Particle Theory
Relativistic Quantum Mechanics
Revision Lecture

Frank Krauss

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In the following slides I'll summarise things you should know, important equations and give you links to example problems.

For the problems I use the format [Chapter.ProblemNumber] and for equations I use (EqnNumber).

Also, I show some example exam questions as a combination of short questions and parts of long questions of prior exams (in yellow boxes) with example solutions (in green boxes). Use them as an indicator of difficulty and expectations of what you are supposed to be able to do.

Classical Field Theory I

- how to derive Euler-Lagrange E.o.M. (89) [3.1, 3.3, 3.4, 3.5, 3.6],

$$0 \equiv \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi}$$

  for any field or field component ϕ;
- how to deal with Lorentz-indices (19, 20) [3.3, 3.4, 3.5], e.g.

$$\frac{\partial(\partial_\mu \phi)}{\partial(\partial^\nu \phi)} = g_{\mu\nu} \,, \quad \frac{\partial(\partial_\mu \phi)}{\partial(\partial_\nu \phi)} = g_\mu^\nu = \delta_\mu^\nu \,, \quad \frac{\partial(\partial_\mu A_\nu)}{\partial(\partial^\rho A^\sigma)} = g_{\mu\rho} g_{\nu\sigma} \,;$$

- how to check for conserved currents and their link to conserved charges [3.6, 3.8]

Classical Field Theory II

Lagrangians for

- real scalar field (96), its E.o.M., the Klein-Gordon Equation (91), and its solutions (93);
- complex scalars (104), E.o.M. (105), and the conserved current (113) and charge (115);
- electromagnetism in terms of fields (125) and of the vector potential/field strength tensors (133)

and how to arrive at Hamiltonians

Classical Field Theory (Typical Exam Question)

Short question 1a) (2022):

Consider the Lagrangian for a classical free complex scalar field ϕ,

$$\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) - m^2 \phi^* \phi \,,$$

and derive and solve the equations of motion for ϕ and ϕ^*. [3 marks]

Calculate the quantity

$$Q = iq \int \mathrm{d}^3x \left(\phi^* \dot{\phi} - \dot{\phi}^* \phi \right) \,.$$

[3 marks]

Classical Field Theory (Typical Exam Question)

Equations of motion:

$$0 = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^{(*)})} - \frac{\partial \mathcal{L}}{\partial \phi^{(*)}} = (\Box + m^2)\phi^{(*)} \qquad \text{[Application, 2 marks]}$$

with solutions

$$\begin{aligned} \phi(x) &= \int \frac{\mathrm{d}^3 k}{(2\pi)^3 2k_0} \left[a(k)e^{-ik\cdot x} + b^*(k)e^{ik\cdot x}\right] \\ \phi^*(x) &= \int \frac{\mathrm{d}^3 k}{(2\pi)^3 2k_0} \left[b(k)e^{-ik\cdot x} + a^*(k)e^{ik\cdot x}\right] \qquad \text{[Application, 1 mark]} \end{aligned}$$

$$\begin{aligned} Q = iq \int \mathrm{d}^3 x \frac{\mathrm{d}^3 k}{(2\pi)^3 2k_0} \frac{\mathrm{d}^3 l}{(2\pi)^3 2l_0} & \\ \left\{ \left[b(k)e^{-ik\cdot x} + a^*(k)e^{ik\cdot x}\right](-il_0)\left[a(l)e^{-il\cdot x} - b^*(l)e^{il\cdot x}\right] \ldots \right\} & \\ = -q \int \frac{\mathrm{d}^3 k}{(2\pi)^3 2k_0} \frac{\mathrm{d}^3 l}{(2\pi)^3 2l_0} (2\pi)^3 \left\{ b(k)a(l)(k_0 - l_0)e^{-i(k+l)_0 x_0} \delta^3(\underline{k} + \underline{l}) \ldots \right\} & \\ = q \int \frac{\mathrm{d}^3 k}{(2\pi)^3 2k_0} \left[a^*(k)a(k) - b^*(k)b(k)\right] \qquad \text{[Application, 3 marks]} & \end{aligned}$$

Classical Field Theory (Typical Exam Question)

Short question 1d) (2021):

Under dilatations, all coordinates and fields are transformed as

$$x \longrightarrow x' = e^{-\rho} x\,, \quad \phi(x) \longrightarrow \phi'(x') = e^{\rho}\phi(x)\,,$$

where ρ is a constant parameter. Show that the (transformed) derivative of the transformed field is given by

$$\partial'_\mu \phi'(x') = \frac{\partial \phi'(x')}{\partial x'^\mu} = e^{2\rho}\partial_\mu \phi(x)\,.$$

Use that the space-time volume element transforms as

$$\mathrm{d}^4x \longrightarrow \mathrm{d}^4x' = e^{-4\rho}\mathrm{d}^4x$$

and calculate the change of action for the Klein-Gordon Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{m^2}{2}\phi^2$$

of a real scalar field in the limit $\rho \to 0$. Comment on the conditions for the existence of a conserved current under dilatations. [5 marks]

