mu + k\alpha, D\rangle$ obtained by multiple application of the ladder operators $\mathbf{E}_{\pm\alpha}$ form an irreducible representation of the $SU(2)$ subalgebra of Eq. (A.90). The condition that $|\mu, D\rangle$ is an eigenstate of $\mathbf{E}_{-\alpha}\mathbf{E}_\alpha$ insures that this state is not merely a linear combination of states with the same μ from different representations D. However, the newly obtained states satisfy

$$\mathbf{E}_3|\mu + k\alpha, D\rangle = \frac{\alpha \cdot \mu + k\alpha^2}{|\alpha|^2}|\mu + k\alpha, D\rangle\,. \tag{A.91}$$

The highest and lowest eigenvalues of $\mathbf{E}_3$ in the representation corresponding to $|\mu + p\alpha, D\rangle$ and $|\mu - q\alpha, D\rangle$, respectively, must then be

$$\begin{aligned} j &= \frac{\alpha \cdot \mu}{\alpha^2} + p \\ -j &= \frac{\alpha \cdot \mu}{\alpha^2} - q \\ 0 &= 2\left[\frac{\alpha \cdot \mu}{\alpha^2} + \frac{p-q}{2}\right]\,. \end{aligned} \tag{A.92}$$

A.2.5 Simple roots and fundamental representations

Generalization of the highest weight

The concept of highest weight already encountered in our treatment of $SU(2)$ can now be extended to arbitrary Lie groups. The basic idea is to define "positivity" for weight vectors. After fixing the Cartan basis, i.e. the H_1, H_2, ..., the components of our weight vectors μ_1, μ_2, ... are fixed, too. Then a weight vector is called positive (negative), it its first non-zero component is positive (negative). Consequently it is dubbed zero, if all components are zero. This is an arbitrary and somewhat sloppy division of the space into two halves, and intuitively it is not clear at all that such a definition is any good. In fact, this definition will turn out to be surprisingly fruitful, allowing to define raising and lowering operators. It will remain to be shown that this arbitrary *ad-hoc* ordering does not induce any prejudices into the notation and spoil it.
Having introduced positivity allows some ordering of weights ($x > y$ if $x - y > 0$, i.e. positive). Now, obviously, the highest weight of any representation is the weight, which is larger than any other weight. According to the definition of positivity here, it is the weight with the largest non–zero first component. Again, the highest weight is of special importance, as was the case for $SU(2)$. It will turn out, that it is the same reason, that makes it unique: Starting from the highest weight the whole representation can be constructed. This concept can also be applied to the roots, the weights of

the adjoint representation. They are either positive or negative and they correspond to raising or lowering operators, respectively. Obviously, when some of the raising operators $E_{\alpha>0}$ act on highest weight in any representation, this must yield zero.
A *simple root* is a positive root, which cannot be decomposed as a sum of two positive roots. The simple roots play a special role, since they determine the whole structure of the group. They enjoy the following properties:

- If α and β are simple roots, then $\beta - \alpha$ is not a root.
  Proof : Assume $\beta - \alpha > 0$ is a root. Then $\beta = \alpha + (\beta - \alpha)$, the sum of two positive roots, ergo β is not simple. The same reasoning holds true for $\alpha - \beta > 0$ and a root.

  Since $\beta - \alpha$ is not a root, $E_{-\alpha}|E_\beta\rangle$ is not a weight in the adjoint representation, i.e. no root. Therefore, $E_{-\alpha}|E_\beta\rangle = 0$, and

  $$\frac{\alpha \cdot \beta}{\alpha^2} = -\frac{p-q}{2} = -\frac{p}{2}\,, \tag{A.93}$$

  see Eq. (A.79). Remember that q equals zero here, because there are exactly zero times $E_{-\alpha}$ has to be applied to annihilate $|E_\beta\rangle$, if both are simple roots.

- Thus, knowing all the integers p is equivalent to knowing all the relative angles between simple roots and their relative length :

  $$\frac{2\alpha \cdot \beta}{\alpha^2} = -p\,, \quad \frac{2\alpha \cdot \beta}{\beta^2} = -p' \tag{A.94}$$

  implies

  $$\cos\theta_{\alpha\beta} = -\frac{1}{2}\sqrt{pp'}\,, \quad \frac{\beta^2}{\alpha^2} = \frac{p}{p'}\,. \tag{A.95}$$

  Since p,$p' > 0$, the angle between any two simple roots satisfies

  $$\frac{\pi}{2} \leq \theta < \pi\,. \tag{A.96}$$

  Simple geometry implies that any set of vectors with this constraint is linearly independent.

Proof: If they were not, constants x_α could be found with

$$\sum_\alpha x_\alpha \alpha = 0 \tag{A.97}$$

for the simple roots α. They could be divided into two sets, $\Gamma_\pm$ with $\alpha \in \Gamma_+$ if $x_\alpha > 0$ and vice versa. Then,

$$y \equiv \sum_{\alpha\in\Gamma_+} x_\alpha \alpha = \sum_{\alpha\in\Gamma_-} (-x_\alpha)\alpha \equiv z\,, \tag{A.98}$$

where now all x_α are non–negative. Both y and z are then positive (or at least non–zero) vectors by construction. But this is impossible, since

$$y^2 = y \cdot z \le 0\,, \tag{A.99}$$

which follows from $\alpha \cdot \beta \le 0$ for any two simple roots α and β.

- Consequently, any positive root ϕ can be written as a sum of simple roots with non–negative integer coefficients k_α.

  Proof : By induction. If ϕ is simple, the proof is complete. If not, ϕ can be split into two positive roots ϕ_1 and ϕ_2. If they are not simple the process of splitting can be continued.

- The number of simple roots is equal to the rank of the group M.

  Proof : Because the roots are vectors with M components, there can not be more than M linearly independent of their kind. Now assume that there are less than M simple roots, so that they do not span the full space of roots. Then, a basis can be chosen such that the first components of all simple roots α vanish. But this implies that the first component of every root vanishes. This in turn has the consequence, that

$$[H_1, E_\phi] = 0 \text{ for all roots } \phi\,. \tag{A.100}$$

  But, since $[H_1, H_i] = 0$ by definition of the Cartan subalgebra, this implies that H_1 commutes with everything – it is an invariant subalgebra in its own rights ! This is in contradiction to the basic assumption that the group is a simple Lie group.

- If it can be determined, which of the linear combinations

$$\sum_{\alpha} k_\alpha \alpha \ , \quad k_\alpha \text{ nonnegative integers} \tag{A.101}$$

are roots, all the roots of a group can be determined, for instance by induction. For $k = \sum k_\alpha = 1$, the sums are just the simple roots. The case $k = 2$ can be constructed by acting on the $k = 1$ configurations with the generators, which correspond to the simple roots. Using Eq. (A.79) distinguishes between new roots or the annihilation of the state. For instance, $\alpha + \beta$ is a root, if $\alpha \cdot \beta < 0$, because then

$$\frac{2\alpha \cdot \beta}{\alpha^2} = -p \implies p > 0 \, , \tag{A.102}$$

where p is the largest integer, so that $\alpha + p\beta$ is a root. In this specific case, $q = 0$ is known, because for $k = 2$, α and β are simple roots. But for more complicated cases, $k = n$ plus the action of a simple root α on it leading to $k = n + 1$, more care has to be taken. In principle, q is not necessarily zero, but it can be determined by examining the roots with $k < n$. Knowing q, then p can be fixed with help of Eq. (A.79),

$$\frac{2\alpha \cdot \gamma}{\alpha^2} = -(p - q) \, . \tag{A.103}$$

If $p \neq 0$, $\gamma + \alpha$ is a root. In this manner all roots associated with $k = n + 1$ can be found, which can be written as sums $\gamma + \alpha$, where γ stems from the step $k = n$ and α is simple. But that's it already for $k = n + 1$. There are just no roots different from $\gamma + \alpha$.

Proof : Suppose, there's some root ρ, which does not have the desired form $\gamma + \alpha$. Then, since $\rho - \alpha$ is not a root, $q = 0$, and

$$\frac{2\alpha \cdot \rho}{\alpha^2} = -p < 0 \, . \tag{A.104}$$

In this case, ρ is linearly independent of all simple roots α leading to a contradiction.

This last point is quite instructive. Having determined all roots inductively starting from the simple roots by using only lengths and angles between them and by employing the master formula Eq. (A.79). Obviously, the full treatment is independent of the basis in root space and of the specific notion of positivity.

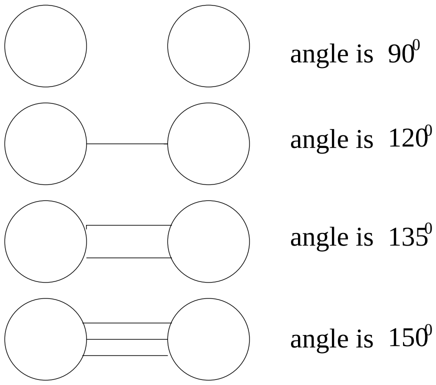


Figure A.1: The rules for the angles between simple roots for Dynkin diagrams.

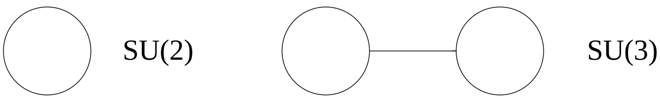


Figure A.2: Dynkin diagrams for $SU(2)$ and $SU(3)$.

Dynkin diagrams

The reasoning so far implies that the simple roots are all that is needed in order to construct a representation of the group. The knowledge of the simple roots can be used to construct all roots, which in turn fix the normalization factors $N_{\alpha,\beta}$ of Eq. (A.65) and thus describe the full algebra.
The treatment of the preceding section allows the construction of a diagrammatic shorthand notation for pinning down the simple roots. This shorthand consists of open circles, representing the simple roots, which are connected by lines. The number of lines linking two circles corresponds to the angle between the simple roots as in Fig.A.1.
As examples consider the Dynkin diagrams for $SU(2)$ and $SU(3)$ as displayed in Fig.A.2.

Fundamental weights

In the following the simple roots are labelled as α^κ with $\kappa = 1, \ldots, M$. Some weight μ of an arbitrary irreducible representation, D, is the highest weight, if and only if $\mu + \phi$ is not a weight in this representation for all positive roots ϕ. Recalling the decomposition of positive roots in terms of simple roots, it is then sufficient to claim that $\mu + \alpha^\kappa$ for any κ is not a weight. Then, acting with E_{α^κ} on $|\mu\rangle$ yields $p = 0$ and hence

$$\frac{2\alpha^\kappa \cdot \mu}{\alpha^{\kappa 2}} = q^\kappa \tag{A.105}$$

with non–negative integers q^κ. But this helps in determining μ, since the α^κ are linearly independent. So every set of q^κ defines the highest weight of some irreducible representation of the group, and starting from the highest weight by applying the lowering operators $E_{-\alpha^\kappa}$ allows to construct the entire representation.

What does this mean for practical purposes? To gain some insight, consider weight vectors μ^λ with

$$\frac{2\alpha^\kappa \mu^\lambda}{\alpha^{\kappa 2}} = \delta^{\kappa\lambda}\,. \tag{A.106}$$

Thus, any μ^λ is the highest weight of an irreducible representation with $q^\lambda = 1$ and $q^{\kappa\neq\lambda} = 0$. Therefore, any highest weight μ can be written as linear combination

$$\mu = \sum_\kappa q^\kappa \mu^\kappa \tag{A.107}$$

and therefore representations with arbitrary highest weight μ can be constructed by merely gluing together representations with highest weight μ^1, μ^2 and so on. This is in full analogy to the way, spin–$n/2$ representations of $SU(2)$ are built. Typically this is achieved by sticking together n spin–$1/2$ representations. The vectors μ^κ are dubbed *fundamental weights* and the m irreducible representations with highest weights given by the fundamental weights are called the *fundamental representations.*

In the treatment above, one big caveat is in order: Please, do not confuse the lower indices which label the components of all the weight and root vectors with the upper indices labeling the simple roots and fundamental representations. Both indices run from 1 to M, but the first ones are vector components and the latter ones just labels for different vectors.

A.2.6 Example: $SU(3)$

Generators

$SU(3)$ is the group of 3×3 unitary matrices with determinant 1. The generators of this group are the 3×3 traceless Hermitian matrices. The tracelessness guarantees hermiticity and that the determinant in fact equals one. The standard basis of $SU(3)$ used in particle physics consists of the *Gell–Mann* λ–matrices

$$\begin{array}{lll} \lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & & \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} & & \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\ \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} & & \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\ \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} & & \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \end{array} \tag{A.108}$$

The generators are given by $\mathbf{T}_a = \lambda_a/2$and they are normalized according to

$$\mathrm{Tr}(\mathbf{T}_a \mathbf{T}_b) = \frac{1}{2}\delta_{ab}\,. \tag{A.109}$$

Notice that $\mathbf{T}_1$, $\mathbf{T}_2$, and $\mathbf{T}_3$ form an $SU(2)$ subgroup of $SU(3)$, which is dubbed the isospin subgroup for obvious reasons.

Introducing $SU(3)$: The isotropic harmonic oscillator

To become acquainted with $SU(3)$ consider first the isotropic harmonic oscillator in three dimensions. Its Hamiltonian is given by

$$\mathcal{H} = \frac{\vec{p}^2}{2m} + \frac{k\vec{r}^2}{2} = \frac{1}{2m}\sum_{j=1}^{3}(p_j^2 + m^2\omega^2 r_j^2) \tag{A.110}$$

A first naive guess is that its symmetry group is $O(3)$. But this guess misses the truth. In fact, the symmetry group of the three–dimensional isotropic harmonic oscillator is $SU(3)$. To see this, switch to the common raising and lowering operators

$$\begin{aligned} \mathbf{a}_j &= \frac{\mathbf{p}_j - i\omega m\mathbf{r}_j}{\sqrt{2m\omega\hbar}} \\ \mathbf{a}_j^\dagger &= \frac{\mathbf{p}_j + i\omega m\mathbf{r}_j}{\sqrt{2m\omega\hbar}} \,. \end{aligned} \tag{A.111}$$

Their commutation relations are

$$[\mathbf{a}_i,\, \mathbf{a}_j^\dagger] = \delta_{ij}\,, \quad [\mathbf{a}_i,\, \mathbf{a}_j] = [\mathbf{a}_i^\dagger,\, \mathbf{a}_j^\dagger] = 0\,, \tag{A.112}$$

and they follow directly from the commutation relations for the position and momentum operators. In fact, they are merely generalizations of the commutation relations for one dimension to the ones for three dimensions. With help of these operators, the Hamiltonian can be rewritten as

$$\mathcal{H} = \hbar\omega \sum_j \left(\mathbf{a}_j^\dagger \mathbf{a}_j + \frac{1}{2} \right) = \frac{\hbar\omega}{2} \sum_j \left\{ \mathbf{a}_j^\dagger, \mathbf{a}_j \right\}\,, \tag{A.113}$$

where the anti-commutator $\{\mathbf{a}, \mathbf{b}\} = \mathbf{ab} + \mathbf{ba}$. The commutators of the raising and lowering operators with the Hamiltonian are

$$\left[\mathcal{H}, \mathbf{a}_i^\dagger\right] = \hbar\omega \mathbf{a}_i^\dagger\,, \quad [\mathcal{H}, \mathbf{a}_i] = -\hbar\omega \mathbf{a}_i \tag{A.114}$$

explaining their names. The occupation number operator

$$\mathcal{N}_i = \mathbf{a}_i^\dagger \mathbf{a}_i \tag{A.115}$$

has integer eigenvalues $n_i \geq 0$. The eigenvalues of the Hamiltonian are thus

$$E_n = \left(n_1 + n_2 + n_3 + \frac{3}{2} \right) \hbar\omega\,, \tag{A.116}$$

see Eq. (A.113). After some bookkeeping it becomes apparent that there are $(n+1)(n+2)/2$ ways, in which any positive integer number n can be divided into three non–negative integers n_1, n_2,and n_3. Therefore, for every energy level labeled by the main quantum number n, there are $(n+1)(n+2)/2$ states, with a corresponding degeneracy. This indeed is labelled by the angular momentum. Thus, now the angular momentum operator is to be constructed. In its components it is given by

$$\mathbf{L}_j = (\vec{\mathbf{r}} \times \vec{\mathbf{p}})_j = \frac{i\hbar}{2} \sum_{k,l=1}^{3} \epsilon_{jkl} (\mathbf{a}_k \mathbf{a}_l^\dagger - \mathbf{a}_k^\dagger \mathbf{a}_l) \,. \tag{A.117}$$

It can be shown that, in general, operators of the form $\mathbf{a}_i^\dagger \mathbf{a}_j$ commute with $\mathcal{H}$, and so does the angular momentum operator. The physical effect of such operators $\mathbf{a}_i^\dagger \mathbf{a}_j$ is to annihilate a quantum in the j–direction and transfer it to the i–direction. It therefore leaves the number of quanta fixed. Therefore, the total quantum number operator

$$\mathcal{N} = \sum_{j=1}^{3} \mathbf{a}_j^\dagger \mathbf{a}_j = \frac{\mathcal{H}}{\hbar\omega} - \frac{3}{2} \tag{A.118}$$

commutes with all nine operators of the form $\mathbf{a}_i^\dagger \mathbf{a}_j$. But from these one might switch to some basis, which is more convenient for further investigations. Starting with $\mathcal{N}$, eight other independent linear combinations of the $\mathbf{a}_i^\dagger \mathbf{a}_j$ can be constructed, namely

$$\begin{array}{rclrcl}
\lambda_1 &=& \mathbf{a}_1^\dagger \mathbf{a}_2 + \mathbf{a}_2^\dagger \mathbf{a}_1 \,, & \lambda_2 &=& -i(\mathbf{a}_1^\dagger \mathbf{a}_2 - \mathbf{a}_2^\dagger \mathbf{a}_1) \,, \\
\lambda_3 &=& \mathbf{a}_1^\dagger \mathbf{a}_1 - \mathbf{a}_2^\dagger \mathbf{a}_2 \,, & \lambda_4 &=& \mathbf{a}_1^\dagger \mathbf{a}_3 + \mathbf{a}_3^\dagger \mathbf{a}_1 \,, \\
\lambda_5 &=& -i(\mathbf{a}_1^\dagger \mathbf{a}_3 - \mathbf{a}_3^\dagger \mathbf{a}_1) \,, & \lambda_6 &=& \mathbf{a}_2^\dagger \mathbf{a}_3 + \mathbf{a}_3^\dagger \mathbf{a}_2 \,, \\
\lambda_7 &=& -i(\mathbf{a}_2^\dagger \mathbf{a}_3 - \mathbf{a}_3^\dagger \mathbf{a}_2) \,, & \lambda_8 &=& \frac{1}{\sqrt{3}}(\mathbf{a}_1^\dagger \mathbf{a}_1 + \mathbf{a}_2^\dagger \mathbf{a}_2 - 2\mathbf{a}_3^\dagger \mathbf{a}_3) \,.
\end{array} \tag{A.119}$$

These operators can be identified with the Gell-Mann matrices. From this, it can immediately be seen that the dynamical symmetry group of the three-dimensional isotropic harmonic oscillator is in fact $SU(3)$ as already advertised.

