Relativistic Physics for Teachers SS 2006

Problems 1 13.4.2006

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1. **Non-relativistic Doppler-effect**
How does the frequency of a wave of sound moving along $\vec{n}$ towards a detector change, when the detector moves with a velocity $\vec{v}$?
Solution:
Consider the wave equation in the frame of the atmosphere S and in the frame of the moving detector, S':

$$\begin{aligned} A_S &= A_0 \cos\left[2\pi\nu\left(t - \frac{\vec{n}\cdot\vec{x}}{c}\right)\right] \\ A_{S'} &= A_0 \cos\left[2\pi\nu'\left(t' - \frac{\vec{n}\cdot\vec{x}'}{c'}\right)\right] \,, \end{aligned}$$

where ν denotes the frequency, c the speed of sound, $\vec{x}$ and t the position and moment in the respective system. The two systems are now connected by a Galilei transformation,

$$t' = t \quad \text{and} \quad \vec{x}' = \vec{x} + \vec{v}t \,.$$

According to Galilei's principle, applying this transformation on A_S should result in $A_{S'}$:

$$A_{S'} = A_0 \cos\left[2\pi\nu\left(t' - \frac{\vec{n}\cdot(\vec{x}' - \vec{v}t')}{c}\right)\right] \,.$$

Comparing coefficients yields

$$\begin{aligned} \nu' &= \nu\left(1 + \frac{\vec{n}\cdot\vec{v}}{c}\right) \\ \frac{\nu'}{c'} &= \frac{\nu}{c} \,, \end{aligned}$$

where the first equation stems from balancing coefficients of t' and the second one comes from adjusting the x'-terms.

2. **Parallel moving light-pulse clock**
Consider a light-pulse clock, similar to the one considered in the script of the lecture, p. 5. In contrast to the case considered in the lecture, let this clock move constantly in the direction of the travelling light-pulse. What is the period (from mirror A to B and back) in this setup?
Solution:
First, it should be noted that the total way for the light consists of $L' + x'_1$ for the way from A to B and $L' - x'_2$ for the way back, if the mirrors move in the $\vec{AB}$-direction. Now, L' is length-contracted, i.e.

$$L' = L\sqrt{1 - v^2/c^2} \,.$$

Similar to the case considered before, $x'_{1,2}$ are given by

$$x'_1 = vt'_1 \quad \text{and} \quad L' + x'_1 = ct'_1 \quad \Longrightarrow \quad x'_1 = \frac{L'v/c}{1-v/c}$$

and

$$x'_2 = vt'_2 \quad \text{and} \quad L' - x'_2 = ct'_2 \quad \Longrightarrow \quad x'_2 = \frac{L'v/c}{1+v/c}\,,$$

respectively. Hence, the total length the light has to propagate is

$$\begin{aligned} S' &= 2L' + x'_1 - x'_2 = L\sqrt{1-v^2/c^2}\left(2 + \frac{v/c}{1-v/c} - \frac{v/c}{1+v/c}\right) \\ &= L\sqrt{1-v^2/c^2}\left(2 + \frac{2v^2/c^2}{1-v^2/c^2}\right) \\ &= \frac{2L}{\sqrt{1-v^2/c^2}} \end{aligned}$$

and the time it takes is

$$T' = \frac{2L}{c} \cdot \frac{1}{\sqrt{1-v^2/c^2}}\,,$$

again dilated by the same amount as in the other mirror configuration.

3. **Muon decay**
The life-time of muons at rest is $\tau \approx 2.2 \cdot 10^{-6}$s. How fast must a muon move in order to travel for 10km before decaying?
Solution:
The muon needs to be time-dilated such that

$$t = \frac{\tau}{\sqrt{1-u^2/c^2}}\,,$$

where

$$ut = d = 10\text{km}\,.$$

Hence

$$t = \frac{d}{u} = \frac{\tau}{\sqrt{1-u^2/c^2}}$$

or

$$\frac{d^2}{u^2} = \frac{\tau^2}{1-u^2/c^2}$$

leading to

$$u = \frac{1}{\sqrt{1+c^2\tau^2/d^2}} \cdot c\,.$$