Classical Field Theory (Typical Exam Question)

Change of field derivative:

$$\frac{\partial\phi'(x')}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial\phi'(x')}{\partial x^\nu} = \frac{\partial x^\nu}{e^{-\rho}\partial x^\mu}\frac{e^\rho\partial\phi(x)}{\partial x^\nu} = e^{2\rho}\phi(x)$$

[Comprehension, 2 marks]
Change of action:

$$\begin{aligned}
\mathcal{S}' - \mathcal{S} &= \int \mathrm{d}^4x'\mathcal{L}' - \int \mathrm{d}^4x\mathcal{L} \\
&= \int \mathrm{d}^4x' \left\{\frac{1}{2}[\partial'_\mu\phi'(x')][\partial'^\mu\phi'(x')] - \frac{m^2}{2}\phi'^2(x')\right\} \\
&\quad - \int \mathrm{d}^4x \left\{\frac{1}{2}[\partial_\mu\phi(x)][\partial^\mu\phi(x)] - \frac{m^2}{2}\phi^2(x)\right\} \\
&= \int \mathrm{d}^4x \left[\frac{e^{-4\rho}e^{4\rho}-1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{(e^{-4\rho}e^{2\rho}-1)m^2}{2}\phi^2\right] \\
&= \frac{(1-e^{-2\rho})m^2}{2}\int \mathrm{d}^4x\phi^2(x) \quad \stackrel{\rho\to 0}{\longrightarrow} \quad \frac{2\rho m^2}{2}\int \mathrm{d}^4x\phi^2(x)
\end{aligned}$$

[Application, 2 marks]
For a massless theory, *i.e.* $m = 0$, $\delta S = 0$, and a conserved current would exist. In contrast, if $m \neq 0$, a massive theory would not have a conserved current under dilatations. [Comprehension, 1 mark]

Second Quantisation I

steps from Figure 1, p62 of lecture notes, in particular [4.3]

- how to obtain conjugate momenta, for example (139, 165), $\pi = \partial\mathcal{L}/\partial\dot{\phi}$;
- the form of the equal-time commutators (141, 167),

$$\left[\hat{\phi}(t,\, \underline{x}), \hat{\pi}(t,\, \underline{y})\right] = i\delta^3(\underline{x} - \underline{y})\,;$$

- the expansion of the field operators in plane waves and creation/annihilation operators (142, 168),

$$\hat{\phi} = \int \frac{\mathrm{d}^3 k}{(2\pi)^3 2k_0} \left[\hat{a}(\underline{k})e^{-ik\cdot x} + \hat{a}^\dagger(\underline{k})e^{ik\cdot x}\right]\;;$$

- and their commutators (149, 171)

$$\left[\hat{a}(\underline{k}),\, \hat{a}^\dagger(\underline{q})\right] \;=\; 2k_0(2\pi)^3\delta^3(\underline{k} - \underline{q})\,.$$

Second Quantisation II

some algebra to evaluate tricky expressions

- four vectors and energy-momentum relation: $m^2 = p^2 = p_0^2 - \underline{p}^3$

  (and therefore quanta with same mass and same $\pm$ momentum have same energy)

- creation/annihilation operators acting on vacuum (152):

$$\hat{a}(\underline{k})|0\rangle = \langle 0|\hat{a}^\dagger(\underline{k}) = 0\ ;$$

- normal-ordering of operators

$$:\hat{a}\hat{a}^\dagger: = :\hat{a}^\dagger\hat{a}: = \hat{a}^\dagger\hat{a}\ ;$$

- representation of δ-function in one/three dimensions (5):

$$\int \mathrm{d}x e^{-ix(k-q)} = (2\pi)\delta(k-q)\ , \quad \int \mathrm{d}^3x e^{-i\underline{x}\cdot(\underline{k}-\underline{q})} = (2\pi)^3\delta^3(\underline{k}-\underline{q})\ ;$$

Second Quantisation III

some typical objects and calculations

- construct states with one or more quanta and calculate their properties [4.1, 4.2]
- express (normal-ordered) Hamilton, momentum and charge operators through creation and annihilation operators, for example (151, 162, 172, 180) [4.3]

$$:\hat{H}:=\int \frac{\mathrm{d}^3 k}{(2\pi)^3 2k_0}\left[k_0 \hat{a}^\dagger(\underline{k})\hat{a}(\underline{k})\right]$$

  for a free scalar field;
- calculate commutators of these operators among themselves and with field operators [4.3, 4.4, 4.5, 4.6, 4.7]

Second Quantisation (Typical Exam Question)

Short question 1c) (2021):

Calculate $\langle \underline{p}|\hat{\phi}(x)|0\rangle$, with $\hat{\phi}$ a quantised scalar field operator, and show that it satisfies the Klein-Gordon equation. [5 marks]