Roots and weights of $SU(3)$

First, the Cartan subalgebra is chosen. It will turn out that a convenient starting point is the generator T_3. The only other generator commuting with T_3 is T_8. Thus,

$$\mathbf{H}_1 = \mathbf{T}_3 \text{ and } \mathbf{H}_2 = \mathbf{T}_8 \tag{A.120}$$

in the language of Sec. A.2.4. Accordingly, $SU(3)$ is of rank 2. A good reason for this choice is that in the representation of Gell–Mann, these two operators already are diagonal. The eigenvectors and the corresponding states labeled by their weights $|\mu_1, \mu_2\rangle$ are

$$\begin{aligned} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} &\to \left|\frac{1}{2}, \frac{1}{2\sqrt{3}}\right\rangle \\ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} &\to \left|-\frac{1}{2}, \frac{1}{2\sqrt{3}}\right\rangle \\ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} &\to \left|0, -\frac{1}{\sqrt{3}}\right\rangle . \end{aligned} \tag{A.121}$$

In the μ_1–μ_2 plane the weight vectors form an equilateral triangle! This can be seen in Fig. A.3 Recalling that the root vectors E_α act as ladder operators on the weights implies that the roots in the μ_1–μ_2 plane are differences of the weights. In fact they are $(1, 0)$, $(1/2, \sqrt{3}/2$, $(-1/2, \sqrt{3}/2)$, and minus these vectors, see Fig. A.4. To find the corresponding generators matrices that mediate the transfer from one state to another have to be constructed. It is not too hard to guess that these matrices are the ones with only one single off–diagonal element. They can easily be composed from the matrices of Eq. (A.108),

$$\begin{aligned} \mathbf{E}_{\pm 1,0} &= \frac{1}{\sqrt{2}}(\mathbf{T}_1 \pm i\mathbf{T}_2) \\ \mathbf{E}_{\pm 1/2, \pm\sqrt{3}/2} &= \frac{1}{\sqrt{2}}(\mathbf{T}_4 \pm i\mathbf{T}_5) \\ \mathbf{E}_{\mp 1/2, \pm\sqrt{3}/2} &= \frac{1}{\sqrt{2}}(\mathbf{T}_6 \pm i\mathbf{T}_7) . \end{aligned} \tag{A.122}$$

These roots form a regular hexagon in the μ_1–μ_2 plane. Therefore, all angles in $SU(3)$ are multiples of $\pi/3$.

Simple roots and fundamental representations of $SU(3)$

Taking a closer look on Eq. (A.122), it immediately becomes apparent that $(1, 0) = (1/2, \sqrt{3}/2) + (1/2, -\sqrt{3}/2)$ is the sum of the two other positive

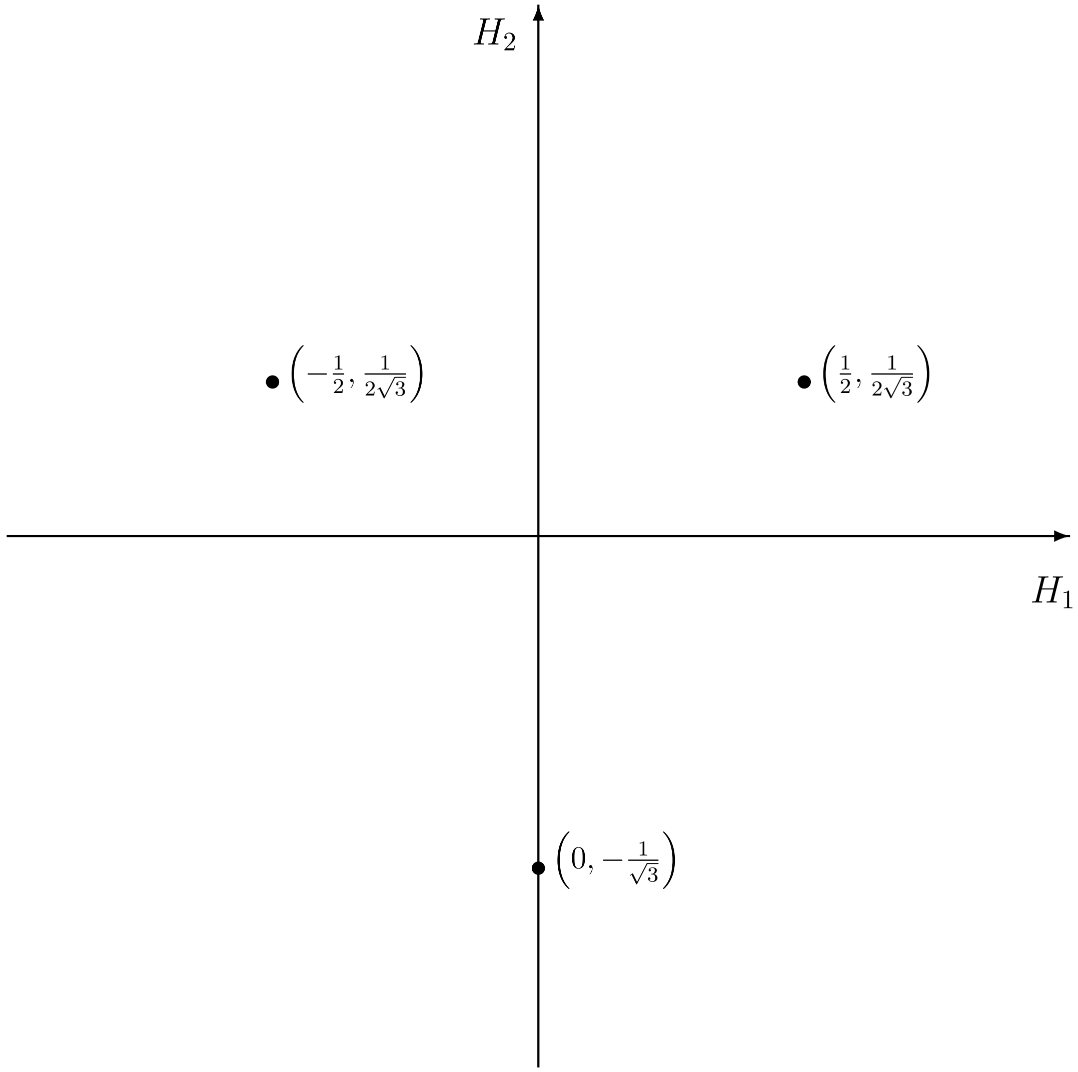


Figure A.3: Weights of $SU(3)$

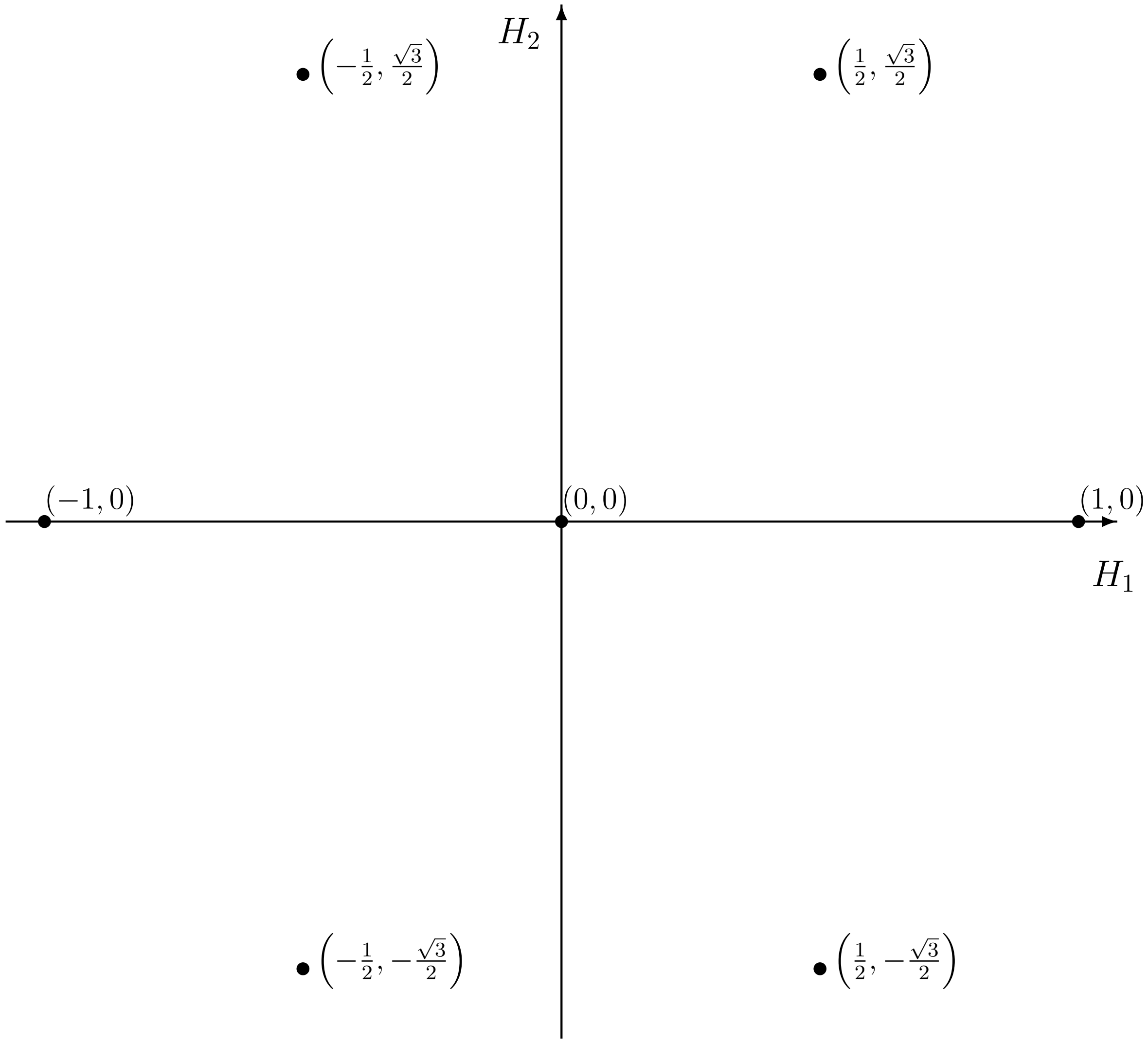


Figure A.4: Roots of $SU(3)$

roots. Thus, the simple roots are

$$\alpha^1 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \;, \; \alpha^2 = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) \tag{A.123}$$

with

$$\alpha^{1^2} = \alpha^{2^2} = 1 \,, \;\; \alpha^1 \cdot \alpha^2 = -\frac{1}{2} \tag{A.124}$$

and hence

$$\frac{2\alpha^1 \cdot \alpha^2}{\alpha^{1^2}} = \frac{2\alpha^1 \cdot \alpha^2}{\alpha^{2^2}} = -1 = -(p-q) = -p \,. \tag{A.125}$$

To illustrate the meaning of the previous chapter, consider $\mathbf{E}^1_\alpha|E^2_\alpha\rangle$. Because $p = 1$, $\alpha^1 + \alpha^2$ is a root, but for example $2\alpha^1 + \alpha^2$ is not.
From the α^i the fundamental weights are obtained as

$$\mu^1 = \left(\frac{1}{2}, \frac{1}{2\sqrt{3}}\right) \;, \; \mu^2 = \left(\frac{1}{2}, -\frac{1}{2\sqrt{3}}\right) \,. \tag{A.126}$$

μ^1 is the highest weight of the defining representation of $SU(3)$ presented before, generated by the $\mathbf{T}_a$ matrices of Eq. (A.108). In turn, μ^2 is the highest weight of a different representation, which will be constructed now. In principle, this representation can be found with a trick, but before talking about tricks, the concise, straightforward way is more rewarding. To this end, first all states are built, which can be obtained by applying lowering operators on the state with the highest weight, $|\mu^2\rangle$. Since for this fundamental representation $\mu^1 = 0$ and $\mu^2 = 1$, it follows that $\mu^2 - \alpha^2$ is a weight, but $\mu^2 - 2\alpha^2$ and $\mu^2 - \alpha^1$ are not. Therefore, there is a state

$$|\mu^2 - \alpha^2\rangle \sim E_{-\alpha^2}|\mu^2\rangle \,, \tag{A.127}$$

which is annihilated when acting on it again with $\mathbf{E}_{-\alpha^2}$, since $\mu^2 - 2\alpha^2$ is not a weight. Acting on it with $\mathbf{E}_{-\alpha^1}$ results in

$$\frac{2\alpha^1 \cdot (\mu^2 - \alpha^2)}{\alpha^1 \cdot \alpha^1} = \frac{-2\alpha^1 \cdot \alpha^2}{\alpha^1 \cdot \alpha^1} = 1 = q - p \,, \tag{A.128}$$

where the explicit form of the roots has been used as well as the fact that $\alpha^1 \cdot \mu^2 \sim \delta^{12}$, cf. Eq. (A.106). To continue, first fix x p. Remember that

p labels the number of times, a raising operator can be applied on a state without annihilating it. Thus consider the effect of $\mathbf{E}_{\alpha^1}$ on the state $|\mu^2-\alpha^2\rangle$. Note that $\mu^2-\alpha^2+\alpha^1$ is not a weight, since

$$\mathbf{E}_{\alpha^1}\mathbf{E}_{-\alpha^2}|\mu^2\rangle = \mathbf{E}_{-\alpha^2}\mathbf{E}_{\alpha^1}|\mu^2\rangle = 0\,, \tag{A.129}$$

because $\alpha^1-\alpha^2$ is not a root. Therefore, $p=0$ in Eq. (A.128), and thus $q=1$ and $\mu^2-\alpha^2-\alpha^1$ is a weight, corresponding to the state

$$|\mu^2-\alpha^2-\alpha^1\rangle \sim \mathbf{E}_{-\alpha^1}\mathbf{E}_{-\alpha^2}|\mu^2\rangle\,. \tag{A.130}$$

This state gets annihilated when acting on it with either $\mathbf{E}_{-\alpha^1}$ or $\mathbf{E}_{-\alpha^2}$. To see this, the same path outlined above with similar arguments must be followed. So, consider first the effect of $\mathbf{E}_{\alpha^1}$ on the state. In analogy to Eq. (A.128),

$$\frac{2\alpha^1\cdot(\mu^2-\alpha^2-\alpha^1)}{\alpha^1\cdot\alpha^1} = \frac{-2\alpha^1\cdot(\alpha^1+\alpha^2)}{\alpha^1\cdot\alpha^1} = -1 = q-p\,. \tag{A.131}$$

Thus, $p=1$ (remember, $\mu^2-\alpha^2$ is a weight), and $q=0$. Hence, $\mathbf{E}_{-\alpha^1}$ indeed annihilates $|\mu^2-\alpha^2-\alpha^1\rangle$. Similarly,

$$\frac{2\alpha^2\cdot(\mu^2-\alpha^2-\alpha^1)}{\alpha^2\cdot\alpha^2} = 0 = q-p\,. \tag{A.132}$$

Here, $p=0$, since $\mu^2-\alpha^1$ is not a weight, and again $q=0$ with the same consequences as above.
Plotting the three weights, μ^2, $\mu^2-\alpha^2$, and $\mu^2-\alpha^2-\alpha^1$, of this representation the inverted equilateral triangle of Fig. A.3 emerges, see Fig. A.5.

Weyl–reflections

The question that naturally arises here, is, whether all three of the weights correspond to unique states in the representation or whether there are some degeneracies, which remain invisible in the neat diagrams encountered so far. This question can be answered in general terms. To this end, consider some arbitrary irreducible representation D with highest weight μ. Let $|\mu\rangle$ be any state with weight μ and form

$$\mathbf{E}_{\xi^1}\mathbf{E}_{\xi^2}\dots\mathbf{E}_{\xi^n}|\mu\rangle\,, \tag{A.133}$$

where the ξ are any roots. Taking into account all n, these states span the full representation. But any state with positive ξ can be dropped, because

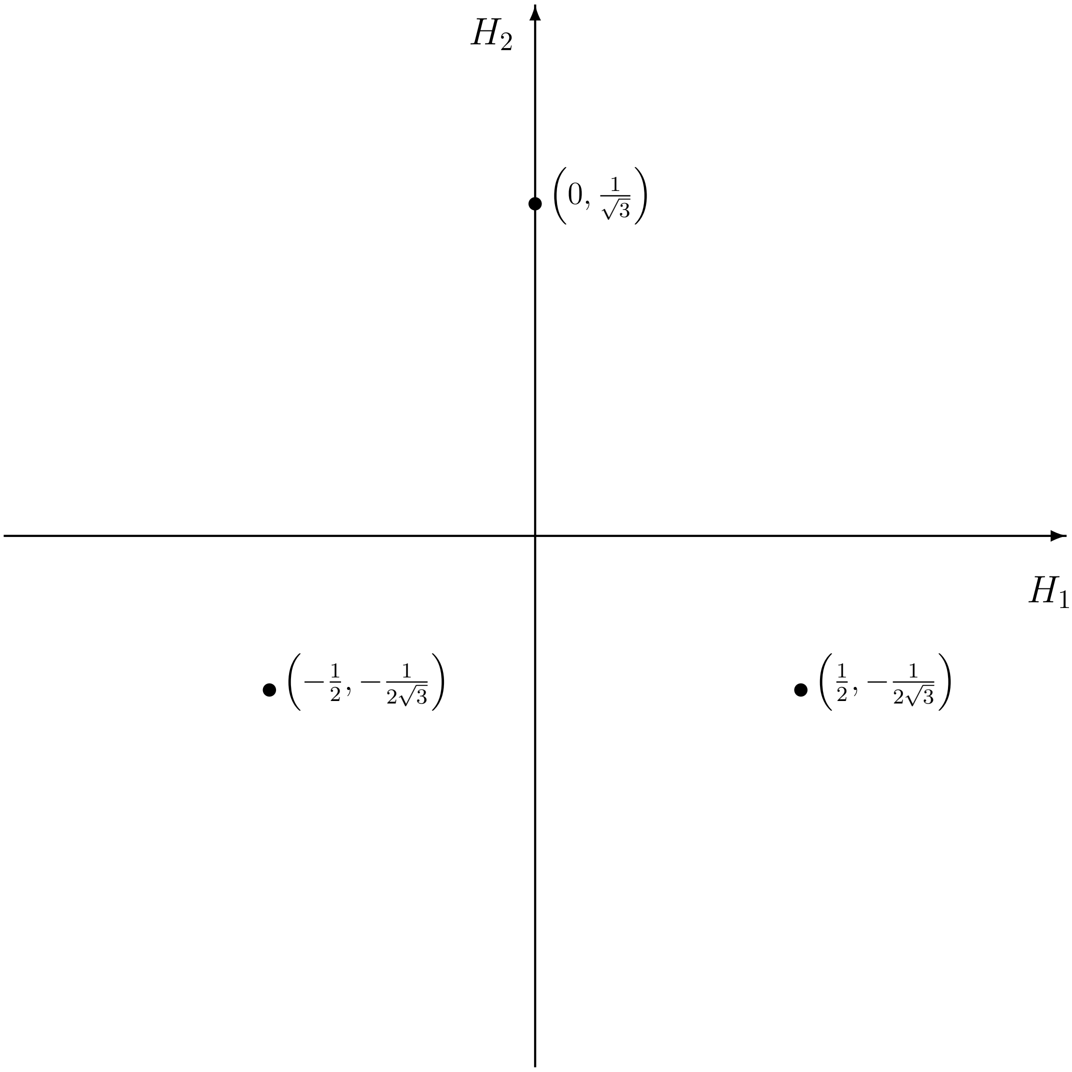


Figure A.5: Weights of $SU(3)$ in the $\mu^2 \equiv (0,\, 1)$–representation.

with help of the commutation relations all raising operators can be moved to the right, until they annihilate $|\mu\rangle$. Therefore, all ξ can be taken negative. These negative roots can be further decomposed into sums of simple roots with non–positive coefficients, thus only

$$E_{-\chi^1} E_{-\chi^2} \dots E_{-\chi^n} |\mu\rangle \tag{A.134}$$

remains, where the χ^i are simple roots. This shows that the state $|\mu\rangle$ is unique, as are states $\mathbf{E}_{-\chi^i}|\mu\rangle$ and $\left[\mathbf{E}_{-\chi^i}\right]^m |\mu\rangle$. Therefore in the example above, $|\mu^2 - \alpha^2\rangle$ in fact is unique. To see whether the other state is unique, some "obvious symmetry of the representation" needs to be checked. This symmetry results from the trivial fact that in each root direction, there is an $SU(2)$ symmetry group, and that representations of $SU(2)$ are symmetric. How can this be seen?
In Sec. A.2.4, the $\mathbf{E}_{\pm\alpha}$ have been related to $SU(2)$, such that each $\mathbf{E}_\alpha$ has an associated $SU(2)$ algebra, where the $\mathbf{E}_{\pm\alpha}$ form the analogs to the ladder operators $\mathbf{J}_\pm$ and some set of operators from the Cartan subalgebra form the analog to $\mathbf{J}_3$. Also, if $|\mu\rangle$ is a state corresponding to the weight μ and an eigenstate of $\mathbf{E}_{-\alpha}\mathbf{E}_\alpha$, then the states obtained by multiple application of $\mathbf{E}_{\pm\alpha}$ from $|\mu\rangle$, $|\mu + k\alpha\rangle$, form an irreducible representation of $SU(2)$. Taking some weight μ and some root α

$$\mu - \frac{2\alpha \cdot \mu}{\alpha^2}\alpha \tag{A.135}$$

is also a weight. This weight is the reflection of μ in the hyper-plane perpendicular to α. Any representation is invariant under all such reflections, which are called *Weyl reflections*. The set of all such reflections for all roots forms a group, which is called the *Weyl group* of the algebra. This can be specified for the case of $SU(3)$. There, μ^2 and $\mu^2 - \alpha^2$ correspond to unique states. The weight $\mu^2 - \alpha^2 - \alpha^1$ can be constructed through a Weyl reflection of μ^2 in the hyper-plane perpendicular to $\alpha^2 + \alpha^1$ (note that the root defining the Weyl reflection does not need to be simple):

$$\mu^2 - \frac{2(\alpha^1 + \alpha^2) \cdot \mu^2}{(\alpha^1 + \alpha^2)^2}(\alpha^1 + \alpha^2) = \mu^2 - \alpha^2 - \alpha^1 \tag{A.136}$$

by the explicit form of the roots and weight. Since there is only one way to construct the root defining the Weyl reflection, the ladder is unique, and so is the state.

This shows that the second fundamental representation is three–dimensional, as is the first. Its weights are just the negatives of the weights already encountered in the first fundamental representation, as you can read of Figs. A.5 and A.3. This connection is so simple that there should be a simple trick to go from the first to the second fundamental representation.

Complex representations

And here is the trick, which, in fact, is pretty simple. The basis of it is that if the $\mathbf{T}_a$ are the generators of some representation D of some Lie algebra, then the matrices $-\mathbf{T}_a^*$ follow exactly the same algebra. Hence, they also generate a representation, which is obviously of the same dimension as D. It should be not a big surprise that this representation is called the *complex conjugate representation*. It is denoted by $\bar{D}$. If a representation is equivalent to its complex conjugate it is called *real*, otherwise it is called *complex*.
Since the Cartan generators of $\bar{D}$ are $-\mathbf{H}_i^*$, the weight of $\bar{D}$ equals $-\mu$ if μ is a weight in D. Consequently the highest weight μ in D is the negative of the lowest weight in $\bar{D}$ and vice versa. Hence, if the highest weight of D equals the negative of its lowest weight, D is real.
What are the consequences for the construction of the μ^2 representation from the μ^1 representation? Quite simple : In the defining (μ^1) representation of $SU(3)$, μ^1 is the highest weight and the lowest is $-\mu^2$. Therefore

$$D_{\mu^2} = \bar{D}_{\mu^1} \, , \tag{A.137}$$

where the subscripts label the highest weight of the representation. In addition, the generators of D_{μ^2} are $-\mathbf{T}_a^*$, where the $\mathbf{T}_a$ are defined in Eq. (A.108).
In physics, there are several notations around to denote representations of $SU(3)$:

1. The most explicit one is to label representations by ordered pairs of integers, (q^1, q^2).

2. Another choice, more common to particle physicists, and quite convenient for small dimensions, is to give just the dimension of the representation and to distinguish between a representation and its complex conjugate with a bar.

Hence

$$(1, 0) = \mathbf{3} \text{ and } (0, 1) = \bar{\mathbf{3}} \, . \tag{A.138}$$

Tensor methods

In this section a useful tool, tensor methods, will be introduced, which allow to explicitly calculate direct products of representations of groups. These tensor techniques will be discussed in close contact with $SU(3)$. To start with, relabel the states of the **3**–representation of $SU(3)$:

$$\begin{aligned}\left|\frac{1}{2}, \frac{1}{2\sqrt{3}}\right\rangle &= |_1\rangle \\ \left|-\frac{1}{2}, \frac{1}{2\sqrt{3}}\right\rangle &= |_2\rangle \\ \left|0, -\frac{1}{\sqrt{3}}\right\rangle &= |_3\rangle \ . \end{aligned} \tag{A.139}$$

Here the explicit form of the eigenvectors, cf. Eq. (A.121), is directly labelled. Note that the indices in the labels are somewhat lower for reasons to become clear in a minute. Next, a set of representation matrices with one upper and one lower index is defined through

$$[T_a]^i_j = \frac{1}{2}\,[\lambda_a]_{ij} \ . \tag{A.140}$$

Here, the λ denote the generators of the group in the representation under consideration, in the case of $SU(3)$ these are the Gell–Mann matrices. The triplet $|_i\rangle$ transforms under the algebra as

$$T_a|_i\rangle = [T_a]^j_i\,|_j\rangle \,, \tag{A.141}$$

where the Einstein convention is used.
Accordingly, for the states of the $\bar{\mathbf{3}}$–representation,

$$\begin{aligned}\left|-\frac{1}{2}, -\frac{1}{2\sqrt{3}}\right\rangle &= |^1\rangle \\ \left|\frac{1}{2}, -\frac{1}{2\sqrt{3}}\right\rangle &= |^2\rangle \\ \left|0, \frac{1}{\sqrt{3}}\right\rangle &= |^3\rangle \end{aligned} \tag{A.142}$$

and

$$T_a|^i\rangle = -\,[T_a]^i_j\,|^j\rangle \,. \tag{A.143}$$

This follows from

$$-\left[T_a^*\right]_i^j = -\left[T_a^T\right]_i^j = -\left[T_a\right]_j^i \,. \tag{A.144}$$

Now states in the product space of n $\mathbf{3}$'s and m $\bar{\mathbf{3}}$'s can be defined as

$$\left|{}^{i_1 \ldots i_n}_{j_1 \ldots j_m}\right\rangle = \left|{}_{j_1}\right\rangle$$

Consider a tensor $|v\rangle$, an arbitrary state in this tensor product space,

$$|v\rangle = v^{j_1 \ldots j_n}_{i_1 \ldots i_m} \left|{}^{i_1 \ldots i_m}_{j_1 \ldots j_n}\right\rangle \,. \tag{A.146}$$

Then following the path of Eq. (**??**), the action of the generators on $|v\rangle$ is given by

$$[T_a v]^{j_1 \ldots j_n}_{i_1 \ldots i_m} \quad = \quad \sum_{l=1}^{n} [T_a]^{j_l}_k \, v^{j_1 \ldots k \ldots j_n}_{i_1 \ldots i_m} - \sum_{l=1}^{m} [T_a]^k_{i_l} \, v^{j_1 \ldots j_n}_{i_1 \ldots k \ldots i_m} \,. \tag{A.147}$$

To pick out the states in this tensor product, which correspond to the irreducible representation $(n\,, m)$, start from the state with highest weight, namely

$$|\mu\rangle = \left|{}^{2\,2\,\ldots}_{1\,1\,\ldots}\right\rangle \,, \tag{A.148}$$

corresponding to the tensor

$$|\mu\rangle \Longleftrightarrow v^{j_1 \ldots j_n}_{i_1 \ldots i_m} = \delta^{j_1 1} \ldots \delta^{j_n 1} \delta_{i_1 2} \ldots \delta_{i_m 2} \,. \tag{A.149}$$

All other states emerge from this one by repeatedly applying lowering operators. In so doing, two important properties come to help:

1. v is symmetric in upper and lower indices and it satisfies

$$\delta^{i_1}_{j_1} v^{j_1 \ldots j_n}_{i_1 \ldots i_m} = 0 \,. \tag{A.150}$$

2. This is preserved under $v \to T_a v$.

The δ^i_j constitutes an invariant tensor, because

$$[T_a]^j_i \delta^i_j = [T_a \delta] = 0 \tag{A.151}$$

and therefore

$$[T_a]_i^j \delta_j^k - [T_a]_j^k \delta_i^j = 0\,. \tag{A.152}$$

Since the lowering operators are just linear combinations of the T_a, all states in the $(n,\, m)$–representation correspond to traceless tensors of the form of v, symmetric in upper and lower indices. This correspondence holds in both directions, i.e. every traceless tensor with the correct number of entries n and m and symmetry in the indices forms a state in the $(n,\, m)$–representation. These tensors are extremely nice and suitable for practical work, it is pretty easy to multiply them to construct tensors with larger numbers of indices, and they are easily decomposed into δ's and ϵ_{ijk}, the other invariant tensor (they are invariant due to the tracelessness of both the ϵ–tensor and the T_a). Taking a bra-state $\langle v|$ instead of the ket-state $|v\rangle$, implies

$$\langle v| = \left\langle {}^{i_1 \ldots i_n}_{j_1 \ldots j_m} \right| \left(v^{j_1 \ldots j_m}_{i_1 \ldots i_n} \right)^* \,. \tag{A.153}$$

However, here care has to be taken, because the bras transform under the algebra with an extra minus sign, so for instance the triplet transforms like

$$-\left\langle {}_i \right| T_a = -\left\langle {}_i \right| T_a \left| {}_j \right\rangle \left\langle {}_j \right| = -\left\langle {}_j \right| [T_a]_j^i \,. \tag{A.154}$$

Hence, bras with upper index transform like kets with lower index. This merely reflects the process of complex conjugation needed to go from kets to bras. Thus,

$$\left(v^{j_1 \ldots j_m}_{i_1 \ldots i_n} \right)^* \equiv \bar{v}^{i_1 \ldots i_n}_{j_1 \ldots j_m} \,. \tag{A.155}$$

Therefore, if $\langle v|$ transforms as $-\langle v|T_a$, then $|\bar{v}\rangle$ transforms as $T_a|\bar{v}\rangle$. To exemplify this behavior, consider the matrix element $\langle u|v\rangle$. Obviously, both $|u\rangle$ and $|v\rangle$ have to live in the same space and must be tensors of the same kind. Then

$$\langle u|v\rangle = \bar{u}^{k_1 \ldots k_m}_{l_1 \ldots l_n} v^{j_1 \ldots j_n}_{i_1 \ldots i_m} = \bar{u}^{i_1 \ldots i_m}_{j_1 \ldots j_n} v^{j_1 \ldots j_n}_{i_1 \ldots i_m} \tag{A.156}$$

to guarantee, that there is no "pending index" left. Only then all indices get contracted.

Dimension of $(n,\, m)$

The dimensionality of a representation $(n,\, m)$ is simple to determine using the tensor concepts developed so far. To do so, only the number of independent tensor components for a symmetric tensor with n lower (or upper) indices must be calculated. For this, solely the number of indices with each value $\{1,\, 2 \text{ or } 3\}$ matters. Therefore the number of independent components equals the number of ways n identical objects can be separated into two identical partitions. Combinatorics assure that this number is $(n+2)!/(n!2!)$. Thus, a tensor with n upper and m lower indices has

$$\frac{(n+2)!}{2!n!}\frac{(m+2)!}{2!m!} = B(n,\, m) \tag{A.157}$$

independent components. But the trace condition implies that out of this a symmetric object with $n-1$ upper and $m-1$ lower indices vanishes. Thus the dimension is given by

$$D(n,\, m) = B(n,\, m) - B(n-1,\, m-1) = \frac{(n+1)(m+1)(n+m+2)}{2} \tag{A.157}$$

for the traceless tensor associated with the representation of $(n\,, m)$.

Clebsch–Gordan decomposition with tensors

The practical implications of the reasoning so far will be exemplified by analyzing the composite representation $\mathbf{3} \otimes \mathbf{3}$. The tensor product is just $v^i u^j$, which can be rewritten as

$$v^i u^j = \frac{1}{2}\left(v^i u^j + v^j u^i\right) + \frac{1}{2}\left(\epsilon^{ijk}\epsilon_{klm} v^l u^m\right) \,. \tag{A.158}$$

With help of this identity, the product has been explicitly decomposed into a symmetric tensor

$$\frac{1}{2}\left(v^i u^j + v^j u^i\right) \,, \tag{A.159}$$

which is a $\mathbf{6}$, and a $\overline{\mathbf{3}}$, namely the lower index object

$$\epsilon_{klm} v^l u^m \,. \tag{A.160}$$

Therefore,

$$\mathbf{3} \otimes \mathbf{3} = \mathbf{6} \oplus \mathbf{3}\,. \tag{A.161}$$

Consider now $\mathbf{3} \otimes \bar{\mathbf{3}}$. Here, the corresponding product can be cast into

$$v^i u_j = \left(v^i u_j - \frac{1}{3}\delta^i_j v^k u_k \right) + \left(\frac{1}{3}\delta^i_j v^k u_k \right)\,. \tag{A.162}$$

The first object denotes traceless tensors with one lower and one upper index, namely the T_a. Therefore it belongs to the **8**-representation. The trivial tensor $v^k u_k$ in contrast has no indices left, therefore it is just $(0,\,0) = \mathbf{1}$. Hence,

$$\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}\,. \tag{A.163}$$

In general, every time, when a tensor is not completely symmetric, two upper indices can be traded for one lower index plus an ϵ–tensor and vice versa.

Young tableaux

The rather abstract ideas above can be represented very elegantly in a diagrammatic language, known as Young tableaux. Their main virtues are that they provide an easy algorithm for the Clebsch–Gordan decomposition of tensor products of two representations, generalizing straightforwardly to the case of $SU(N)$.
The algorithm is based on the observation, that in $SU(3)$ the $\bar{\mathbf{3}}$ is an antisymmetric combination of two $\mathbf{3}$'s. Ergo, an arbitrary representation can be written as a tensor product of $\mathbf{3}$'s with appropriate symmetry properties. To illustrate this point, consider the $(n,\,m)$ representation. Basically it is a tensor with components

$$A^{i_1 i_2 \dots i_n}_{j_1 j_2 \dots j_m}\,, \tag{A.164}$$

symmetric in upper and lower indices and traceless. The lower indices can now be raised with the help of an appropriate number of ϵ tensors, yielding an object with $n + 2m$ upper indices, namely

$$\tilde{A}^{i_1 i_2 \dots i_n k_1 k_2 \dots k_m l_1 l_2 \dots l_m} = A^{i_1 i_2 \dots i_n}_{j_1 j_2 \dots j_m} \epsilon^{j_1 k_1 l_1} \epsilon^{j_2 k_2 l_2} \dots \epsilon^{j_m k_m l_m}\,. \tag{A.165}$$

Obviously, the $\tilde{A}$ is antisymmetric in each pair of indices $k_x \leftrightarrow l_x$ with $x = 1, \ldots, m$. To explore the symmetry properties of this object consider the states

$$\tilde{A}^{i_1 i_2 \ldots i_n k_1 k_2 \ldots k_m l_1 l_2 \ldots l_m} \left|_{i_1 i_2 \ldots i_n k_1 k_2 \ldots k_m l_1 l_2 \ldots l_m}\right\rangle \tag{A.166}$$

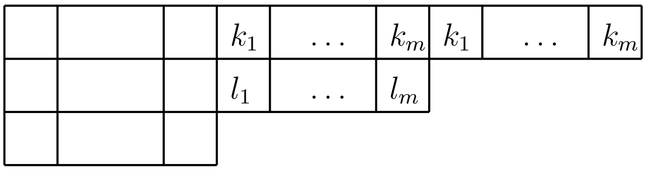
forming the $(n,\, m)$ representation. Remember that in $SU(3)$ the state $|_1\rangle$ has the highest weight $(1/2,\, 1/2\sqrt{3})$ in the **3** representation. The state with next highest weight in this representation is $|_1\rangle$ with weight $(0,\, -1/\sqrt{3})$. Hence, the highest weight in the $(n,\, m)$ representation is obtained, when all i's equal 1, and when each pair of k, l takes values 1 and 3. For the state with highest weight non–zero tensor components $\tilde{A}$ are obtained from Eq. (A.165) by setting all i's and all k's to 1 and all l's to 3 and by anti–symmetrization in all k, l pairs. Starting from this state, all other states are accessed by applying lowering operators.
Now translate this into Young tableaux. In this formulation, all indices are associated with boxes, which are arranged according to

$$\begin{array}{|c|c|c|c|c|c|}\hline k_1 & \ldots & k_m & i_1 & \ldots & i_n \\ \hline l_1 & \ldots & l_m \\ \cline{1-3}\end{array} \tag{A.167}$$

This Young tableau then translates into the tensor in question by first symmetrizing in each row and by finally anti-symmetrizing the indices in each column. After such a treatment, the resulting tensor corresponds to a state in the $(n, \,, m)$ representation. This anti-symmetrizing in the columns has the direct consequence, that columns with four or more boxes vanish, since there are only three indices to anti-symmetrize in (N in $SU(N)$). Moreover, every column of three boxes simply yields an ϵ tensor in the three corresponding indices and gives a factor of 1,

$$\epsilon^{ijk} = \begin{array}{|c|}\hline i \\ \hline j \\ \hline k \\ \hline\end{array} \tag{A.166}$$

Thus, tableaux of the form



(A.165)

are equivalent to the one depicted in Eq. (A.167).
With help of the Young tableaux the Clebsch–Goredan decomposition of tensor products is easily performed. The algorithm is as follows:

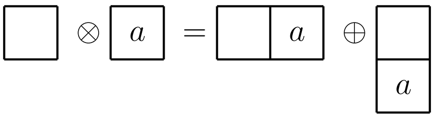
1. Start by multiplying two tableaux A and B corresponding to the product of representations $\alpha \otimes \beta$. So, B is to the left of A.

2. Denote all boxes in the top row of B by b_1 and all boxes in the second row by b_2. Remember, that an eventual third row is obsolete.

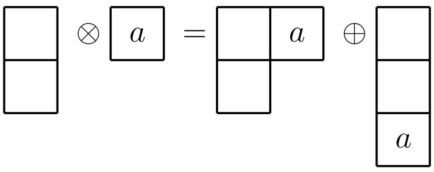
3. Take the boxes of B's top row and add them to A. Here, each of the b_1's has to be placed in a different column. Repeat the same procedure for the second row of B.

4. Neglect all of the newly formed tableaux, in which, reading from right to left or from top to bottom the number of b_1's is smaller than the number of b_2's. With this step, double counting of tensors is prevented.