To show that this is the case, we need to show that

$$
\begin{aligned}
0 = (\partial^\mu\partial_\mu + m^2)\langle \underline{p}|\hat{\phi}(x)|0\rangle \\
&= (\partial^\mu\partial_\mu + m^2)\int\frac{\mathrm{d}^3k}{(2\pi)^3 2k_0}\left\langle 0\left|\hat{a}(\underline{p})\left[\hat{a}(\underline{k})e^{-ik\cdot x} + \hat{a}^\dagger(\underline{k})e^{ik\cdot x}\right]\right|0\right\rangle \\
&= (\partial^\mu\partial_\mu + m^2)\int\frac{\mathrm{d}^3k}{(2\pi)^3 2k_0}\left\langle 0\left|\hat{a}(\underline{p})\hat{a}^\dagger(\underline{k})e^{ik\cdot x}\right|0\right\rangle \\
&= (\partial^\mu\partial_\mu + m^2)\int\frac{\mathrm{d}^3k}{(2\pi)^3 2k_0}\left\langle 0\left|\left[\hat{a}(\underline{p}),\ \hat{a}^\dagger(\underline{k})\right]e^{ik\cdot x}\right|0\right\rangle \\
&= (\partial^\mu\partial_\mu + m^2)\int\frac{\mathrm{d}^3k}{(2\pi)^3 2k_0}\left\langle 0\left|(2\pi)^3 2k_0\delta^3(\underline{p}-\underline{k})e^{ik\cdot x}\right|0\right\rangle \\
&= (\partial^\mu\partial_\mu + m^2)e^{ip\cdot x}\langle 0|0\rangle = (-p^2 + m^2)e^{ip\cdot x}\langle 0|0\rangle = 0
\end{aligned}
$$

[Application, 4 marks] because of the relativistic energy-momentum relation. [Application, 1 mark]

Second Quantisation (Typical Exam Question)

Long question 2b) (2019):

Derive the conjugate momenta $\pi(x)$ and $\pi^*(x)$ of the fields and show that the Hamiltonian is given by

$$H = \int d^3x \mathcal{H} = \int d^3x \left[\pi^* \pi + (\underline{\nabla}\phi^*)(\underline{\nabla}\phi) + m^2 \phi^* \phi\right] \, .$$

[2 marks]

Application: Conjugate momenta:

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}^* \text{ and } \quad \pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}^*$$

[1 mark]

Hamiltonian:

$$\begin{aligned} \mathcal{H} &= \pi\dot{\phi} + \pi^*\dot{\phi}^* - \mathcal{L} \\ &= 2\dot{\phi}^*\dot{\phi} - \dot{\phi}^*\dot{\phi} + \underline{\nabla}\phi^* \cdot \underline{\nabla}\phi + m^2\phi^*\phi \\ &= \dot{\phi}^*\dot{\phi} + \underline{\nabla}\phi^* \cdot \underline{\nabla}\phi + m^2\phi^*\phi \end{aligned}$$

[1 mark]

Second Quantisation (Typical Exam Question)

Short question 1d) (2020):

The (normal-ordered) Hamilton and charge operators for a complex scalar theory are given by

$$
\begin{aligned}
:\hat{H}: &= \int \frac{\mathrm{d}^3 k}{(2\pi)^3 2k_0}\, k_0 \left[a^\dagger(\underline{k})a(\underline{k}) + b^\dagger(\underline{k})b(\underline{k})\right] \\
:\hat{Q}: &= \int \frac{\mathrm{d}^3 k}{(2\pi)^3 2k_0} \left[a^\dagger(\underline{k})a(\underline{k}) - b^\dagger(\underline{k})b(\underline{k})\right] .
\end{aligned}
$$

Calculate the comutator $[:\hat{H}:,\ :\hat{Q}:]$ ad interpret the result. [5 marks]

Second Quantisation (Typical Exam Question)

$[:\hat{H}:,\ :\hat{Q}:]$

$$
\begin{aligned}
&= \int \frac{\mathrm{d}^3 k k_0}{(2\pi)^3 2k_0} \frac{\mathrm{d}^3 q}{(2\pi)^3 2q_0} \left[a^\dagger(\underline{k})a(\underline{k}) + b^\dagger(\underline{k})b(\underline{k}),\ a^\dagger(\underline{k})a(\underline{k}) - b^\dagger(\underline{k})b(\underline{k})\right] \\
&= \int \frac{\mathrm{d}^3 k k_0}{(2\pi)^3 2k_0} \frac{\mathrm{d}^3 q}{(2\pi)^3 2q_0} \left\{\left[a^\dagger(\underline{k})a(\underline{k}),\ a^\dagger(\underline{k})a(\underline{k})\right] - \left[b^\dagger(\underline{k})b(\underline{k}),\ b^\dagger(\underline{k})b(\underline{k})\right]\right\} \\
&= \int \frac{\mathrm{d}^3 k k_0}{(2\pi)^3 2k_0} \frac{\mathrm{d}^3 q}{(2\pi)^3 2q_0} \left\{a^\dagger(\underline{k})a(\underline{k})a^\dagger(\underline{q})a(\underline{q}) - a^\dagger(\underline{q})a(\underline{q})a^\dagger(\underline{k})a(\underline{k})\right. \\
&\qquad \left. - b^\dagger(\underline{k})b(\underline{k})b^\dagger(\underline{q})b(\underline{q}) + b^\dagger(\underline{q})b(\underline{q})b^\dagger(\underline{k})b(\underline{k})\right\} \\
&= \int \frac{\mathrm{d}^3 k k_0}{(2\pi)^3 2k_0} \frac{\mathrm{d}^3 q}{(2\pi)^3 2q_0} (2q_0)(2\pi)^3\delta(\underline{k} - \underline{q}) \\
&\qquad \times \left\{\left[a^\dagger(\underline{k})a(\underline{q}) - a^\dagger(\underline{q})a(\underline{k})\right] - \left[b^\dagger(\underline{k})b(\underline{q}) - b^\dagger(\underline{q})b(\underline{k})\right]\right\} = 0\,,
\end{aligned}
$$