To illustrate this, consider some trivial examples:



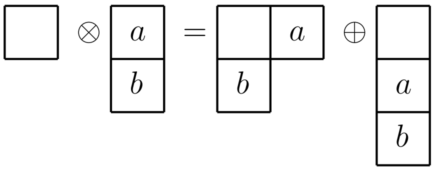
$$\mathbf{3} \otimes \mathbf{3} = \mathbf{6} \oplus \bar{\mathbf{3}}$$



$$\bar{\mathbf{3}} \otimes \mathbf{3} = \mathbf{8} \oplus \mathbf{1}$$

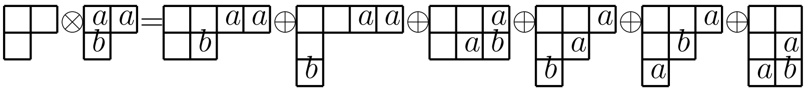
(A.160)

A little bit less trivial, but with the same result as the last product is



$$\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1} \tag{A.159}$$

Finally, something more involved:



$$\mathbf{8} \otimes \mathbf{8} = \mathbf{27} \oplus \mathbf{10} \oplus \overline{\mathbf{10}} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1} \tag{A.158}$$

Note that the two **8**'s occurring in this last example are different, since they have a different pattern of a's and b's filled into the boxes. Note also that using the formalism of Young tableaux, it is quite easy to show that

$$\mathbf{d} \in \mathbf{8} \otimes \mathbf{d} \quad \text{for all} \quad \mathbf{d}\,. \tag{A.159}$$

Appendix B

Path integral formalism

In this chapter an alternative representation of quantum field theory will be presented, the path integral formalism. The discussion will start by first reviewing the formalism for quantum mechanics. First, it will be defined through the construction of total transition amplitudes and its usage will be exemplified by the calculation of expectation values. Then, the path integral for a simple harmonic oscillator will be discussed. After that the focus shifts on field theories, and free quantum field theories for scalar fields will be evaluated once more, i.e. the corresponding free field propagators will be constructed. Through the introduction of sources, interacting field theories will be made accesible for perturbation theory, and, accordingly, the first terms of the perturbative series for $\lambda\phi^4$ theory will be given for some n-point functions.

B.1 Path integral in quantum mechanics

B.1.1 Constructive definition

Total amplitude in quantum mechanics

Consider a system with one particle contained in a potential $V(x)$. The time-dependent state vector of the particle is $|\psi(t)\rangle$. The probability density for this particle to be at position x at time t is

$$\mathcal{P}(x,\, t) = |\psi(x,\, t)|^2 = |\langle x|\psi(x,\, t)\rangle|^2\ . \tag{B.1}$$

Here, $\psi(x,\, t)$ is the wave function, representing $|\psi(t)\rangle$ in terms of position eigenvectors. $|\psi(t)\rangle$ satisfies the Schrödinger equation,

$$i\frac{\partial}{\partial t}|\psi(t)\rangle = \mathcal{H}|\psi(t)\rangle\,. \tag{B.2}$$

Its formal solution is

$$|\psi(t)\rangle = e^{-i\mathcal{H}(t-t_0)}|\psi(t_0)\rangle\,, \tag{B.3}$$

and in position space this becomes

$$\begin{aligned}\psi(x,\, t) = \langle x|\psi(t)\rangle &= \langle x|e^{-i\mathcal{H}(t-t_0)}|\psi(t_0)\rangle \\ &= \int \mathrm{d}x_1 \langle x|e^{-i\mathcal{H}(t-t_0)}|x_1\rangle\langle x_1|\psi(t_0)\rangle \\ &= \int \mathrm{d}x_1 G(x,\, t;\, x_1,\, t_0)\psi(x_1,\, t_0)\,. \end{aligned} \tag{B.2}$$

Here $G(x,\, t;\, x_0,\, t_0)$ denotes the total amplitude in position space. Again, the term $\exp\left[-i\mathcal{H}(t-t_0)\right]$ is the time-evolution operator, which propagates the state $|\psi(t_0)\rangle$ forward in time. The basic observation, which leads to the path integral formalism, is that the total amplitude can be represented as a path integral. To see this, suppose the particle is localised at x_0 at time t_0, i.e. $|\psi(t_0)\rangle = |x_0\rangle$. Then,

$$\psi(x_1,\, t_0) = \langle x_1|\psi(t_0)\rangle = \langle x_1|x_0\rangle = \delta(x_1 - x_0)\,. \tag{B.3}$$

Consequently,

$$\psi(x,\, t) = \int \mathrm{d}x_1 G(x,\, t;\, x_1,\, t_0)\delta(x_1 - x_0) = G(x,\, t;\, x_0,\, t_0)\,, \tag{B.4}$$

and, hence, $G(x,\, t;\, x_0,\, t_0)$ is the quantum mechanical amplitude for the particle to go from $(x_0,\, t_0)$ to $(x,\, t)$. However, in this definition it is not clear, how the particle reached its destination. In fact, without performing any further measurement, quantum mechanics postulates that the particle took all paths (weighted by their amplitudes) and that the resulting total amplitude is a superposition of the individual amplitudes of all paths. Stated again, in other words: The total amplitude is nothing but the sum over all amplitudes of all paths.

One can now break up the propagation into smaller steps, i.e. one can decompose $G(x,\, t;\, x_0,\, t_0)$ such that it is a product of amplitudes for the particle to go from $(x_0,\, t_0)$ to $(x_1,\, t_1)$ and then to $(x,\, t)$. With similar arguments as above one can write

$$G(x,\, t;\, x_0,\, t_0) = \int \mathrm{d}x_1 G(x,\, t;\, x_1,\, t_1) G(x_1,\, t_1;\, x_0,\, t_0)\,, \tag{B.5}$$

where the (implicit) time ordering $t_0 < t_1 < t$ is understood.

Construction of a total amplitude

This procedure can be repeated over and over again and one finally finds

$$\begin{aligned} &G(x,\, t;\, x_0,\, t_0) \\ &= \int \mathrm{d}x_N \ldots \mathrm{d}x_1 G(x,\, t;\, x_N,\, t_N) G(x_N,\, t_N;\, x_{N-1},\, t_{N-1}) \ldots G(x_1,\, t_1;\, x_0,\, t_0)\,. \end{aligned} \tag{B.4}$$

Taking the limit $N \to \infty$ one can identify the individual integrals over the x_i as the functional integral with respect to the paths $x(t)$,

$$\lim_{N\to\infty} \int \prod_{i=1}^{N} \mathrm{d}x_1 \to \int \mathcal{D}x(t)\,. \tag{B.5}$$

The task left is to re-write the integrand, i.e. the product of single amplitudes in infinitesimal distances in a better, more compact form. To do so, assume that $t_{i+1} - t_i = \epsilon$ is small. Then, one can expand

$$\begin{aligned} G(x_{i+1},\, t_{i+1};\, x_i,\, t_i) &= \left\langle x_{i+1} \left| e^{-i\mathcal{H}(t_{i+1}-t_i)} \right| x_i \right\rangle \\ &= \langle x_{i+1} \left| [1 - i\epsilon\mathcal{H} + \ldots] \right| x_i \rangle\,. \end{aligned} \tag{B.5}$$

Now, analyse the matrix elements to find

$$\begin{aligned} \langle x_{i+1} | x_i \rangle &= \delta(x_{i+1} - x_i) = \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{2\pi} e^{ip(x_{i+1}-x_i)} \\ \langle x_{i+1} \left| \mathcal{H} \right| x_i \rangle &= \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{2\pi} \mathcal{H}(p_i,\, \bar{x}_i) e^{ip(x_{i+1}-x_i)}\,, \end{aligned} \tag{B.5}$$

where $\bar{x}_i$ is the average x, in this case

$$\bar{x}_i = \frac{x_{i+1} + x_i}{2}\,. \tag{B.6}$$

Therefore,

$$\begin{aligned} G(x_{i+1},\, t_{i+1};\, x_i,\, t_i) &= \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{2\pi} \left(1 - i\epsilon\mathcal{H}(p_i,\, \bar{x}_i) + \ldots\right) e^{ip(x_{i+1}-x_i)} \\ &\approx \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{2\pi} e^{-i\epsilon\mathcal{H}(p_i,\, \bar{x}_i)} e^{ip(x_{i+1}-x_i)}\,, \end{aligned} \tag{B.6}$$

and the product of amplitudes can be cast into the following form, representing one specific path out of the set of all paths,

$$\begin{aligned} &\lim_{N\to\infty} \prod_{i=0}^{N} G(x_{i+1},\, t_{i+1},\, x_i,\, t_i) \\ &= \lim_{N\to\infty} \int_{-\infty}^{\infty} \prod_{j=0}^{N} \frac{\mathrm{d}p_j}{2\pi} \exp\left\{ \left[i \sum_{j=0}^{N} p_j(x_{j+1} - x_j) \right] - \left(\frac{t - t_0}{N+1} \right) \mathcal{H}(p_j,\, \bar{x}_j) \right\} \\ &= \int \mathcal{D}p \exp\left[i \int_{t_0}^{t} \mathrm{d}t' \left(p\dot{x} - H(p,\, x)\right) \right]\,. \end{aligned} \tag{B.5}$$

The entire amplitude therefore is

$$G(x,\, t;\, x_0,\, t_0) \;=\; \int \mathcal{D}\bar{x}\mathcal{D}p \exp\left[i \int_{t_0}^{t} \mathrm{d}t' \left(p\dot{\bar{x}} - H(p,\, \bar{x})\right) \right] \tag{B.6}$$

Assume now the standard form of a Hamiltonian with no unorthodox powers of the momentum, $\mathcal{H} = p^2/2 + V(x)$. Then, completing the squares in one of the mini-pieces of the amplitude yields

$$\begin{aligned} G(x_{i+1},\, t_{i+1};\, x_i,\, t_i) &= \int_{-\infty}^{\infty} \frac{\mathrm{d}p_i}{2\pi} e^{ip_i(x_{i+1}-x_i)} e^{-i\epsilon(p_i^2/2 + V(\bar{x}_i))} \\ &= \sqrt{\frac{1}{2\pi i\epsilon}} \exp\left[i\epsilon \left(\frac{\dot{\bar{x}}_i^2}{2} - V(\bar{x}_i) \right) \right]\,, \end{aligned} \tag{B.6}$$

where $\dot{\bar{x}}_i = (x_{i+1} - x_i)/\epsilon$. Then the total amplitude becomes

$$\begin{aligned} G(x,\, t;\, x_0,\, t_0) &= \int \mathcal{D}\bar{x} \exp\left[i \int\limits_{t_0}^{t} \mathrm{d}t' \mathcal{L}(\dot{\bar{x}},\, \bar{x},\, t) \right] \\ &= \int \mathcal{D}\bar{x} \exp\left[i\mathcal{S}[\bar{x}] \right] , \qquad \text{(B.6)} \end{aligned}$$

where $\mathcal{S}$ is the classical action functional of the theory. The interpretation of this is that the amplitude of a single path in quantum mechanics is the exponential of i times the classical action related to it. A comment concerning the average $\bar{x}$ and operator ordering is in order here. Straightforward calculation shows that taking the average amounts to

$$\left\langle x_{i+1} \left| \frac{1}{2} (xp + px) \right| x_i \right\rangle = \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}p_i}{2\pi} e^{ip_i(x_{i+1} - x_i)} p_i \bar{x}_i \, . \qquad \text{(B.7)}$$

Transition amplitudes and expectation values

The very nature of the path integral renders it a useful representation to calculate transition matrix elements, vacuum expectation values and coordinate or momentum space representations of matrix elements. As one easily anticipates, it is also extremely useful in quantum field theory. However, the calculation of transition matrix elements is a straightforward generalisation of what has been achieved so far. If the system is in the state $|\psi\rangle$ at some time t_0 then it will evolve to the state $|\phi\rangle$ at some later time t, where

$$|\phi\rangle = e^{-i\mathcal{H}(t-t_0)} |\psi\rangle \, . \qquad \text{(B.8)}$$

The amplitude to be in the state $|\chi\rangle$ instead at time t is $\langle\chi|\phi\rangle$,

$$\begin{aligned} \langle\chi|\phi\rangle &= \left\langle \chi \left| e^{-i\mathcal{H}(t-t_0)} \right| \psi \right\rangle \\ &= \int \mathrm{d}x_0 \mathrm{d}x \left\langle \chi \left| x \right.\right\rangle \left\langle x \left| e^{-i\mathcal{H}(t-t_0)} \right| x_0 \right\rangle \left\langle x_0 \right| \psi \rangle \\ &= \int \mathrm{d}x_0 \mathrm{d}x \chi^*(x) G(x,\, t;\, x_0,\, t_0) \phi(x_0) \\ &= \int \mathrm{d}x_0 \mathrm{d}x \int \mathcal{D}\bar{x}\, \chi^*(x) \exp\left(i\mathcal{S}[\bar{x}] \right) \phi(x_0) \, , \qquad \text{(B.6)} \end{aligned}$$

where all paths satisfy $\bar{x}(t_0) = x_0$ and $\bar{x}(t) = x$.
To arrive at the path integral representation of a vacuum expectation value of an operator, one has to rewrite the total amplitude in terms of energy eigenstates, $|\phi_n\rangle$, of the Hamiltonian. Then

$$\mathcal{H}|\phi_n\rangle = E_n|\phi_n\rangle \tag{B.7}$$

and it is assumed that the ϕ_n form a complete set, i.e.

$$\sum_n |\phi_n\rangle\langle\phi_n| = 1\,. \tag{B.8}$$

Using this completeness relation to insert some suitable "1"s, one can write

$$\begin{aligned} G(x,\, t;\, x_0,\, t_0) &= \left\langle x \left| e^{-i\mathcal{H}(t-t_0)} \right| x_0 \right\rangle \\ &= \sum_{n,m} \phi_n(x) e^{-iE_n t} \langle\phi_n|\phi_m\rangle e^{iE_m t_0} \phi_m^*(x_0) \\ &= \sum_n e^{-iE_n(t-t_0)} \phi_n^*(x_0)\phi_n(x)\,. \end{aligned} \tag{B.7}$$

Setting $t_0 = 0$ and letting t go to infinity, all contributions vanish apart from the one related to the lowest energy state, $n = 0$, provided that there is a gap between E_1 and E_0. In contrast, any other term with $E_{n\geq 1}$ in the sum will oscillate as t becomes large, and each term above the ground state will decay exponentially. Therefore

$$\begin{aligned} \lim_{t\to\infty} G(x,\, t;\, x,\, 0) &= \phi_0^*(x)\phi_0(x)\,, \\ \lim_{t\to\infty} G(x,\, t;\, 0,\, 0) &\approx \phi_0(x)\phi_0^*(0)\,. \end{aligned} \tag{B.7}$$

This can be used to evaluate the vacuum expectation value of any operator $\mathcal{O}(x)$ as

$$\begin{aligned} \langle\mathcal{O}\rangle &= \lim_{t\to\infty} \int \mathrm{d}x\, G(x\, t;\, x\,, 0)\mathcal{O}(x) \\ &= \lim_{t\to\infty} \int \mathrm{d}x\, G(x\, t;\, x\,, -t)\mathcal{O}(x) \\ &= \int \mathrm{d}x \int \mathcal{D}\bar{x}\, \mathcal{O}(\bar{x}) \exp\left[i \int_{-\infty}^{\infty} \mathrm{d}t' \mathcal{L}(\dot{\bar{x}},\, \bar{x},\, t') \right]\,, \end{aligned} \tag{B.6}$$

where all paths satisfy $\bar{x}(-\infty) = \bar{x}(\infty) = x$.

Simple example: Free particle

In the following the simple example of total amplitude for a free particle will be calculated explicitly. For a free particle of mass m, the Hamiltonian is simply

$$\mathcal{H} = \frac{p^2}{2m} \,. \tag{B.7}$$

Hence,

$$G(x,\, t;\, x_0,\, t_0) \;=\; \int \mathcal{D}p\mathcal{D}\bar{x}\; \exp\left[i \int\limits_{t_0}^{t} \mathrm{d}t' \left(p\dot{\bar{x}} - \frac{p^2}{2m} \right) \right] \,, \tag{B.8}$$

where all paths start at $(x_0,\, t_0)$ and end at $(x,\, t)$. Using the definition of Eq. (B.6) for the total amplitude and plugging in the expression of Eq. (B.7) for a single path, and completing the squares in p_j one ends up with

$$\begin{aligned} &G(x,\, t;\, x_0,\, t_0) \\ &= \lim_{N\to\infty} \int \left(\prod_{k=0}^{N} \frac{\mathrm{d}p_k}{2\pi} \right) \left(\prod_{i=1}^{N} \mathrm{d}\bar{x}_i \right) \exp\left\{ i \sum_{j=0}^{N} \epsilon p_j \left(\frac{\bar{x}_{j+1} - \bar{x}_j}{\epsilon} \right) - \epsilon \frac{p_j^2}{2m} \right\} \\ &= \lim_{N\to\infty} \int \left(\prod_{i=1}^{N} \mathrm{d}\bar{x}_i \right) \left(\frac{m}{2\pi i \epsilon} \right)^{\frac{N+1}{2}} \exp\left\{ i \frac{m\epsilon}{2} \sum_{j=0}^{N} \left(\frac{\bar{x}_{j+1} - \bar{x}_j}{\epsilon} \right)^2 \right\} \,. \end{aligned} \tag{B.6}$$

This is the discretised form of the free particle action,

$$S = \int\limits_{t_0}^{t} \mathrm{d}t' \, \frac{m\dot{x}^2}{2} \,. \tag{B.7}$$

Now, the remaining N Gaussian integrals over the $\bar{x}_i$ have to be done before the limit $N \to \infty$ is to be taken. This can be achieved by successively completing the squares. Starting with $i = 1$ one finds

$$
\begin{aligned}
& G(x,\, t;\, x_0,\, t_0) \\
&= \lim_{N\to\infty} \left(\frac{m}{2\pi i\epsilon}\right)^{\frac{N+1}{2}} \int \prod_{i=2}^{N} \mathrm{d}\bar{x}_i \exp\left\{ i\frac{m\epsilon}{2} \sum_{j=2}^{N} \left(\frac{\bar{x}_{j+1}-\bar{x}_j}{\epsilon}\right)^2 \right\} \\
&\qquad \int_{-\infty}^{\infty} \mathrm{d}\bar{x}_1 \exp\left[\frac{im\epsilon}{2}\left((\bar{x}_2-\bar{x}_1)^2 - (\bar{x}_1-\bar{x}_0)^2\right)\right] \\
&= \lim_{N\to\infty} \left(\frac{m}{2\pi i\epsilon}\right)^{\frac{N+1}{2}} \left(\frac{\pi\epsilon}{im}\right)^{\frac{1}{2}} \\
&\quad \int \prod_{i=2}^{N} \mathrm{d}\bar{x}_i \exp\left\{ i\frac{m\epsilon}{2} \sum_{j=2}^{N} \left(\frac{\bar{x}_{j+1}-\bar{x}_j}{\epsilon}\right)^2 \right\} \exp\left\{\frac{im}{4\epsilon}\left(\bar{x}_2-\bar{x}_0\right)^2\right\}
\end{aligned}
\tag{B.3}
$$

Taking out the $\bar{x}_2$-integral in a similar fashion one finds

$$
\begin{aligned}
G(x,\, t;\, x_0,\, t_0) &= \lim_{N\to\infty} \left(\frac{m}{2\pi i\epsilon}\right)^{\frac{N+1}{2}} \left(\frac{1}{2}\cdot\frac{2\pi\epsilon}{im}\right)^{\frac{1}{2}} \left(\frac{2}{3}\cdot\frac{2\pi\epsilon}{im}\right)^{\frac{1}{2}} \\
& \int \prod_{i=3}^{N} \mathrm{d}\bar{x}_i \exp\left\{ i\frac{m\epsilon}{2} \sum_{j=3}^{N} \left(\frac{\bar{x}_{j+1}-\bar{x}_j}{\epsilon}\right)^2 \right\} \exp\left\{\frac{im}{2(3\epsilon)}\left(\bar{x}_3-\bar{x}_0\right)^2\right\} .
\end{aligned}
\tag{B.2}
$$

Here, the funny factor 2/3 stems from completing the squares of the $\bar{x}_2$-integral in Eq. (B.8),

$$
\begin{aligned}
& \frac{im}{4\epsilon}\left(\bar{x}_2-\bar{x}_0\right)^2 + \frac{im}{2\epsilon}\left(\bar{x}_3-\bar{x}_2\right)^2 \\
&\quad = \frac{im}{2\epsilon}\left\{\frac{3}{2}\left[\bar{x}_2 - \frac{2\bar{x}_3+\bar{x}_0}{3}\right]^2 + \frac{1}{3}\left[\bar{x}_3-\bar{x}_0\right]^2\right\}
\end{aligned}
\tag{B.2}
$$

This also gives some hindsight on the emerging pattern. After all N integrals have been done, the final result reads

$$
\begin{aligned}
& G(x,\, t;\, x_0,\, t_0) \\
&= \lim_{N\to\infty} \left(\frac{m}{2\pi i\epsilon}\right)^{\frac{N+1}{2}} \prod_{j=1}^{N} \left[\frac{j}{j+1}\cdot\frac{2\pi\epsilon}{im}\right]^{\frac{1}{2}} \exp\left\{\frac{im}{2(N+1)\epsilon}\left(\bar{x}_{N+1}-\bar{x}_0\right)^2\right\} .
\end{aligned}
\tag{B.1}
$$

Identifying $\bar{x}_{N+1} = \bar{x}$, $(N+1)\epsilon = t - t_0$ and

$$\left(\frac{m}{2\pi i\epsilon}\right)^{\frac{N+1}{2}} \prod_{j=1}^{N} \left[\frac{j}{j+1} \cdot \frac{2\pi\epsilon}{im}\right]^{\frac{1}{2}} = \left(\frac{m}{2\pi i\epsilon}\right)^{\frac{N+1}{2}} \left(\frac{-2\pi i\epsilon}{m}\right)^{\frac{N}{2}} \left(\frac{1}{N+1}\right)^{\frac{1}{2}} \tag{B.2}$$

one finally has

$$G(x,\, t;\, x_0,\, t_0) = \left(\frac{m}{2i\pi(t-t_0)}\right)^{\frac{1}{2}} \exp\left(\frac{im(\bar{x}-\bar{x}_0)^2}{2(t-t_0)}\right). \tag{B.3}$$

Note that the remaining oscillatory term $(-1)^{N/2}$ has been supressed in the equation above.