[Application, 3 marks]
which shows that the charge is a conserved quantity. [Comprehension, 2 marks]

Fermions I

Dirac equation:

- free Dirac equation with α_i, β-matrices (184) and with γ-matrices (191)
- the commutation relations of the matrices (186, 190) [5.1, 5.2, 5.5, 5.8a)];
- Dirac equation (E.o.M.) for the spinor and its conjugate (191, 194) [5.7];
- how to "bar" a spinor: $\bar{\psi} = \psi^\dagger \gamma^0$;
- the conserved current (199);
- expansion of the Dirac fields in plane waves and u and v spinors and the form of the u and v spinors at rest (204) and with momentum(208);
- completenss relations of u and v spinors (213, 214) [5.4].

Fermions II

Quantisation of the Dirac Equation:

- conjugate momenta (215),
- expansion of the fields (219), e.g.

$$\psi(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2p_0} \sum_{i=1}^{2} \left[e^{-ip\cdot x}\, \hat{b}_i(\underline{p}) u^{(i)}(\underline{p}) + e^{ip\cdot x}\, \hat{d}_i^\dagger(\underline{p}) v^{(i)}(\underline{p}) \right]$$

- ani-commutation relations of the field operators (217) and of the creation/annihilation operators (220)

$$\left\{ \hat{b}_i(\underline{p}),\, \hat{b}_j^\dagger(\underline{q}) \right\} = \left\{ \hat{d}_i(\underline{p}),\, \hat{d}_j^\dagger(\underline{q}) \right\} = 2p_0 (2\pi)^3 \delta^3(\underline{p} - \underline{q}) \delta_{ij}$$

Fermions III

States and Operators of the Dirac Equation:

- anti-commutator of creation/annihilation operators encodes Pauli principle (222-224);
- how to derive the (normal-ordered) Hamilton operator (229) [5.6] and charge operator (233)

Fermions (Typical Exam Question)

Short question 1b) (2021):

Prove that

$$\bar{u}(\underline{p}')\sigma^{\mu\nu}(p+p')_\nu u(\underline{p}) = i\bar{u}(\underline{p}')(p-p')^\mu u(\underline{p})\,,$$

where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu,\,\gamma^\nu]$ is the commutator of two gamma matrices and the spinors have identical mass m. [5 marks]

Evaluate by using, repeatedly, the equation of motion for spinors, $(\not{p}-m)u(\underline{p}) = 0$ and the anti-commutator relation for the γ-matrices,

$$\begin{aligned}
\bar{u}(\underline{p}')\sigma^{\mu\nu}(p+p')_\nu u(\underline{p}) &= i\bar{u}(\underline{p}')\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right)(p+p')_\nu u(\underline{p}) \\
&= \frac{i}{2}\bar{u}(\underline{p}')\left[\gamma^\mu(\not{p}+\not{p}') - (\not{p}+\not{p}')\gamma^\mu\right]u(\underline{p}) \qquad \text{[Application, 1 mark]} \\
&= \frac{i}{2}\bar{u}(\underline{p}')\left[\gamma^\mu m + \gamma^\mu\not{p}' - \not{p}\gamma^\mu - m\gamma^\mu\right]u(\underline{p}) \\
&= \frac{i}{2}\bar{u}(\underline{p}')\left[2p'^\mu - \not{p}'\gamma^\mu - 2p^\mu + \gamma^\mu\not{p}\right]u(\underline{p}) \\
&= \frac{i}{2}\bar{u}(\underline{p}')\left[2p'^\mu - m/\gamma^\mu - 2p^\mu + \gamma^\mu m\right]u(\underline{p}) = i\bar{u}(\underline{p}')(p'-p)^\mu u(\underline{p})\,.
\end{aligned}$$

[Application, 4 marks]

Fermions (Typical Exam Question)

Short question 1b) (2022):