Next simple example: Harmonic oscillator

Turning to the next simple example, the harmonic oscillator. Its Hamiltonian reads

$$\mathcal{H} = \frac{p^2}{2m} + \frac{m}{2}\omega^2 x^2 . \tag{B.4}$$

The total amplitude is

$$G(x,\, t;\, x_0,\, t_0) = \int \mathcal{D}\bar{x} \exp\left\{i\mathcal{S}[\bar{x}]\right\} = \int \mathcal{D}\bar{x} \exp\left\{ i \int_{t_0}^{t} \mathrm{d}t' \frac{m}{2}\left[\dot{\bar{x}}^2 - \omega^2 \bar{x}^2\right] \right\}, \tag{B.4}$$

where, as usual, $\bar{x}(t_0) = x_0$ and $\bar{x}(t) = x$. Consider now a Taylor expansion of the action functional around the classical path, $\bar{x}_{\mathrm{cl.}}$.

$$\mathcal{S}[\bar{x}] = \mathcal{S}[\bar{x}_{\mathrm{cl.}}] + \left.\frac{\delta\mathcal{S}}{\delta\bar{x}}\right|_{\bar{x}=\bar{x}_{\mathrm{cl.}}} (\bar{x}-\bar{x}_{\mathrm{cl.}}) + \left.\frac{\delta^2\mathcal{S}}{\delta\bar{x}^2}\right|_{\bar{x}=\bar{x}_{\mathrm{cl.}}} \frac{(\bar{x}-\bar{x}_{\mathrm{cl.}})^2}{2}. \tag{B.5}$$

Terms higher than second order vanish, due to the orders present in the action. To do a stationary phase approximation one chooses the classical path such that

$$\left.\frac{\delta\mathcal{S}}{\delta\bar{x}}\right|_{\bar{x}=\bar{x}_{\mathrm{cl.}}} = 0 . \tag{B.6}$$

This is a consistent choice, since evaluation of the variation $\delta\mathcal{S}/\delta\bar{x} = 0$ just yields the classical equations of motion. For an harmonic oscillator starting at $(x_0,\, t_0)$ and going through $(x_1,\, t_1)$ this path is just given by

$$\bar{x}_{\mathrm{cl.}}(t) = x_0 \frac{\sin\left(\omega(t_1 - t)\right)}{\sin\left(\omega(t_1 - t_0)\right)} + x_1 \frac{\sin\left(\omega(t - t_0)\right)}{\sin\left(\omega(t_1 - t_0)\right)}\,. \tag{B.7}$$

Hence, the first term in the Taylor expansion reads

$$\mathcal{S}[\bar{x}_{\mathrm{cl.}}] = \frac{m}{2}\left[\frac{\omega}{\sin\left(\omega(t_1 - t_0)\right)}\right]\left[(x_1 + x_0)^2 \cos\left(\omega(t_1 - t_0)\right) - 2x_1 x_0\right]\,. \tag{B.8}$$

Plugging the expansion of the action into the expression of the total amplitude, the latter becomes

$$\begin{aligned} &G(x,\, t;\, x_0,\, t_0) \\ &= \exp\left\{i\mathcal{S}[\bar{x}_{\mathrm{cl.}}]\right\} \int \mathcal{D}\bar{x}\, \exp\left\{\frac{1}{2}\int\limits_{t_0}^{t} \mathrm{d}t'\, \left[(\bar{x} - \bar{x}_{\mathrm{cl.}})\left(\partial_t^2 + \omega^2\right)(\bar{x} - \bar{x}_{\mathrm{cl.}})\right]\right\} \end{aligned} \tag{B.7}$$

To do this remaining integral one switches from paths $\bar{x}$ to fluctuations $y = \bar{x} - \bar{x}_{\mathrm{cl.}}$ around the classical path. In this notation, $y(t) = y(t_0) = 0$ and one can think of using periodic boundary conditions. Hence, one may consider paths with period $t - t_0$. Using some little functional calculus leads to

$$\begin{aligned} G(x,\, t;\, x_0,\, t_0) &= \det{}^{-\frac{1}{2}}\left[\frac{im(\partial_t^2 + \omega^2)}{2\pi}\right] \\ &\cdot \exp\left\{\frac{im}{2}\left[\frac{\omega}{\sin\left(\omega(t_1 - t_0)\right)}\right]\left[(x_1 + x_0)^2\cos\left(\omega(t_1 - t_0)\right) - 2x_1x_0\right]\,.\right\} \end{aligned} \tag{B.6}$$

The determinant of an operator is the product of its eigenvalues. Since periodic paths with period $T = t - t_0$ are integrated over, the eigenfunctions of $\partial_t^2 + \omega^2$ are

$$\sin\frac{n\pi t}{T} \quad \text{with} \quad n = 1,\, 2,\, \ldots \tag{B.7}$$

leading to eigenvalues

$$\lambda_n = -\left(\frac{n\pi}{T}\right)^2 + \omega^2\,. \tag{B.8}$$

Hence,

$$\det\left[\frac{im(\partial_t^2+\omega^2)}{2\pi}\right] = \prod_n \frac{m}{2\pi i}\left[\left(\frac{n\pi}{T}\right)^2-\omega^2\right]$$
$$= \prod_n\left[\frac{m}{2\pi i}\left(\frac{n\pi}{T}\right)^2\right]\prod_n\left[1-\frac{\omega^2T^2}{(n\pi)^2}\right]. \quad \text{(B.8)}$$

The maybe surprising fact of this equation is that, if a free particle is considered, $\omega = 0$, and plugging in the result for this case, one finds

$$\prod_n\left[\frac{m}{2\pi i}\left(\frac{n\pi}{T}\right)^2\right] = \frac{2\pi iT}{m}\,, \quad \text{(B.9)}$$

exactly as before. However, switching back to oscillators, according to one of Euler's identities, the remaining piece is nothing but

$$\prod_n\left[1-\frac{\omega^2T^2}{(n\pi)^2}\right] = \frac{\sin(\omega T)}{\omega T}\,. \quad \text{(B.10)}$$

Taking everything together the total amplitude for a harmonic oscillator is given by

$$G(x,\,t;\,x_0,\,t_0) = \left(\frac{m\omega}{2\pi i\sin[\omega(t-t_0)]}\right)^{\frac{1}{2}}$$
$$\cdot\exp\left\{\frac{im}{2}\left[\frac{\omega}{\sin\left(\omega(t_1-t_0)\right)}\right]\left[(x_1+x_0)^2\cos\left(\omega(t_1-t_0)\right)-2x_1x_0\right]\,.\right\} \quad \text{(B.9)}$$

As a first application one can calculate the energy eigenfunctions from this total amplitude. First of all, to alleviate the task of taking the limit $iT\to\infty$, one may rewrite the total amplitude such that an exponential of "iT times something" appears in front.

$$G(x,\,T;\,x_0,\,0)$$
$$= \left(\frac{m\omega}{\pi}\right)^{\frac{1}{2}} e^{-\frac{i\omega T}{2}}\left(1-e^{-2i\omega T}\right)^{-\frac{1}{2}}$$
$$\exp\left\{-\frac{m\omega}{2}\left[(x^2+x_0^2)\left(\frac{1+e^{-2i\omega T}}{1-e^{-2i\omega T}}-\frac{4xx_0e^{-i\omega T}}{1-e^{-2i\omega T}}\right)\right]\right\}$$
$$\overset{iT\to\infty}{\longrightarrow} \left(\frac{m\omega}{\pi}\right)^{\frac{1}{2}} e^{-\frac{i\omega T}{2}}\exp\left(-\frac{m\omega(x^2+x_0^2)}{2}\right)$$
$$= e^{-iE_0T}\phi_0^*(x)\phi_0(x_0)\,. \quad \text{(B.6)}$$

Therefore,

$$E_0 = \frac{\omega}{2} \quad \text{and} \quad \phi_0(x) = \left(\frac{m\omega}{\pi}\right)^{\frac{1}{4}} \exp\left(-\frac{m\omega x^2}{2}\right) \tag{B.7}$$

Adding external forces

When an external force $F(t)$ is added to the harmonic oscillator, its action becomes

$$\mathcal{S}[x(\bar{t})] = \int_{t_0}^{t} \mathrm{d}\bar{t} \left[\frac{1}{2}m^2\left(\dot{x}^2 - \omega^2 x^2\right) + xF(\bar{t})\right] \tag{B.8}$$

leading to the equation of motion

$$\frac{\delta\mathcal{S}}{\delta x} = m\left(\partial_t^2 + \omega^2\right)x(t) - F(t) = 0\,, \tag{B.9}$$

as expected. Using the classical boundary conditions $x_{\mathrm{cl}}^F(t_0) = x_0$ and $x_{\mathrm{cl}}^F(t) = x$ one finds the general solution

$$x_{\mathrm{cl}}^F(t') = x_{\mathrm{cl}}(t') - \int_{t_0}^{t'} \mathrm{d}\bar{t} g(t', \bar{t})F(\bar{t})\,, \tag{B.10}$$

where, again, x_{cl} is the classical path in the absence of the force. It has been calculated already in Eq. (B.7). In the equation, above, $g(t', \bar{t})$ is the Green's function of the free harmonic oscillator and satisfies

$$(\partial_{t'}^2 + \omega^2)g(t', \bar{t}) = -\delta(t' - \bar{t})\,. \tag{B.11}$$

An explicit expression for the Green's function is

$$g(t', \bar{t}) = \begin{cases} \dfrac{\sin[\omega(t - t')]\sin[\omega(\bar{t} - t_0)]}{\omega\sin[\omega(t - t_0)]} & \text{for} \quad t' \leq \bar{t} \\ \dfrac{\sin[\omega(t - \bar{t})]\sin[\omega(t' - t_0)]}{\omega\sin[\omega(t - t_0)]} & \text{for} \quad \bar{t} \leq t'\,. \end{cases} \tag{B.12}$$

As one can see $g(t',\, \bar{t})$ is an admixture of advanced and retarded signals, a precursor of the Feynman propagator. The total amplitude then becomes

$$\begin{aligned} G_F(x,\, t;\, x_0,\, t_0) &= \int \mathcal{D}x \exp\left\{i\mathcal{S}[x]\right\} \\ &= \int \mathcal{D}x \exp\left\{ i \int\limits_{t_0}^{t} \mathrm{d}t' \left[\frac{m}{2}\left(\dot{x}^2 - \omega^2 x^2\right) + F(t')x(t')\right]\right\} . \end{aligned} \tag{B.11}$$

As it stands, the integrand is purely oscillatory, to restore convergence one might add a term of the form

$$\exp\left\{-\frac{\epsilon}{2}\int\limits_{t_0}^{t} \mathrm{d}t' x^2(t')\right\} \tag{B.12}$$

with $\epsilon > 0$. Letting ϵ go to zero at the end of the calculation then ensures convergence (and, in fact, yields exactly the term $\sim i\epsilon$ in the denominators of the propagators). To calculate the transition amplitude, it is convenient to do a Fourier transform w.r.t. the time,

$$\begin{aligned} &\left[\dot{x}^2(t) - (\omega^2 - i\epsilon)x^2(t)\right] \\ &= \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}E}{\sqrt{2\pi}}\frac{\mathrm{d}E'}{\sqrt{2\pi}} e^{i(E+E')t}\left[-EE' - \omega^2 + i\epsilon\right] x(E)x(E') \end{aligned} \tag{B.12}$$

and

$$F(t)x(t) = \frac{1}{2}\int\limits_{-\infty}^{\infty} \frac{\mathrm{d}E}{\sqrt{2\pi}}\frac{\mathrm{d}E'}{\sqrt{2\pi}} e^{i(E+E')t}\left[x(E)F(E') + x(E')F(E)\right] . \tag{B.12}$$

Using the integral representation of the δ-function and after a change of variables,

$$\begin{aligned} x'(E) &= x(E) + \frac{F(E)}{E^2 - \omega^2 + i\epsilon} , \\ x'(t) &= x(t) + \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}E}{\sqrt{2\pi}} e^{iEt} \frac{F(E)}{E^2 - \omega^2 + i\epsilon} , \end{aligned} \tag{B.12}$$

which basically amounts to completing the squares, one finds

$$G_F(x,\,+\infty;\,x_0,\,-\infty) = \exp\left\{-\frac{i}{2}\int\limits_{-\infty}^{\infty} \mathrm{d}E\frac{F(E)F(-E)}{E^2-\omega^2+i\epsilon}\right\}$$
$$\int \mathcal{D}x \exp\left\{\frac{i}{2}\int\limits_{-\infty}^{\infty} \mathrm{d}E\,\left[x'(E)(E^2-\omega^2+i\epsilon)x'(-E)\right]\right\}\,. \qquad \text{(B.12)}$$

The magic of the path integral formulation is that the Jacobian of this transformation, i.e. of completing the squares is one, or, in other words, that

$$\mathcal{D}x = \mathcal{D}x'\,. \qquad \text{(B.13)}$$

Then, one realizes immediately that the last term of the expression in Eq. (B.13) is nothing but the free oscillator transition amplitude. Therefore one can express the transition amplitude for the oscillator with an external force, G_F, through the free one, G, as

$$\begin{aligned} G_F(x,\,+\infty;\,x_0,\,-\infty) &= \langle x,\,+\infty|x_0,\,-\infty\rangle \\ &= \exp\left\{-\frac{i}{2}\int\limits_{-\infty}^{\infty} \mathrm{d}E\frac{F(E)F(-E)}{E^2-\omega^2+i\epsilon}\right\} G(x,\,+\infty;\,x_0,\,-\infty)\,. \end{aligned} \qquad \text{(B.13)}$$

This can be massaged into a form that is a bit more intuitive,

$$\exp\left\{-\frac{i}{2}\int\limits_{-\infty}^{\infty} \mathrm{d}E\frac{F(E)F(-E)}{E^2-\omega^2+i\epsilon}\right\} = \exp\left\{-\frac{i}{2}\int\limits_{-\infty}^{\infty} \mathrm{d}t\mathrm{d}t'F(t)g(t,\,t')F(-t')\right\}\,, \qquad \text{(B.13)}$$

where, expressed through energies,

$$g(t,\,t') = \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}E}{2\pi}\frac{e^{-iE(t-t')}}{E^2-\omega^2+i\epsilon}\,. \qquad \text{(B.14)}$$

To understand the physical meaning of Eq. (B.14), assume that for $t\to\pm\infty$ there is no driving force F. Then the vacuum state will not depend on the existence of F. Let $|\phi_{\pm\infty}\rangle$ be the vacuum states in the infinite future and past. Then

$$\begin{aligned}
&\langle x,\, +\infty | x_0,\, -\infty\rangle_F \\
&= \int \mathrm{d}\phi \mathrm{d}\phi' \left\langle x,\, +\infty | \phi'_{+\infty}\right\rangle \left\langle \phi'_{+\infty} | \phi_{-\infty}\right\rangle_F \left\langle \phi_{-\infty} | x_0,\, -\infty\right\rangle \\
&= \int \mathrm{d}\phi \mathrm{d}\phi' \left\langle x,\, +\infty | \phi'_{+\infty}\right\rangle \left\langle \phi'_{+\infty} | \phi_{-\infty}\right\rangle \left\langle \phi_{-\infty} | x_0,\, -\infty\right\rangle \\
&\qquad \exp\left\{ -\frac{i}{2} \int\limits_{-\infty}^{\infty} \mathrm{d}t \mathrm{d}t' F(t) g(t,\, t') F(-t') \right\} . \qquad \text{(B.12)}
\end{aligned}$$

Identifying

$$\left\langle \phi'_{+\infty} | \phi_{-\infty}\right\rangle_{F=0} = \langle x,\, +\infty | x_0,\, -\infty\rangle_{F=0} \qquad \text{(B.13)}$$

allows to rewrite

$$\begin{aligned}
&\langle x,\, +\infty | x_0,\, -\infty\rangle_F \\
&= \langle x,\, +\infty | x_0,\, -\infty\rangle_{F=0} \exp\left\{ -\frac{i}{2} \int\limits_{-\infty}^{\infty} \mathrm{d}t \mathrm{d}t' F(t) g(t,\, t') F(-t') \right\} \text{(B.13)}
\end{aligned}$$

In other words,

$$\exp\left\{ -\frac{i}{2} \int\limits_{-\infty}^{\infty} \mathrm{d}t \mathrm{d}t' F(t) g(t,\, t') F(-t') \right\} \qquad \text{(B.14)}$$

is the transition amplitude in the presence of an external driving force to go from the ground state in the far past to the ground state in the far future. Defining

$$Z[F] \;:=\; \exp\left\{ -\frac{i}{2} \int\limits_{-\infty}^{\infty} \mathrm{d}t \mathrm{d}t' F(t) g(t,\, t') F(-t') \right\} := e^{i\mathcal{W}[F]} \qquad \text{(B.15)}$$

one sees immediately that

$$g(t_1,\, t_2) = \frac{-i\delta}{\delta F(t_1)} \frac{-i\delta}{\delta F(t_2)} \mathcal{W}[F] \bigg|_{F=0} . \qquad \text{(B.16)}$$

This provides a way to extract Green's functions from the vacuum-to-vacuum transition amplitude in the presence of a source: just take the according functional derivative w.r.t. the source.
It should be noted that

$$g(t_1,\, t_2) = \frac{1}{2i\omega}\left[\theta(t_1 - t_2)e^{-i\omega(t_1-t_2)} + \theta(t_2 - t_1)e^{i\omega(t_1-t_2)}\right] . \qquad \text{(B.17)}$$