The Dirac Hamiltonian can be expressed by the α and β matrices as

$$\hat{\mathcal{H}} = \underline{\alpha} \cdot \underline{p} + \beta m$$

with $\left\{\alpha_i,\, \alpha_j\right\} = 2\delta_{ij}$, $\{\alpha_i,\, \beta\} = 0$, and $\alpha_i^2 = \beta^2 = \mathbf{1}$. Calculate the commutators $[\hat{\mathcal{H}},\, \hat{\Sigma}_i]$ and $[\hat{\mathcal{H}},\, \frac{\hat{\underline{\Sigma}}\cdot \underline{p}}{|\underline{p}|}]$, where the spin operator is given by

$$\hat{\underline{\Sigma}} = \frac{i}{2}\underline{\gamma} \times \underline{\gamma} \quad \longleftrightarrow \quad \hat{\Sigma}_i = \frac{i}{2}\epsilon_{ijk}\alpha_j\alpha_k\,.$$

[4 marks]

Interpret the results. [2 marks]

Fermions (Typical Exam Question)

Direct calculation yields

$$
\begin{aligned}
[\hat{H},\,\hat{\Sigma}_i] &= \frac{i}{2}\epsilon_{ijk}\left(p_l[\alpha_l,\,\alpha_j\alpha_k] + m[\beta,\,\alpha_j\alpha_k]\right)\\
&= \frac{i\epsilon_{ijk}p_l}{2}\left[\alpha_l\alpha_j\alpha_k - \alpha_j\alpha_k\alpha_l\right] + \frac{i\epsilon_{ijk}m}{2}\left[\beta\alpha_j\alpha_k - \alpha_j\alpha_k\beta\right]\\
&= \frac{i\epsilon_{ijk}p_l}{2}\left[(2\delta_{lj} + \alpha_j\alpha_l)\alpha_k - \alpha_j(2\delta_{kl} + \alpha_l\alpha_k)\right] + \frac{i\epsilon_{ijk}m}{2}\left[-\alpha_j\beta\alpha_k + \alpha_j\beta\alpha_k\right]\\
&= \frac{i\epsilon_{ijk}}{2}\left[2\alpha_k p_j - 2\alpha_j p_k\right] = i\epsilon_{ijk}\left[\alpha_k p_j - \alpha_j p_k\right] = -2i(\underline{\alpha}\times\underline{p})_i
\end{aligned}
$$

[Application, 2 marks]

and

$$
\left[\hat{H},\,\frac{\hat{\underline{\Sigma}}\cdot\underline{p}}{|\underline{p}|}\right] = \frac{i\epsilon_{ijk}p_i}{|\underline{p}|}\left[\alpha_k p_j - \alpha_j p_k\right] = \frac{i\epsilon_{ijk}p_i p_j\alpha_k}{|\underline{p}|} = 0
$$

[Application, 2 marks]

While the spin operator does not commute with the Hamiltonian, and spin therefore is not a conserved quantity, the helicity $\underline{\Sigma}\cdot\underline{p}$ is. [Synthesis, 2 marks]

Fermions (Typical Exam Question)

Long question 2c) (2021):

Consider the two-particle state

$$|\underline{p}_1, s_1; \underline{p}_2, s_2\rangle = \hat{b}^{\dagger}_{s_1}(\underline{p}_1)\hat{b}^{\dagger}_{s_2}(\underline{p}_2)|0\rangle$$

and calculate its energy and charge. [4 marks]

Fermions (Typical Exam Question)

To calculate the energy we have to evaluate

$$
\begin{aligned}
:\hat{H}:\ |\underline{p}_1, s_1; \underline{p}_2, s_2\rangle
&= \int \frac{\mathrm{d}^3 q}{(2\pi)^3 2q_0}\, q_0 \sum_{s=1}^{2} \left[\hat{b}_s^\dagger(\underline{q})\hat{b}_s(\underline{q}) + \hat{d}_s^\dagger(\underline{q})\hat{d}_s(\underline{q})\right] \hat{b}_{s_1}^\dagger(\underline{p}_1)\hat{b}_{s_2}^\dagger(\underline{p}_2)|0\rangle \\
&= \int \frac{\mathrm{d}^3 q}{(2\pi)^3 2q_0}\, q_0 \sum_{r=1}^{2} \left[\hat{b}_r^\dagger(\underline{q})\hat{b}_r(\underline{q})\hat{b}_{s_1}^\dagger(\underline{p}_1)\hat{b}_{s_2}^\dagger(\underline{p}_2)\right] |0\rangle \\
&= \int \frac{\mathrm{d}^3 q}{(2\pi)^3 2q_0}\, q_0 \sum_{r=1}^{2} \hat{b}_r^\dagger(\underline{q}) \left[\{\hat{b}_r(\underline{q}),\ \hat{b}_{s_1}^\dagger(\underline{p}_1)\} - \hat{b}_{s_1}^\dagger(\underline{p}_1)\hat{b}_r(\underline{q})\right] \hat{b}_{s_2}^\dagger(\underline{p}_2)|0\rangle \\
&= \int \frac{\mathrm{d}^3 q}{(2\pi)^3 2q_0}\, q_0 \sum_{r=1}^{2} \Big[(2\pi)^3 2q_0\, \delta^3(\underline{q} - \underline{p}_1)\delta_{rs_1}\, \hat{b}_r^\dagger(\underline{q})\hat{b}_{s_2}^\dagger(\underline{p}_2) \\
&\qquad -\hat{b}_r^\dagger(\underline{q})\hat{b}_{s_1}^\dagger(\underline{p}_1)\left(\{\hat{b}_r(\underline{q}),\ \hat{b}_{s_2}^\dagger(\underline{p}_2)\} - \hat{b}_{s_2}^\dagger(\underline{p}_2)\hat{b}_r(\underline{q})\right)\Big] |0\rangle \\
&= \int \frac{\mathrm{d}^3 q}{(2\pi)^3 2q_0}\, q_0 \sum_{r=1}^{2} \Big[(2\pi)^3 2q_0\, \delta^3(\underline{q} - \underline{p}_1)\delta_{rs_1}\, \hat{b}_r^\dagger(\underline{q})\hat{b}_{s_2}^\dagger(\underline{p}_2) \\
&\qquad -(2\pi)^3 2q_0\, \delta^3(\underline{q} - \underline{p}_2)\delta_{rs_2}\, \hat{b}_r^\dagger(\underline{q})\hat{b}_{s_1}^\dagger(\underline{p}_1)\Big] |0\rangle \\
&= \left[E_1 \hat{b}_{s_1}^\dagger(\underline{p}_1)\hat{b}_{s_2}^\dagger(\underline{p}_2) - E_2 \hat{b}_{s_2}^\dagger(\underline{p}_2)\hat{b}_{s_1}^\dagger(\underline{p}_1)\right] |0\rangle = (E_1 + E_2)|\underline{p}_1, s_1; \underline{p}_2, s_2\rangle
\end{aligned}
$$

[3 marks]

Fermions (Typical Exam Question)

Long question 2d) (2021):

Assume you had quantised the Dirac field with commutators instead of anti-commutators, *i.e.* demanding

$$[\hat{\psi}_\alpha(t,\underline{x}),\,\hat{\pi}_\beta(t,\underline{y})] = i\delta_{\alpha\beta}\delta^3(\underline{x}-\underline{y})$$

as the only non-vanishing equal-time commutator. What would be the energy of the field? Comment on the result. [6 marks]

Fermions (Typical Exam Question)

First we have to determine the Hamiltonian, using the commutators instead of the anti-communtators. Inspecting the calculation of the non-normal-ordered Hamiltonian in (a) it becomes obvious that commutator relations have not been invoked, and therefore

$$\hat{H} = \int \frac{\mathrm{d}^3 q}{(2\pi)^3 2q_0}\, q_0 \sum_{i=1}^{2} \left[\hat{b}_i^\dagger(\underline{q})\hat{b}_i(\underline{q}) \;-\; \hat{d}_i(\underline{q})\hat{d}_i^\dagger(\underline{q})\right] .$$

[Comprehension, 2 marks]
In the absence of Fermi-Dirac statistics, normal-ordering of the creation and annihilation operators does nor involve a sign change,

$$:\hat{d}_i(\underline{q})\hat{d}_i^\dagger(\underline{q}) := \hat{d}_i^\dagger(\underline{q})\hat{d}_i(\underline{q})$$

and therefore

$$\hat{H} = \int \frac{\mathrm{d}^3 q}{(2\pi)^3 2q_0}\, q_0 \sum_{i=1}^{2} \left[\hat{b}_i^\dagger(\underline{q})\hat{b}_i(\underline{q}) \;-\; \hat{d}_i^\dagger(\underline{q})\hat{d}_i(\underline{q})\right] .$$

[Comprehension, 2 marks]
As a result, the energy of the field would not be bound from below, a physically unacceptable situation. [Comprehension, 2 marks]

Electrodynamics

know and be able to calculate

- classical Lagrangian (125, 133) and gauge invariance (120, 235)
- fixing the gauge: Coulomb vs. Lorentz (240, 241)
- polarisation vectors and their properties (243-245) [6.1]
- creation/annihilation operators in Coulomb (249, 254-255) [6.2] and Lorentz gauge (263, 275-276) [6.3]
- combining polarisation vectors and creation/annihiliation operators for more complicated operators (259-260, 273, 277) [6.2, 6.3]
- physical states based on transverse polarisations [e.g. 6.3]

be able to **describe**

- why canonical quantisation is tricky and the logic of quantisation in Coulomb or Lorentz gauge

Electrodynamics (Typical Exam Question)

Short question 1c) (2019):

Write down the Lagrangian for free electromagnetic fields, as expressed by the vector potential A^μ. Sketch the process of quantizing the electromagnetic field and explain why naively following the standard procedure of canonical quantization fails. [5 marks]

Knowledge: Lagrangian

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} = -\frac{1}{4}\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)$$

The standard procedure of determining conjugate momenta and demanding commutation relations fails because

1. the conjugate momentum to A^0, $\pi^0 = 0$. Therefore the equal-time commutator relation $[A^0(\underline{x},\ t),\ \pi^0(\underline{y},\ t)] = \delta^3(\underline{x} - \underline{y})$ cannot be satisfied. [2 marks]
2. gauge invariance eliminates another dynamic d.o.f., implying that also the equal-time commutators for the space-like components cannot all be satisfied. In effect, the physical photon field has two degrees of freedom, but the vector potential naively seems to have four degrees of freedom; so two of these d.o.f. cannot be quantized consistently. [3 marks]