B.1.2 Perturbation theory

In a next step, a potential $V(x)$ can be added. Only in very few special cases such a case can be computed exactly, one of the most beloved examples is the harmonic oscillator. In other cases, a typical strategy is to separate the Hamiltonian $\mathcal{H}$ into two pieces: a part $\mathcal{H}_0$, which can be solved exactly, and an interaction part $\lambda V(x)$. The solution of the exactly known piece, like for instance, a free particle Hamiltonian, will be denoted as G_0 in the following. However, when λ is small, one can do a perturbative expansion in λ to find an approximate solution for G, the full Hamiltonians total amplitude, in terms of G_0. To do so, start with the path integral representation of G,

$$G(x,\, t;\, x_0,\, t_0) = \int \mathcal{D}\bar{x}\, \exp\left[\int_{t_0}^{t} \mathrm{d}t'\, (\mathcal{S}_0[\bar{x}] - \lambda V(\bar{x}))\right] , \qquad \text{(B.18)}$$

where $\mathcal{S}_0$ is the action of the system described by $\mathcal{H}_0$, i.e. for $\lambda = 0$. Assuming again that the Hamiltonian does not contain any unorthodox dependence on the momenta, one can carry out the corresponding Gaussian integrals. Also, one can then separate the unperturbed piece and the potential and do an expansion of the exponential containing it,

$$\begin{aligned} &G(x,\, t;\, x_0,\, t_0) \\ &= \int \mathcal{D}\bar{x}\, \exp\left[\int_{t_0}^{t} \mathrm{d}t' \mathcal{S}_0[\bar{x}]\right] \sum_{n=0}^{\infty} \left[\frac{(-i\lambda)^n}{n!} \left(\int_{t_0}^{t} \mathrm{d}t' V(\bar{x})\right)^n\right] \\ &= \sum_{n=0}^{\infty} \left[\frac{(-i\lambda)^n}{n!} \int \mathcal{D}\bar{x} \left(\int_{t_0}^{t} \mathrm{d}t' V(\bar{x})\right)^n \exp\left(\int_{t_0}^{t} \mathrm{d}t' \mathcal{S}_0[\bar{x}]\right)\right] \\ &=: \sum_{n=0}^{\infty} G^{(n)}(x,\, t;\, x_0,\, t_0)\,, \end{aligned} \qquad \text{(B.16)}$$

where n labels the number of interactions, and, obviously

$$G^{(0)}(x,\, t;\, x_0,\, t_0) = G(x,\, t;\, x_0,\, t_0)\,. \tag{B.17}$$

The next term in this expansion yields

$$\begin{aligned} G^{(1)}(x,\, t;\, x_0,\, t_0) &= -i\lambda \int \mathcal{D}\bar{x}(\bar{t}) \int\limits_{t_0}^{t} \mathrm{d}t' V(\bar{x}(t')) \exp\left(\int\limits_{t_0}^{t} \mathrm{d}t'' \mathcal{S}_0[\bar{x}(t'')] \right) \\ &= -i\lambda \int\limits_{t_0}^{t} \mathrm{d}t' \int \mathcal{D}\bar{x}(\bar{t}) V(\bar{x}(t')) \exp\left(\int\limits_{t_0}^{t} \mathrm{d}t'' \mathcal{S}_0[\bar{x}(t'')] \right)\,. \end{aligned} \tag{B.16}$$

Here the various time-dependences have been exhibited explicitly to clarify the situation. Now, the propagation of the particle between $(x_0,\, t_0)$ and $(x,\, t)$ can be broken up such that it passes $(x_1,\, t_1)$. Then, to first order in λ, the amplitude reads

$$-i\lambda G^{(0)}(x,\, t;\, x_1,\, t_1) V(x_1,\, t_1) G^{(0)}(x_1,\, t_1;\, x_0,\, t_0) \tag{B.17}$$

To obtain the full amplitude, one has to sum over all possible $(x_1,\, t_1)$, where $t_1 \in [t_0,\, t]$. In other words

$$\begin{aligned} &G^{(1)}(x,\, t;\, x_0,\, t_0) \\ &= \int\limits_{t_0}^{t} \mathrm{d}t_1 \int\limits_{-\infty}^{\infty} \mathrm{d}x_1 (-i\lambda) G^{(0)}(x,\, t;\, x_1,\, t_1) V(x_1,\, t_1) G^{(0)}(x_1,\, t_1;\, x_0,\, t_0) \\ &= (-i\lambda) \int\limits_{t_0}^{t} \mathrm{d}t_1 \int\limits_{-\infty}^{\infty} \mathrm{d}x_1 \int \mathcal{D}\bar{x}\, \exp\left(\int\limits_{t_0}^{t} \mathrm{d}t' \mathcal{S}_0[\bar{x}(t')] \right) V(x_1,\, t_1) \\ &\qquad \int \mathcal{D}\bar{y}\, \exp\left(\int\limits_{t_0}^{t} \mathrm{d}t' \mathcal{S}_0[\bar{y}(t')] \right)\,, \end{aligned} \tag{B.15}$$

clarifying the notation above. Expanding the total amplitude in this fashion lends itself to a diagrammatic representation of the terms, cf. Fig.B.1. There, the free field total amplitudes are represented as straight lines and the interaction points are depicted as black blobs. To translate back from a diagram

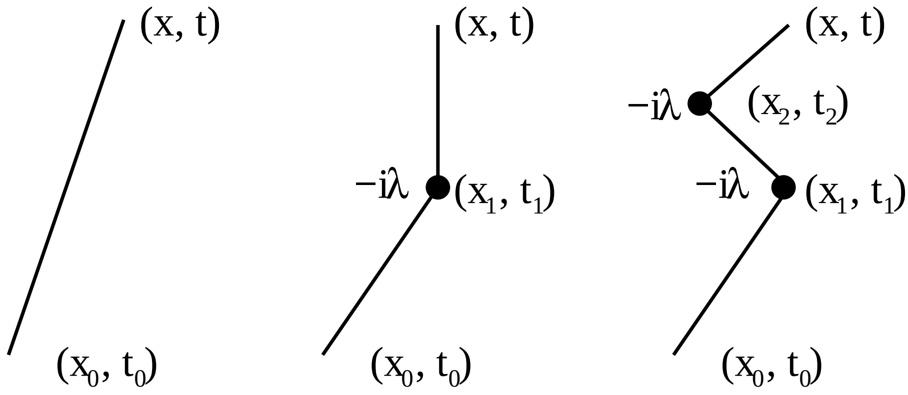


Figure B.1: Diagrammatic representation of the first three terms of $G^{(n)}(x,\, t;\, x_0,\, t_0)$.

to an expression one starts at the top of a diagram, i.e. at its endpoint, and works the way back to its starting point. Each straight line between two points $(x_i,\, t_i)$ and $(x_{i-1},\, t_{i-1})$ then relates to a factor $G^{(0)}(x_i,\, t_i;\, x_{i-1},\, t_{i-1})$, and each blob at a position $(x_i,\, t_i)$ corresponds to a factor $-i\lambda V(x_i,\, t_i)$, where the position is to be integrated over. For the last diagram in the figure one then has

$$
\begin{aligned}
&G^{(2)}(x,\, t;\, x_0,\, t_0) \\
&= (-i\lambda)^2 \int_{t_0}^{t} \mathrm{d}t_2 \int_{-\infty}^{\infty} \mathrm{d}x_2 G^{(0)}(x,\, t;\, x_2,\, t_2) V(x_2,\, t_2) \\
&\quad \int_{t_0}^{t} \mathrm{d}t_1 \int_{-\infty}^{\infty} \mathrm{d}x_1 G^{(0)}(x_2,\, t_2;\, x_1,\, t_1) V(x_1,\, t_1) G^{(0)}(x_1,\, t_1;\, x_0,\, t_0)
\end{aligned}
\tag{B.13}
$$

for its contribution to the total amplitude $G^{(n)}$.

B.2 Scalar field theory

The concepts developed in the previous section are now translated to the case of quantum field theories. Instead of integrating over the position and momentum x and p of a single particle, now the integration is over the field ϕ and its conjugate momentum π. In this respect it is useful to remember that the quantum theory of a free field can be interpreted as a collection of infinitely many harmonic oscillators, where one wavefunction represents one mode of the field.

B.2.1 Free scalar fields

Exploring analogies

To dwell on these analogy, let $|\Psi(t)\rangle$ be a state of the (field theoretical) system, which evolves through the Schrödinger equation as

$$i\frac{\partial}{\partial t}|\Psi(t)\rangle = \mathcal{H}|\Psi(t)\rangle\,. \tag{B.14}$$

Here, the Hamiltonian is that of a free scalar field,

$$\mathcal{H} = \frac{1}{2}\int \mathrm{d}^3x\left(\pi^2 + \left|\vec{\nabla}\phi\right|^2 + m^2\phi^2\right)\,. \tag{B.15}$$

To construct a time-evolution operator, one has to project the states onto ϕ-space and define corresponding wave functionals through

$$\Psi[\phi(\vec{x}),\, t] = \langle\phi|\Psi(t)\rangle\,. \tag{B.16}$$

Then, the evolution of such an object is given by

$$\begin{aligned}\Psi[\phi(\vec{x}),\, t] &= \int \mathcal{D}\phi_0\left\langle\phi\left|e^{-i\mathcal{H}(t-t_0)}\right|\phi_0\right\rangle\Psi[\phi_0(\vec{x}),\, t_0] \\ &=: \int \mathcal{D}\phi_0 G[\phi,\, t;\, \phi_0,\, t_0]\,\Psi[\phi_0(\vec{x}),\, t_0]\,.\end{aligned} \tag{B.16}$$

The functional integration

In the following paragraph some light will be shed on this nomenclature. Assuming real scalar fields, the field ϕ is a function connecting the real three-dimensional space with real numbers, $R^3 \to R$. Thus, the functional integral $\int\mathcal{D}\phi$ denotes an integral over a set of such functions. Therefore, to draw the evolution of $\Psi[\phi(\vec{x}),\, t]$, one would have to stack layers of such functions

onto each other. These layers are labelled through the discretised time, i.e. through the t_i, where $t_i \in [t_0,\, t]$.
Hence, to obtain the path integral representation of $G[\phi,\, t;\, \phi_0,\, t_0]$, one again divides the interval $[t_0,\, t]$ into $N+1$ equal parts. Next, for each division, i.e. at each layer, a factor "1" is inserted in the form

$$1 = \int \mathcal{D}\phi_i\, |\phi_i\,\rangle\langle\,\phi_i| \quad \text{with} \quad \phi_i = \phi_i(\vec{x},\, t_i)\,. \tag{B.17}$$

Then, with $N \to \infty$,

$$G[\phi,\, t;\, \phi_0,\, t_0] = \lim_{N\to\infty} \left[\int \prod_{i=1}^{N} \mathcal{D}\phi_i\right] \left[\prod_{i=0}^{N} \left\langle \phi_{i+1} \left| e^{i\mathcal{H}(t_{i+1}-t_i)} \right| \phi_i \right\rangle\right]\,, \tag{B.18}$$

Again, with $N \to \infty$, the time intervals become small, allowing to expand the exponentials,

$$\begin{aligned} G[\phi_{i+1},\, t_{i+1};\, \phi_i,\, t_i] &= \left\langle \phi_{i+1} \left| e^{i\mathcal{H}(t_{i+1}-t_i)} \right| \phi_i \right\rangle \\ &\approx \langle \phi_{i+1} \left| 1 - i\mathcal{H}(t_{i+1}-t_i) \right| \phi_i \rangle \\ &= \delta\left[\phi_{i+1} - \phi_i\right] - i\epsilon \left\langle \phi_{i+1} \left| \mathcal{H} \right| \phi_i \right\rangle\,, \end{aligned} \tag{B.17}$$

where $\epsilon = t_{i+1} - t_i = (t_{N+1} - t_0)/(N+1)$ is assumed. In fact, the δ-functional can be understood as a product of infinitely many δ-functions, each of which is taken at one point $\vec{x}$ in space. Rewriting each of them as a Fourier-transform one has

$$\begin{aligned} \delta\left[\phi_{i+1} - \phi_i\right] &= \prod_{\vec{x}} \delta\left(\phi_{i+1}(\vec{x}) - \phi_i(\vec{x})\right) \\ &= \prod_{\vec{x}} \int_{-\infty}^{\infty} \frac{\mathrm{d}\pi_i}{2\pi} \exp\left\{i\pi_i(\vec{x})\left[\phi_{i+1}(\vec{x}) - \phi_i(\vec{x})\right]\right\} \\ &=: \int \mathcal{D}\pi_i \exp\left\{i \int \mathrm{d}^3\vec{x}\, \pi_i(\vec{x})\left[\phi_{i+1}(\vec{x}) - \phi_i(\vec{x})\right]\right\}. \end{aligned} \tag{B.16}$$

In a similar way, and in full analogy to the path integral in quantum mechanics considered before,

$$\langle \phi_{i+1} \left| \mathcal{H} \right| \phi_i \rangle = \int \mathcal{D}\pi_i\, \mathcal{H}[\pi_i,\, \bar{\phi}_i]\, \exp\left\{i \int \mathrm{d}^3\vec{x}\, \pi_i(\vec{x})\left[\phi_{i+1}(\vec{x}) - \phi_i(\vec{x})\right]\right\}\,, \tag{B.16}$$

where, again, an average has been introduced. This time it is over the fields,

$$\bar{\phi}_i = \frac{\phi_{i+1} + \phi_i}{2}\,. \tag{B.17}$$

In the equation above, $\mathcal{H}[\pi_i,\, \bar{\phi}_i]$ is a functional, not an operator, whereas in the sandwich, $\mathcal{H}$ with no arguments whatsoever denotes an operator. Repeating the same tricks as above, i.e. re-exponentiating of the Hamiltonian functional and identifying derivatives of the fields, one finds

$$\begin{aligned}
&G[\phi,\, t;\, \phi_0,\, t_0] \\
&\quad= \lim_{N\to\infty} \left[\int \prod_{i=1}^{N} \mathcal{D}\phi_i\right] \left[\int \prod_{j=0}^{N} \mathcal{D}\pi_j\right] \\
&\qquad \exp\left\{ i\epsilon \sum_{j=0}^{N} \int \mathrm{d}^3\vec{x} \left[\pi_j(\vec{x}) \frac{\phi_{j+1}(\vec{x}) - \phi_j(\vec{x})}{\epsilon} - H[\pi_j(\vec{x}),\, \bar{\phi}_j(\vec{x})]\right]\right\} \\
&\quad= \int \mathcal{D}\bar{\phi}\mathcal{D}\pi \, \exp\left[i \int_{t_0}^{t} \mathrm{d}t' \int \mathrm{d}^3\vec{x} \left(\pi(\vec{x})\dot{\bar{\phi}}(\vec{x}) - H[\pi,\, \bar{\phi}]\right)\right]\,.
\end{aligned} \tag{B.15}$$

Carrying out the Gaussian integration over π (again, $H[\pi_j(\vec{x}),\, \bar{\phi}_j(\vec{x})]$ is assumed to be quadratic in π), one ends up with

$$\begin{aligned}
G[\phi,\, t;\, \phi_0,\, t_0] &= \int \mathcal{D}\bar{\phi} \exp\left[i \int_{t_0}^{t} \mathrm{d}t' \int \mathrm{d}^3\vec{x} \mathcal{L}(\dot{\bar{\phi}},\, \bar{\phi},\, t')\right] \\
&= \int \mathcal{D}\bar{\phi} \exp\left[i\mathcal{S}(\bar{\phi})\right]
\end{aligned} \tag{B.15}$$

Again, $\mathcal{L}$ and $\mathcal{S}$ are the classical Lagrangian and action, respectively. It should be stressed that in the path formalism for the fields,

$$\mathcal{D}\bar{\phi} = \lim_{N\to\infty} \left[\int \prod_{i=1}^{N} \prod_{\vec{x}} d\phi(\vec{x},\, t_i) \left(\frac{1}{2\pi i\epsilon}\right)\right]\,. \tag{B.16}$$

In most cases, however, the normalisation will be thrown into the overall normalisation of the total amplitude and forgotten.
This total amplitude is over all paths in the space of field configurations, described by $\phi(\vec{x},\, t)$ and $\pi(\vec{x},\, t)$. the path is fixed at the two endpoints

$\phi_0(\vec{x}_0,\, t_0)$ and $\phi(\vec{x},\, t)$. In quantum mechanics in contrast, the total amplitude is over all eigenvalues $x(t)$ and $p(t)$ of the position and momentum operators of the single particle.
Taking the Fourier transform of the fields in the total amplitude, Eq. (B.16), and of the action, one has

$$
\begin{aligned}
&G[\phi,\, t;\, \phi_0,\, t_0] \\
&= \int \mathcal{D}\bar{\phi} \exp\left\{ i \int\limits_{t_0}^{t} \mathrm{d}t' \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \frac{1}{2} \left[(\partial_t \bar{\phi}(-\vec{k}))(\partial_t \bar{\phi}(\vec{k})) - \omega_{\vec{k}} \bar{\phi}(-\vec{k}) \bar{\phi}(\vec{k}) \right] \right\} .
\end{aligned} \tag{B.15}
$$

This is just the infinite sum over actions of harmonic oscillators, one for each mode $\vec{k}$. It should be noted that this identification works so well, since the individual actions are diagonal in the modes, i.e. there is no mixing between two modes $\vec{k}_1$ and $\vec{k}_2$. Therefore, one has

$$
\begin{aligned}
&G[\phi,\, t;\, \phi_0,\, t_0] \\
&= \prod_{\vec{k}} \left(\frac{\omega_{\vec{k}}}{2\pi i \sin[\omega_{\vec{k}}(t-t_0)]} \right)^{\frac{1}{2}} \\
&\quad \exp\left\{ \frac{i}{2} \frac{1}{(2\pi)^3} \left(\frac{\omega_{\vec{k}}}{\sin[\omega_{\vec{k}}(t-t_0)]} \right) \left[\left(\phi_{\vec{k}}^2 + \phi_{0,\vec{k}}^2 \right) \cos[\omega_{\vec{k}}(t-t_0)] - 2\phi_{\vec{k}} \phi_{0,\vec{k}} \right] \right\} \\
&= \prod_{\vec{k}} \left(\frac{\omega_{\vec{k}}}{2\pi i \sin[\omega_{\vec{k}}(t-t_0)]} \right)^{\frac{1}{2}} \exp\left\{ \frac{i}{2} \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \frac{\omega_{\vec{k}}}{\sin[\omega_{\vec{k}}(t-t_0)]} \right. \\
&\qquad \left. \left[\left(\phi^2(\vec{k}) + \phi_0^2(\vec{k}) \right) \cos[\omega_{\vec{k}}(t-t_0)] - 2\phi(\vec{k})\phi_0(\vec{k}) \right] \right\} ,
\end{aligned} \tag{B.12}
$$

where the difference between the first and second expression is that in the second expression the product over all $\vec{k}$ has been replaced by the integration. The ground state wave functional can be recovered by taking $T = t - t_0 \to \infty$ and considering the leading term only. Using

$$
\sin(\omega_k T) \to -\frac{i}{2} e^{i\omega_k T} \quad \text{and} \quad \frac{\cos(\omega_k T)}{\sin(\omega_k T)} \to i \tag{B.13}
$$

for $T \to \infty$ one finds

$$G[\phi,\, \phi_0,\, T] \longrightarrow \left[\prod_{\vec{k}} \left(\frac{\omega_k}{\pi}\right)^{\frac{1}{2}} e^{-\frac{i\omega_k T}{2}}\right] \exp\left[-\int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\frac{\omega_k}{2}\left(\phi^2(\vec{k}) + \phi_0^2(\vec{k})\right)\right] \tag{B.13}$$