Electrodynamics (Typical Exam Question)

Long question 2a) (2022):

The spatial components of the (normal-ordered) spin operator of the electromagnetic field are given by

$$:\hat{S}^i := \epsilon^{ilm} \int \mathrm{d}^3x \, :\partial_t \hat{A}^m(x)\hat{A}^l(x):$$

Show that in Coulomb gauge

$$:\hat{S}^i: \;=\; \frac{i\epsilon^{ilm}}{2} \sum_{\lambda,\kappa} \int \frac{\mathrm{d}^3k}{(2\pi)^3 2k_0} \Big\{ \epsilon^m(\lambda,\, \underline{k})\epsilon^l(\kappa,\, \underline{k}) \Big[\hat{a}^\dagger(\lambda,\, \underline{k})\hat{a}(\kappa,\, \underline{k}) - \hat{a}^\dagger(\kappa,\, \underline{k})\hat{a}(\lambda,\, \underline{k})\Big] \Big\}$$

The polarisation vectors are orthonormal w.r.t. each other and orthogonal to the momentum, $\underline{\epsilon}(1,\, \underline{k}) \times \underline{\epsilon}(2,\, \underline{k}) = -\underline{\epsilon}(2,\, \underline{k}) \times \underline{\epsilon}(1,\, \underline{k}) = \underline{k}/|\underline{k}|$. Use this to confirm that

$$:\hat{\underline{S}} := i \int \frac{\mathrm{d}^3k}{(2\pi)^3 2k_0} \frac{\underline{k}}{|\underline{k}|} \Big[\hat{a}^\dagger(2,\, \underline{k})\hat{a}(1,\, \underline{k}) - \hat{a}^\dagger(1,\, \underline{k})\hat{a}(2,\, \underline{k})\Big]$$

[8 marks]

Hint: You may assume that the polarisation vectors in Coulomb gauge have real components only.

Electrodynamics (Typical Exam Question)

$$\int \mathrm{d}^3x :\partial_t \hat{A}^m(x)\hat{A}^l(x):= \ldots \qquad \text{[Application, 3 marks]}$$

$$= \quad \frac{i}{2}\sum_{\lambda,\,\kappa}\int\frac{\mathrm{d}^3k}{(2\pi^3 2k_0)} :\Big[- e^{-2ik_0\cdot x_0}\epsilon^m(\lambda,\,\underline{k})\epsilon^l(\kappa,\,-\underline{k})\hat{a}(\lambda,\,\underline{k})\hat{a}(\kappa,\,-\underline{k}) + \ldots\Big]:$$

The first and last lines are symmetric in l and m under the integration, so when multipliying with the Levi-Civita tensor they will vanish. Therefore [Application, 2 marks]

$$:\hat{S}^i: \quad = \quad \frac{i\epsilon^{ilm}}{2}\sum_{\lambda,\,\kappa}\int\frac{\mathrm{d}^3k}{(2\pi^3 2k_0)}\Big\{\epsilon^m(\lambda,\,\underline{k})\epsilon^l(\kappa,\,\underline{k})\left[\hat{a}^\dagger(\lambda,\,\underline{k})\hat{a}(\kappa,\,\underline{k}) - \hat{a}^\dagger(\kappa,\,\underline{k})\hat{a}(\lambda,\,\underline{k})\right]\Big\}$$

In a first step we realise that we must have $\lambda \neq \kappa$, and therefore

$$:\hat{S}^i:= \cdots = i\int\frac{\mathrm{d}^3k}{(2\pi^3 2k_0)}\Big\{\frac{k^i}{|\underline{k}|}\left[\hat{a}^\dagger(2,\,\underline{k})\hat{a}(1,\,\underline{k}) - \hat{a}^\dagger(1,\,\underline{k})\hat{a}(2,\,\underline{k})\right]\Big\}$$

[Application, 3 marks]

Electrodynamics (Typical Exam Question)

Long question 2b) (2022):

Defining (complex) circular polarisations as

$$\underline{\epsilon}(\pm,\,\underline{k}) = \frac{1}{\sqrt{2}}\left[\underline{\epsilon}(1,\,\underline{k}) \pm i\underline{\epsilon}(2,\,\underline{k})\right]$$

show that the spin operator becomes diagonal,

$$:\hat{\underline{S}}:= \int \frac{\mathrm{d}^3k}{(2\pi)^3 2k_0}\,\frac{\underline{k}}{|\underline{k}|}\,\left[\hat{a}^\dagger(+,\,\underline{k})\hat{a}(+,\,\underline{k}) - \hat{a}^\dagger(-,\,\underline{k})\hat{a}(-,\,\underline{k})\right]\ .$$

We define the helicity operator as

$$:\hat{\underline{\Lambda}}:= \int \frac{\mathrm{d}^3k}{(2\pi)^3 2k_0}\,\left[\hat{a}^\dagger(+,\,\underline{k})\hat{a}(+,\,\underline{k}) - \hat{a}^\dagger(-,\,\underline{k})\hat{a}(-,\,\underline{k})\right]\ .$$