From this one can read off the vacuum energy,

$$\mathcal{E}_0 = \frac{1}{2}\int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\omega_k\,. \tag{B.14}$$

Adding sources

To add dynamics to the system, one couples the fields to external sources, each for one field. For a real scalar field this translates to coupling the field $\phi(x)$ locally to a source J by adding a term $J(t,\, \vec{x})\phi(t,\, \vec{x})$ to the action. For free fields this is not particularly interesting, but as will be seen later on, for the interacting case this will be the path towards vacuum expectation values of time-ordered products, i.e. n-point functions. In fact, as it will turn out, the so modified actions give rise to generating functionals of the n-point functions. When adding such a source, the action becomes

$$\begin{aligned} \mathcal{S} &= \int_{t_0}^{t} \mathrm{d}t' \int \mathrm{d}^3\vec{x} \left\{\frac{1}{2}\left[\partial_\mu\phi(t',\, \vec{x})\partial^\mu\phi(t',\, \vec{x}) - m^2\phi(t',\, \vec{x})^2\right] + J(t',\, \vec{x})\phi(t',\, \vec{x})\right\} \\ &= \int_{t_0}^{t} \mathrm{d}t' \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \left\{\frac{1}{2}\left[\partial_{t'}\phi(t',\, -\vec{k})\partial^{t'}\phi(t',\, \vec{k}) - \omega_k^2\phi(t',\, -\vec{k})\phi(t',\, \vec{k})\right]\right. \\ &\qquad \left. + J(t',\, -\vec{k})\phi(t',\, \vec{k})\right\} \end{aligned} \tag{B.12}$$

Therefore, the total amplitude in presence of the sources reads

$$\begin{aligned} G_J[\phi,\, t;\, \phi_0,\, t_0] = \prod_{\vec{k}} \int \mathcal{D}\phi_k(t) \\ \exp\left\{\frac{i}{(2\pi)^3}\int_{t_0}^{t} \mathrm{d}t' \left[\frac{1}{2}\left(\partial_{t'}^2\phi_{\vec{k}}^2(t') - \omega_{\vec{k}}\phi_{\vec{k}}^2(t')\right) + J_{\vec{k}}(t')\phi_{\vec{k}}(t')\right]\right\} \end{aligned} \tag{B.11}$$

Completing the squares yields

$$G_J[\phi,\, t;\, \phi_0,\, t_0] = G[\phi,\, t;\, \phi_0,\, t_0]$$
$$\prod_{\vec{k}} \exp\left\{ \frac{i}{2(2\pi)^3} \int\limits_{t_0}^{t} \mathrm{d}t' \int\limits_{t_0}^{t} \mathrm{d}\bar{t} \left[J(t',\, -\vec{k}) g(t',\, \bar{t}) J(\bar{t},\, \vec{k}) \right] \right\} \tag{B.11}$$

Here, again, $g(t',\, \bar{t})$ is the Green's function of an harmonic oscillator,

$$g(t',\, \bar{t}) = \begin{cases} \dfrac{\sin[\omega_k(t-t')]\sin[\omega_k(\bar{t}-t_0)]}{\omega_k \sin[\omega_k(t-t_0)]} & \text{for} \quad t' \le \bar{t} \\ \dfrac{\sin[\omega_k(t-\bar{t})]\sin[\omega_k(t'-t_0)]}{\omega_k \sin[\omega_k(t-t_0)]} & \text{for} \quad \bar{t} \le t' \,. \end{cases} \tag{B.12}$$

In a scattering experiment, as outlined before, two particles that are essentially free are collided and interact in a very tiny region. Then, the kinematics of the outgoing particles is measured, such that both the incoming and outgoing particles can be considered as “asymptotic”, i.e. as represented by plane waves. In this situation, one is interested in the limit $t \to +\infty$ and $t_0 \to -\infty$. Only the vacuum contributions to G_J will survive in this limits, therefore $G_J(\phi,\, +\infty,\, \phi_0,\, -\infty)$ is called the vacuum-to-vacuum amplitude in the presence of the source J, conventionally denoted by $Z[J]$. Taking this limit explicitly, i.e. considering $t = -t_0 = T \to \infty$one finds for the $t' \le \bar{t}$-piece of the Green's function $g(\bar{t},\, t')$

$$\begin{aligned} g(\bar{t},\, t') &\overset{t' \le \bar{t}}{\equiv} \frac{\sin[\omega_k(t-\bar{t})]\sin[\omega_k(t'-t_0)]}{\omega_k \sin[\omega(t-t_0)]} \\ &= \frac{1}{\omega_k \sin(2\omega_k T)} [\sin(\omega_k T)\cos(\omega_k \bar{t}) - \sin(\omega_k \bar{t})\cos(\omega_k T)] \\ &\qquad [\sin(\omega_k t')\cos(\omega_k T) - \sin(\omega_k T)\cos(\omega_k t')] \\ &= \frac{1}{2\omega_k} \Bigg[\sin(\omega_k t')\cos(\omega_k \bar{t}) - \sin(\omega_k \bar{t})\cos(\omega_k t') \\ &\qquad + \cos(\omega_k \bar{t})\cos(\omega_k t') \frac{\sin(\omega_k T)}{\cos(\omega_k T)} - \sin(\omega_k \bar{t})\sin(\omega_k t') \frac{\cos(\omega_k T)}{\sin(\omega_k T)} \Bigg] \\ &\overset{iT \to \infty}{\longrightarrow} \frac{1}{2\omega_k} [\sin[\omega_k(\bar{t}-t')] - i\cos[\omega_k(\bar{t}-t')]] = \frac{i}{2\omega_k} e^{-i\omega_k(\bar{t}-t')} \,. \end{aligned} \tag{B.8}$$

The other piece of the Green's function, i.e for $\bar{t} \le t'$ then is given by

$$g(\bar{t},\, t') \overset{t' \ge \bar{t}}{\equiv} \frac{i}{2\omega_k} e^{i\omega_k(\bar{t}-t')} \,. \tag{B.9}$$

Taking together, therefore the Green's function reads

$$g(\bar{t},\, t') = \frac{i}{2\omega_k}\left[e^{-i\omega_k(\bar{t}-t')}\theta(\bar{t}-t') + e^{i\omega_k(\bar{t}-t')}\theta(t'-\bar{t})\right] = -\Delta_F(\vec{k},\, \bar{t}-t')\,. \tag{B.9}$$

This is nothing but the Feynman propagator which has been encountered before. Hence, for free field theories (denoted by the subscript 0), one finds

$$Z_0[J] = Z_0[0]\exp\left\{-\frac{i}{2}\int\frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\int\limits_{-\infty}^{\infty}\mathrm{d}\bar{t}\mathrm{d}t'\,J(-\vec{k},\,\bar{t})\Delta_F(\vec{k},\,\bar{t}-t')J(\vec{k},\,t')\right\}\,, \tag{B.9}$$

or, in position space,

$$Z_0[J] = Z_0[0]\exp\left\{-\frac{i}{2}\int\mathrm{d}^4\bar{x}\mathrm{d}^4x'\,J(\bar{x})\Delta_F(\bar{x}-x')J(x')\right\}\,. \tag{B.10}$$

Here, $Z_0[0]$ denotes the vacuum-to-vacuum amplitude, it contains the limiting value of $G(\phi,\,\infty;\,\phi_0,\,-\infty)$. In general, it is not necessary to know the exact form of $Z_0[0]$, since it cancels out in the construction of n-point functions. Therefore, it won't be discussed any further.
Recall from previous considerations that the boundary conditions of the propagator, i.e. the selection of terms propagating forward and backward in time, was fixed through the $i\epsilon$-prescription (or $i0^+$). It has been added to the path integral formulation of quantum mechanics by the addition of an extra term, which, in field theory, amounts to adding

$$G = \int\mathcal{D}\phi\,\exp\{i\mathcal{S}[\phi]\}\exp\left\{-\epsilon\int\mathrm{d}^4x\phi^2(x)\right\}\,. \tag{B.11}$$

As before, this term is interpreted as a suitable convergence factor. A heuristic argument of how this can be understood is as follows: the path integral is a sum over all paths, contributing as a sum of corresponding phases. These phases emerge due to the factor i in front of the action. This means that the integrand oscillates horribly when changing the path and there is no reason to assume that the full procedure then converges to anything meaningful. Only the addition of the real small convergence factor ensures proper behaviour.

n-point functions

In a previous section it has been shown that the path integral of the forced harmonic oscillator can be used as generating functional for Green's functions. By pretty much the same reasoning, the vacuum-to-vacuum amplitude in the presence of sources is the generating functional for time-ordered products of the fields. It is the desire to calculate such Green's functions and input them into the reduction formula that makes $Z[J]$ the object of central interest in the path integral representation. For the sake of illustration, a number of time-ordered products will be calculated, using the explicit form of $Z_0[J]$ as written in Eq. (B.10).

$$\langle 0 \,| T\left[\phi(x_1),\, \phi(x_2)\right] |\, 0\rangle = \frac{-i\delta}{\delta J(x_1)} \frac{-i\delta}{\delta J(x_2)} \frac{Z_0[j]}{Z_0[0]}\Bigg|_{J=0} = i\Delta_F(x_1 - x_2)\,. \quad \text{(B.12)}$$

$$\begin{aligned}
&\langle 0 \,| T\left[\phi(x_1),\, \phi(x_2),\, \phi(x_3),\, \phi(x_4)\right] |\, 0\rangle \\
&= \frac{-i\delta}{\delta J(x_1)} \frac{-i\delta}{\delta J(x_2)} \frac{-i\delta}{\delta J(x_3)} \frac{-i\delta}{\delta J(x_4)} \frac{Z_0[j]}{Z_0[0]}\Bigg|_{J=0} \\
&= i\Delta_F(x_1 - x_2)\Delta_F(x_3 - x_4) + i\Delta_F(x_1 - x_3)\Delta_F(x_2 - x_4) \\
&\quad + i\Delta_F(x_1 - x_4)\Delta_F(x_2 - x_3)\,. \qquad \text{(B.10)}
\end{aligned}$$

It is clear that without interactions, n-point functions with n higher than two just decompose into sums of products of two-point functions. Diagrammatically this is just a set of disconnected lines linking two points each. Defining a new functional, $\mathcal{W}_0[J]$ through

$$Z_0[J] = Z_0[0]\, \exp\left\{i\mathcal{W}_0[J]\right\} \qquad \text{(B.11)}$$

one immediately understands that

$$\begin{aligned}
\frac{i\delta}{\delta J(x_1)} \mathcal{W}_0[J]\Bigg|_{J=0} &= 0\,, \\
\frac{i\delta}{\delta J(x_1)} \frac{i\delta}{\delta J(x_2)} \mathcal{W}_0[J]\Bigg|_{J=0} &= \Delta_F(x_1 - x_2)\,, \\
\prod_{j=1}^{n>2} \frac{i\delta}{\delta J(x_j)} \mathcal{W}_0[J]\Bigg|_{J=0} &= 0\,, \qquad \text{(B.10)}
\end{aligned}$$

i.e. that this new functional just generates the two-point function and, when introducing interactions later on, this functional generates the connected diagrams only.

B.2.2 Adding interactions:$\lambda\phi^4$ theory

So far the focus of the discussion was on the calculation of vacuum-to-vacuum amplitudes of free field theories. Such amplitudes act, as has been demonstrated, as generating functionals for vacuum expectation values of time-ordered products of free fields. It was also shown that for bosonic fields the path integral representation factorised into infinite products of harmonic oscillators.
For interacting fields, however, it was shown before that vacuum expectation values of time-ordered products can be expanded as a perturbative series, where the first term was the corresponding free field result. The question addressed in this section is how one can construct higher order terms of such a perturbative expansion from the path integral formulation. The objects of interest there are, as mentioned before, time-ordered products of the fields, the Green's functions. From the considerations in the previous chapter **??** it is known that a reduction formula is employed to break n-point functions down into Green's functions with fewer legs and to construct S-matrix elements accordingly. In the following, using the examples of $\lambda\phi^4$ theory, such n-point functions will be constructed from the path integral.

Basic trick

In a first step, define the path integral for interacting fields by replacing the free field action with the interacting field action in the exponential. This leads to

$$Z[J] = \int \mathcal{D}\phi \exp\left\{i\int \mathrm{d}^4x\left[\frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m_0^2\phi^2\right) - \frac{\lambda_0}{4!}\phi^4 + J(x)\phi(x)\right]\right\}. \tag{B.10}$$

It is expected that this is the generating functional for the vacuum expectation values of time-ordered products of the fields. At first sight, this sounds good but it has to be taken with a (the usual) grain of salt: $Z[J]$ cannot be calculated in closed form. Even more, due to the quartic potential it does definitely not resemble anything that can be calculated - only Gaussian integrals can be treated in functional integration. So the question is how to proceed. If λ_0 is "small", one can expect to do a perturbative expansion in

λ_0. Extracting it, one finds

$$
\begin{aligned}
Z[J] &= \int \mathcal{D}\phi \, \exp\left\{-i\int \mathrm{d}^4x \left[\frac{\lambda_0}{4!}\phi^4\right]\right\} \\
&\qquad \exp\left\{i\int \mathrm{d}^4x \left[\frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m_0^2\phi^2\right) + J(x)\phi(x)\right]\right\} . \\
&= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{-i\lambda_0}{4!}\right)^n \int \mathcal{D}\phi \left[\int \mathrm{d}^4x \, \phi^4\right]^n \\
&\qquad \exp\left\{i\int \mathrm{d}^4x \left[\frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m_0^2\phi^2\right) + J(x)\phi(x)\right]\right\} .
\end{aligned}
\tag{B.7}
$$

The key step in simplifying this expression is to notice that fields occuring can be replaced by differentiation w.r.t. the source J. For the quartic term this translates to

$$
\left[\int \mathrm{d}^4x \, \phi^4(x)\right]^n \longrightarrow \left[\int \mathrm{d}^4x \left(\frac{-i\delta}{\delta J(x)}\right)^4\right]^n . \tag{B.8}
$$

These functional derivatives can be brought outside the functional integral,

$$
\begin{aligned}
Z[J] &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{-i\lambda_0}{4!}\right)^n \left[\int \mathrm{d}^4x \left(\frac{-i\delta}{\delta J(x)}\right)^4\right]^n \\
&\qquad \int \mathcal{D}\phi \, \exp\left\{i\int \mathrm{d}^4x \left[\frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m_0^2\phi^2\right) + J(x)\phi(x)\right]\right\} . \\
&= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{-i\lambda_0}{4!}\right)^n \left[\int \mathrm{d}^4x \left(\frac{-i\delta}{\delta J(x)}\right)^4\right]^n Z_0[J] \\
&= Z_0[0]\sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{-i\lambda_0}{4!}\right)^n \left[\int \mathrm{d}^4x \left(\frac{-i\delta}{\delta J(x)}\right)^4\right]^n \\
&\qquad \exp\left\{-\frac{i}{2}\int \mathrm{d}^4x\mathrm{d}^4x' \left[J(x)\Delta_F(x-x')J(x')\right]\right\} ,
\end{aligned}
\tag{B.5}
$$

making the computation of vacuum expectation values of time-ordered products in terms of a perturbative series incredibly simple: just functionally differentiate. In other words,

$$
\langle 0 \,| T\phi(x_1)\phi(x_2)\ldots\phi(x_N)|\, 0\rangle = \frac{-i\delta}{\delta J(x_1)}\frac{-i\delta}{\delta J(x_2)}\cdots\frac{-i\delta}{\delta J(x_N)}\left.\frac{Z[J]}{Z[0]}\right|_{J=0} . \tag{B.6}
$$

Example: Two-point function to first order

To exemplify these findings consider the two-point function. Writing

$$\langle|T\phi(x_1)\phi(x_2)|\,0\rangle = G_2^{(0)}(x_1,\,x_2) + \lambda_0 G_2^{(1)}(x_1,\,x_2) + \lambda_0^2 G_2^{(2)}(x_1,\,x_2) + \ldots \tag{B.6}$$

one has

$$\begin{aligned} G_2^{(0)}(x_1,\,x_2) &= \frac{-i\delta}{\delta J(x_1)}\frac{-i\delta}{\delta J(x_2)}\left.\frac{Z[J]}{Z[0]}\right|_{J=0} \\ &= \frac{-i\delta}{\delta J(x_1)}\frac{-i\delta}{\delta J(x_2)}\exp\left\{-\frac{i}{2}\int \mathrm{d}^4x\mathrm{d}^4x'\,[J(x)\Delta_F(x-x')J(x')]\right\}\Bigg|_{J=0} \\ &= i\Delta_F(x_1-x_2)\,. \end{aligned} \tag{B.5}$$

This is, not surprisingly, the free field result. Going to first order, one finds

$$\begin{aligned} G_2^{(1)}(x_1,\,x_2) &= \frac{-i\delta}{\delta J(x_1)}\frac{-i\delta}{\delta J(x_2)}\left[-i\int \mathrm{d}^4y\left(\frac{-i\delta}{\delta J(y)}\right)^4\right] \\ &\qquad \exp\left\{-\frac{i}{2}\int \mathrm{d}^4x\mathrm{d}^4x'\,[J(x)\Delta_F(x-x')J(x')]\right\}\Bigg|_{J=0} \\ &= -\frac{1}{2}\int \mathrm{d}^4y\Delta_F(x_1-y)\Delta_F(y-y)\Delta_F(y-x_2)\,, \end{aligned} \tag{B.4}$$

which is nothing but the "tadpole" diagram encountered before.