Calculate the helicity of single photon states with polarisations $\pm$ and momentum q, $\hat{a}^\dagger(\pm,\,\underline{q})|0\rangle$, and interpret the result. [6 marks]

Electrodynamics (Typical Exam Question)

Inverting the definitions of the circular polarisations, we have

$$\{\hat{a}(1,\,\underline{k})\,,\quad \hat{a}(2,\,\underline{k})\} = \frac{\hat{a}(\pm,\,\underline{k}) \pm \hat{a}(\mp,\,\underline{k})}{\sqrt{2}}$$

and simple replacements of $\hat{a}(1,2)$ and $\hat{a}^\dagger(1,2)$ yields the desired result. [Application, 1 mark]
In a first step we have to calculate the commutators of the $\hat{a}(\pm)$: [Comprehension, 1 marks]

$$\begin{aligned}\left[\hat{a}(\pm,\,\underline{k}),\ \hat{a}^\dagger(\pm,\,\underline{q})\right] &= \cdots = (2\pi)^2 2k_0\,\delta^3(\underline{k}-\underline{q})\\ \left[\hat{a}(\pm,\,\underline{k}),\ \hat{a}^\dagger(\mp,\,\underline{q})\right] &= 0\,.\end{aligned}$$

Therefore

$$\begin{aligned}&\hat{\Lambda}\hat{a}^\dagger(\pm,\,\underline{q})|0\rangle\\ &= \int\frac{\mathrm{d}^3k}{(2\pi)^3 2k_0}\left[\hat{a}^\dagger(+,\,\underline{k})\hat{a}(+,\,\underline{k}) - \hat{a}^\dagger(-,\,\underline{k})\hat{a}(-,\,\underline{k})\right]\hat{a}^\dagger(\pm,\,\underline{q})|0\rangle\\ &= \pm\int\frac{\mathrm{d}^3k}{(2\pi)^3 2k_0}\hat{a}^\dagger(\pm,\,\underline{k})\left[\hat{a}(\pm,\,\underline{k}),\ \hat{a}^\dagger(\pm,\,\underline{q})\right]|0\rangle\\ &= \pm\hat{a}^\dagger(\pm,\,\underline{q})|0\rangle\end{aligned}$$

The circular polarised states are eigenstates of helicity with eigenvalue $\pm$. [Comprehension, 4 marks]

Propagators

- propagators/Greens functions as solutions to free E.o.M.'s and their definition through δ-functions (286, 300, 307, 314)
- construction of propagators through Fourier transformation of the E.o.M. (291-292, 301, 308, 315-318) [7.2, 7.3]

Propagators (Typical Exam Question)

Short question 1c) (2020):

Consider the Lagrangian for a free massive vector field V_μ. given by

$$\mathcal{L} = -\frac{1}{4} V_{\mu\nu} V^{\mu\nu} - \frac{m^2}{2} V_\mu V^\mu \,,$$

with the field strength tensor $V_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$. Derive the equation of motion for V^μ and use it to construct the propagator for the vector field in momentum space. [5 marks]

Hint: Remember the case of the photon field in Lorentz gauge where we made an *ansatz* for the two tensors in the propagator numerator. Use a similar logic here and write the numerator as $Ak^2 g^{\mu\nu} + Bk^\mu k^\nu$.

Propagators (Typical Exam Question)

Equations of motion: [Application, 2 marks]

$$0 = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu V_\nu)} - \frac{\partial \mathcal{L}}{\partial V_\nu} = \cdots = -\Box V^\nu + \partial^\nu(\partial \cdot V) - m^2 V^\nu$$

Propagator/Green's function in position space from kernel of E.o.M.:

$$\left[\Box g_{\mu\nu} - \partial_\mu \partial_\nu + m^2 g_{\mu\nu}\right] G_0^{\nu\rho}(x,\, y) \;=\; i\delta^4(x-y) g_\mu^\rho$$

and therefore in momentum space [Application, 1 mark]

$$\left[(k^2 - m^2) g_{\mu\nu} - k_\mu k_\nu\right] \tilde{G}_0^{\nu\rho}(k) \;=\; -i g_\mu^\rho$$

Use the *ansatz* [Comprehension, 1 mark] $\tilde{G}_0^{\nu\rho}(k) = i\frac{Ak^2 g^{\nu\rho} - Bk^\nu k^\rho}{k^2 - m^2}$ and solve for A and B.

$$\begin{aligned} -i g_\mu^\rho &\stackrel{!}{=} \left[(k^2 - m^2) g_{\mu\nu} - k_\mu k_\nu\right] \frac{iAk^2 g^{\nu\rho} - iBk^\nu k^\rho}{k^2 - m^2} \\ &= \frac{iAk^2(k^2 - m^2) g_\mu^\rho + i(Bm^2 - Ak^2) k_\mu k^\rho}{k^2 - m^2} \end{aligned}$$

and therefore $A = -\frac{1}{k^2}$ and $B = -\frac{1}{m^2}$.