Generating functional for connected diagrams

$Z[J]$, as in the free field case, generates all diagrams, connected and disconnected ones. Similarly to the free field case, one can define another functional $W[J]$ such that it generates the connected diagrams only. In fact the respective definition is identical,

$$Z[J] = Z[0]\,\exp(iW[J])\,. \tag{B.5}$$

In terms of the free field result,

$$W_0[J] = -\frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4x'\,J(x)\Delta_F(x-x')J(x') \tag{B.6}$$

one has for interacting fields

$$\exp(iW[J]) = \frac{1}{Z[0]}\exp\left[-\frac{i\lambda_0}{4!}\int d^4y\left(\frac{-i\delta}{\delta J(y)}\right)^4\right]\exp\left[iW_0[J]\right]. \tag{B.6}$$

B.3 Path integral for the free Dirac field

B.3.1 Functional calculus for Grassmann numbers

Recalling properties of Grassmann numbers

Assume x and η to be an ordinary, commuting, and a Grassmann, anticommuting number, respectively. Then

$$[x,\, x] = \{\eta,\, \eta\} = 0\,, \tag{B.7}$$

and, in full analogy,

$$\left[\frac{\mathrm{d}}{\mathrm{d}x},\, x\right] = \left\{\frac{\mathrm{d}}{\mathrm{d}\eta},\, \eta\right\} = 1\,. \tag{B.8}$$

For any function $f(\eta)$, the power series expansion in η must terminate in the linear term, i.e.

$$f(\eta) = a_0 + a_1\eta\,. \tag{B.9}$$

Since $\mathrm{d}^2 f/\mathrm{d}\eta^2 = 0$,

$$\left\{\frac{\mathrm{d}}{\mathrm{d}\eta},\, \frac{\mathrm{d}}{\mathrm{d}\eta}\right\} = 0\,. \tag{B.10}$$

So far, nothing really shocking has happened. But now, consider this. Assume that the a_i in the series expansion of f are ordinary numbers. Then

$$\frac{df}{\mathrm{d}\eta} = a_1\,. \tag{B.11}$$

However, if the a's are anticommuting numbers, the differential operator has to be swapped with a_1 before it can act on η. In such a case,

$$\frac{df}{\mathrm{d}\eta} = -a_1\,. \tag{B.12}$$

In other words: some care has to be spent to ensure the correct statistics in differentiation with respect to Grassmann numbers. For such numbers one has to fix integration through definition. To this end, one defines

$$\begin{aligned}\int \mathrm{d}\eta \ &:= \ 0 \\ \int \mathrm{d}\eta\, \eta \ &:= \ 1\,, \end{aligned} \tag{B.12}$$

chosen such that the operation $\int \mathrm{d}\eta$ is translationally invariant and linear. It has to be stressed that integration is not the inverse operation of differentiation, in fact, integration acts just like differentiation. Again, the order of the operations and variables is important. To exemplify this consider

$$\int \mathrm{d}\eta_1 \mathrm{d}\eta_2\, \eta_1\eta_2 = -\int \mathrm{d}\eta_1 \mathrm{d}\eta_2\, \eta_2\eta_1 = -\int \mathrm{d}\eta_2 \mathrm{d}\eta_1\, \eta_1\eta_2 = -1\,. \tag{B.13}$$

Functional calculus

To proceed, switch to the infinite-dimensional limit, i.e. to functions $\xi(x)$ and $\eta(x)$ in the space of ordinary function or in the space of Grassmann-valued functions. There, the analogy of bosonic and fermionic functions, differing only by the usage of commutators and anti-commutators, respectively, is further employed to define the corresponding functional derivatives as

$$\left[\frac{\delta}{\delta\xi(x)},\, \xi(y)\right] = \left\{\frac{\delta}{\delta\eta(x)},\, \eta(y)\right\} = \delta(x-y)\,. \tag{B.14}$$

The functional integrals are given by

$$\begin{aligned}\mathcal{D}\xi \ &= \ \prod_x \mathrm{d}\xi(x) \\ \mathcal{D}\eta \ &= \ \prod_x \mathrm{d}\eta(x)\,, \end{aligned} \tag{B.14}$$

but for the fermionic variables the ordering of the $\mathrm{d}\eta$ is assumed to be opposite to the ordering in the functional integral. This will ensure that no extra minus signs will show up when the integral is evaluated. Similarly, the δ-functionals are defined such that

$$\int \mathcal{D}\xi \delta[\xi] = \int \mathcal{D}\eta \delta[\eta] = 1 \tag{B.15}$$

leading to the following representations of the δ-functional

$$\begin{aligned}\delta[\xi - a] &= \prod_x \delta(\xi(x) - a(x)) \\ \delta[\eta - \sigma] &= \prod_x \delta(\eta(x) - \sigma(x))\,. \end{aligned} \tag{B.15}$$

Then,

$$\begin{aligned}\int \mathcal{D}\xi F(\xi)\delta[\xi - a] &= \int \prod_x \mathrm{d}\xi(x) F[\xi] \prod_y \delta(\xi(y) - a(y)) \\ &= \prod_x \int \mathrm{d}\xi(x)\delta(\xi(x) - a(x)) = F[a] \\ \int \mathcal{D}\eta F(\eta)\delta[\eta - \sigma] &= \int \prod_x \mathrm{d}\eta(x) F[\eta] \prod_y \delta(\eta(y) - \sigma(y)) \\ &= \prod_x \int \mathrm{d}\eta(x)\delta(\eta(x) - \sigma(x))F[\eta] = F[\sigma]\,. \end{aligned} \tag{B.12}$$

Finally, consider the Gaussian integral, first for bosonic variables. From

$$\begin{aligned}\int \mathcal{D}\xi \exp\left[-\int \mathrm{d}x\xi^2(x)\right] &\longrightarrow \int \prod_x \mathrm{d}\xi(x) \exp\left[-\sum_x \xi^2(x)\right] \\ &= \int \prod_x \mathrm{d}\xi(x) \prod_x \exp\left[-\xi^2(x)\right] \\ &= \prod_x \int \mathrm{d}\xi(x) \exp\left[-\xi^2(x)\right] = \prod_x \sqrt{\pi} \end{aligned} \tag{B.10}$$

it follows

$$\begin{aligned}\int \mathcal{D}\xi \exp\left[-\int \mathrm{d}x\alpha\xi^2(x)\right] &= \prod_x \sqrt{\frac{\pi}{\alpha}}\,, \\ \int \mathcal{D}\xi \exp\left[-\int \mathrm{d}x\alpha(x)\xi^2(x)\right] &= \prod_x \sqrt{\frac{\pi}{\alpha(x)}}\,. \end{aligned} \tag{B.10}$$

Extend the consideration above to a case, where α is a function that depends on two variables, $\alpha(x) \to \alpha(x,\, y)$. This allows to evaluate

$$\int \mathcal{D}\xi \exp\left[-\int \mathrm{d}x\mathrm{d}y \xi(x)\alpha(x,\, y)\xi(y)\right] = \prod_x \sqrt{\frac{\pi}{\alpha(x,\, y)}} = \frac{1}{\sqrt{\det\left[\alpha(x,\, y)\right]}} \prod_x \sqrt{\pi}\,. \tag{B.10}$$

The result can be easily justified by considering the finite-dimensional case, for instance on a lattice, and taking the continuum limit. To extend this even further, add a source term to the Gaussian. By completing the squares one finds

$$\int \mathcal{D}\xi \exp\left[-\int \mathrm{d}x\mathrm{d}y \xi(x)\alpha(x,\, y)\xi(y) - \int \mathrm{d}x J(x)\xi(x)\right] = \frac{\prod_x \sqrt{\pi}}{\sqrt{\det\left[\alpha(x,\, y)\right]}} \exp\left[\frac{1}{4}\int \mathrm{d}x\mathrm{d}y J(x)\alpha^{-1}(x,\, y)J(y)\right]\,, \tag{B.10}$$

where $\alpha^{-1}(x,\, y)$ denotes the inverse operator matrix of $\alpha(x,\, y)$, defined through

$$\int \mathrm{d}z \alpha^{-1}(x,\, z)\alpha(z,\, y) = \delta(x-y)\,. \tag{B.11}$$

The source term in the Gaussian allows to compute the functional integral of any moment of the Gaussian through differentiation w.r.t. the sources. Especially,

$$\begin{aligned} &\int \mathcal{D}\xi \xi(x_1)\xi(x_2)\ldots\xi(x_n) \exp\left[-\int \mathrm{d}x\mathrm{d}y \xi(x)\alpha(x,\, y)\xi(y)\right] \\ &= \int \mathcal{D}\xi \frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_2)}\cdots\frac{\delta}{\delta J(x_n)} \\ &\quad \exp\left[-\int \mathrm{d}x\mathrm{d}y \xi(x)\alpha(x,\, y)\xi(y) - \int \mathrm{d}x J(x)\xi(x)\right]\Bigg|_{J=0}\,. \end{aligned} \tag{B.10}$$

For Grassmann numbers the Gaussian integral reads

$$\int \mathcal{D}\eta \exp\left[-\int \mathrm{d}x\mathrm{d}y \eta(x)\alpha(x,\, y)\eta(y)\right]\,, \tag{B.11}$$

where α should be antisymmetric in its arguments, $\alpha(x,\, y) = \alpha(y,\, x)$ in order to yield a non-vanishing result. To gain some insight into how such integrals are evaluated, consider the finite-dimensional case first. There, due to the antisymmetry of α

$$\begin{aligned}\int \mathrm{d}\eta_1 \mathrm{d}\eta_2\, e^{-\eta_1\alpha_{12}\eta_2 - \eta_2\alpha_{21}\eta_1} &= \int \mathrm{d}\eta_1 \mathrm{d}\eta_2\, e^{-2\eta_1\alpha_{12}\eta_2} \\ = \int \mathrm{d}\eta_1 \mathrm{d}\eta_2\, (1 - 2\eta_1\alpha_{12}\eta_2) &= 2\alpha_{12}\,. \end{aligned} \tag{B.10}$$

The series terminates due to $\eta_1^2 = \eta_2^2 = 0$. For an antisymmetric 2×2 matrix,

$$\alpha_{12}^2 = -\alpha_{12}\alpha_{21} = \det \alpha \tag{B.11}$$

and therefore

$$\int \mathrm{d}\eta_1 \mathrm{d}\eta_2\, e^{-2\eta_1\alpha_{12}\eta_2} = 2\sqrt{\det\,[\alpha]}\,, \tag{B.12}$$

just opposite to the bosonic case. It is easy to check that for higher dimensions

$$\int \left[\prod_{j=1}^{N} \mathrm{d}\eta_j\right] \exp\left[-2\sum_{j=1}^{N-1}\sum_{i=j+1}^{N} \eta_j\alpha_{ji}\eta_i\right] = 2^{\frac{N}{2}}\sqrt{\det\,[\alpha]}\,. \tag{B.13}$$

This can be simply generalised to infinite-dimensional integrals, yielding

$$\int \mathcal{D}\eta \exp\left[-\int \mathrm{d}x\mathrm{d}y\, \eta(x)\alpha(x,\, y)\eta(y)\right] = \left(\prod_x \sqrt{2}\right)\sqrt{\det\,[\alpha]}\,. \tag{B.14}$$

As a last extension, consider complex fields. For commuting, ordinary numbers this generalisation is nothing but straightforward. To underline this consider $\xi(x) = [\xi_1(x) + i\xi_2(x)]/\sqrt{2}$ to be such an ordinary complex function with ξ_1 and ξ_2 real-valued. Switching from integration variables ξ^* and ξ to ξ_1 and ξ_2, the Gaussian integral

$$\int \mathcal{D}\xi^*\mathcal{D}\xi\, \exp\left[-\int \mathrm{d}x\mathrm{d}y\, \xi * (x)a(x,\, y)\xi(y)\right] \tag{B.15}$$

with symmetric function $a(x,\, y) = a(y,\, x)$ transforms into two copies of Eq. (B.11). Thus

$$\int \mathcal{D}\xi^* \mathcal{D}\xi \, \exp\left[-\int \mathrm{d}x\mathrm{d}y \, \xi * (x) a(x,\, y) \xi(y)\right] = \frac{(i\pi)^\infty}{\det[a]}\,, \tag{B.16}$$

where the numerator hindsights the infinite normalisation. The Gaussian integral over complex Grassmann numbers is more complicated. To understand this, define the complex anticommuting numbers η

$$\eta = \frac{\chi_1 + i\chi_2}{\sqrt{2}} \quad \text{and} \quad \eta^* = \frac{\chi_1 - i\chi_2}{\sqrt{2}}\,, \tag{B.17}$$

where $\chi_{1,\,2}$ are real-valued Grassmann numbers. Demanding that integration is a linear and translationally invariant operation one has

$$\begin{aligned} 1 &= \int \mathrm{d}\eta\eta = \int \mathrm{d}\eta^* \eta^* \\ 0 &= \int \mathrm{d}\eta = \int \mathrm{d}\eta^* = \int \mathrm{d}\eta\eta^* = \int \mathrm{d}\eta^* \eta\,, \end{aligned} \tag{B.17}$$

translating into

$$\mathrm{d}\eta = \frac{\mathrm{d}\chi_1 - i\mathrm{d}\chi_2}{\sqrt{2}}\,, \quad \mathrm{d}\eta^* = \frac{\mathrm{d}\chi_1 + i\mathrm{d}\chi_2}{\sqrt{2}} \quad \text{and} \quad \mathrm{d}\eta\mathrm{d}\eta^* = -i\mathrm{d}\chi_1\mathrm{d}\chi_2\,. \tag{B.18}$$

Using these integration rules and the expansion of the exponential one easily finds

$$\int \mathrm{d}\eta^*\mathrm{d}\eta \, \exp\left[-\eta^* a \eta\right] = \int \mathrm{d}\eta^*\mathrm{d}\eta \, \left[1 - \eta^* a \eta\right] = a = \det[a]\,, \tag{B.19}$$

where, this time, a is an ordinary commuting number. Adding one more dimension,

$$\begin{aligned} \int \mathrm{d}\eta_1^*\mathrm{d}\eta_2^*\mathrm{d}\eta_1\mathrm{d}\eta_2 \, \exp\left[-\eta_i^* a_{ij} \eta_j\right] &= \frac{1}{2!} \int \mathrm{d}\eta_1^*\mathrm{d}\eta_2^*\mathrm{d}\eta_1\mathrm{d}\eta_2 \, \left[\eta_i^* a_{ij} \eta_j\right]^2 \\ = \int \mathrm{d}\eta_1^*\mathrm{d}\eta_2^*\mathrm{d}\eta_1\mathrm{d}\eta_2 \, &\left[\eta_2^*\eta_1^*\eta_2\eta_1 \left(a_{11}a_{22} - a_{21}a_{12}\right)\right] = -\det[a]\,. \end{aligned} \tag{B.19}$$

This looks very good, apart from the sign-difference in front of the determinant. To resolve this one can add yet another integration variable, leading to a 3 instead of a 2×2-matrix a. Then one finds

$$\begin{aligned}
&\int \mathrm{d}\eta_1^*\mathrm{d}\eta_2^*\mathrm{d}\eta_3^*\mathrm{d}\eta_1\mathrm{d}\eta_2\mathrm{d}\eta_3 \, \exp\left[-\eta_i^* a_{ij}\eta_j\right] \\
&= -\frac{1}{3!}\int\int \mathrm{d}\eta_1^*\mathrm{d}\eta_2^*\mathrm{d}\eta_3^*\mathrm{d}\eta_1\mathrm{d}\eta_2\mathrm{d}\eta_3 \, \left[\eta_i^* a_{ij}\eta_j\right]^3 = -\det[a]\,. \qquad \text{(B.19)}
\end{aligned}$$

For the general, N-dimensional case the result reads

$$\int \mathrm{d}\eta_1^* \dots \mathrm{d}\eta_N^* \mathrm{d}\eta_1 \dots \mathrm{d}\eta_N \, \exp\left[-\eta_i^* a_{ij}\eta_j\right] = (-1)^{\frac{N(N+3)}{2}} \det[a]\,. \qquad \text{(B.20)}$$

Due to this oscillating sign, the limit $N \to \infty$ and the corresponding functional integral are ill-defined. In principle, one could try to bury this sign in some overall normalisation, but this might not always lead to unambiguous definitions of objects derived from path integrals. To get all signs correct there, one has to keep track of all factors -1, possibly an unsurmountable task. The resolution to this apparent problem lies in the fact that the overall sign is defined by the ordering of the integration variables w.r.t. their ordering in the exponential. Since the ordering in the exponential seems to be somewhat more fixed one could therefore try to fiddle around with the ordering of the integration parameters hoping to gain a reasonable result in the end. It can be shown that, in order to fix the overall sign to be positive, the ordering of choice in the integration variables is $\mathrm{d}\eta_1^*\mathrm{d}\eta_1\mathrm{d}\eta_2^*\mathrm{d}\eta_2 \dots \mathrm{d}\eta_N^*\mathrm{d}\eta_N$. Defining the functional integration in the same fashion, i.e. defining

$$\mathcal{D}\eta^*\mathcal{D}\eta := \prod_x \left[\mathrm{d}\eta^*(x)\mathrm{d}\eta(x)\right] \qquad \text{(B.21)}$$

one has, for the functional integral,

$$\int \mathcal{D}\eta^*\mathcal{D}\eta \exp\left[-\int \mathrm{d}x\mathrm{d}y \, \eta^*(x)a(x,\,y)\eta(y)\right] = \det[a]\,. \qquad \text{(B.22)}$$

In fact, applying the same trick, just with reversed sign, leads to

$$\int \mathcal{D}\eta^*\mathcal{D}\eta \exp\left[+\int \mathrm{d}x\mathrm{d}y \, \eta^*(x)a(x,\,y)\eta(y)\right] = \det[a]\,, \qquad \text{(B.23)}$$

for the hyperbolic fermionic integral. To gain this result, however, one has to define

$$\mathcal{D}\eta^*\mathcal{D}\eta := \prod_x \left[\mathrm{d}\eta(x)\mathrm{d}\eta^*(x)\right] \qquad \text{(B.24)}$$

in contrast to the ordering before.

B.3.2 Transition amplitude for spinor fields

For the construction of the transition amplitude for spinor fields one cannot rely entirely on the experiences gained so far for bosonic fields. First of all, as outlined above, integration proceeds in a different fashion; on top of that, the interpretation through the harmonic oscillator path integral won't work. Nevertheless, one might start by simply copying the structures used for the bosonic case and see what happens. A good guess for the path integral for fermionic fields can be obtained by taking the exponential of the free spinor action, as described by the Dirac equation. Thus, one has for field configurations ψ and ψ_0 at times t and t_0, respectively

$$G[\psi,\, t;\, \psi_0,\, t_0] = \int \mathcal{D}\psi^\dagger \mathcal{D}\psi \exp\left[i \int\limits_{t_0}^{t} \mathrm{d}t' \int \mathrm{d}^3\vec{x}\, \bar{\psi}(t',\, \vec{x})\, (i\partial\!\!\!/ - m)\, \psi(t',\, \vec{x})\right] \tag{B.25}$$

where the ψ are, as expected, four-component (Dirac) spinors, out of which each component represents an anti-commuting variable. Obviously this makes up for the big difference between the bosonic examples encountered so far and this case: the integration is over the space of anticommuting functions. However, in a first step, it proves to be useful to decouple the spin information by switching to the momentum representation, Eq. (**??**),

$$\psi(t,\, \vec{x}) = \sum_{r=1}^{4} \int \frac{\mathrm{d}^3\vec{p}}{\sqrt{(2\pi)^3 (2E_p)}} b(t,\, \vec{p},\, r) w(\vec{p},\, r) e^{i\vec{p}\vec{x}}\,, \tag{B.26}$$

where the w denote the momentum spinors u and v, cf. Eqs. (**??**) and (**??**) and $E_p^2 = \vec{p}^2 + m^2$. Here, also the spin-functions b collectively stand for both functions b and d from Eq. (**??**). Plugging this expansion into the path integral and making use of the spinor identities, one obtains

$$\begin{aligned}&\int \mathrm{d}^3\vec{x}\, \bar{\psi}(t',\, \vec{x})\, (i\partial\!\!\!/ - m)\, \psi(t',\, \vec{x}) \\ &= \sum_{r=1}^{4} \int \mathrm{d}^3\vec{p}\, b^*(t',\, \vec{p},\, r)\, (i\partial_{t'} - \delta_r E_p)\, b(t',\, \vec{p},\, r)\,,\end{aligned} \tag{B.26}$$

where the sign factor $\delta_r = 1$ for $r = 1,\, 2$ and $\delta_r = -1$ for $r = 3,\, 4$. Under this transformation, the path integral is cast into a manageable form, i.e.

factorised into an infinite product of single-particle fermionic path integrals, one for each spin component r. One can therefore write

$$G[b_f,\, t;\, b_i,\, t_0] = \prod_{r=1}^{4}\prod_{\vec{p}}\int \mathcal{D}b^*\mathcal{D}b\,\exp\left[i\int_{t_0}^{t} \mathrm{d}t' b^*(t',\,\vec{p},\,r)\,(i\partial_{t'} - \delta_r E_p)\,b(t',\,\vec{p},\,r)\right]. \tag{B.25}$$

Concentrating on one mode $\vec{p}$ for positive energy E_p only and denoting it by η one has

$$G[\eta,\, t;\, \eta_0,\, t_0] = \int \mathcal{D}\eta^*(t')\mathcal{D}\eta(t')\,\exp\left[i\int_{t_0}^{t} \mathrm{d}t'\eta^*(t')\,(i\partial_{t'} - E_p)\,\eta(t')\right] \tag{B.26}$$

Using the definitions above [1]