Lecture 7: Percolation
Cluster growth models

- Last lecture: Random walkers and diffusion
- Related process: Cluster growth
  - Start from a small seed
  - Add cluster sites according to different rules
  - Larger structures emerge
- Examples: Growth of tumors, snowflakes, etc..
- In the following: two different models/operational descriptions of the modalities of cluster growth, main difference in “how do new sites dock to the cluster”
- N.B.: only two-dimensional clusters on a square lattice
Eden clusters: Tumors

- Start with a seed at \( x, y = (0, 0) \).
- Any unoccupied nearest neighbour eligible for added site - pick one of these “perimeter sites” at random.
- Repeat until cluster is finished (predefined size, number of sites included or similar)
- Note: As the cluster grows, perimeter sites could also be “inside” the cluster, surrounded by four occupied sites at the nearest neighbours.
- Resulting cluster roughly circular (spherical), with a somewhat “fuzzy” edge and, eventually, some holes in it.
- Model also known as “cancer model”.
An example Eden cluster
DLA clusters: Snowflakes

- Diffusion-limited aggregation (DLA) adds sites from the outside:
  - Again, start with a seed at (0, 0)
  - Initialise a random walker at a large enough distance and let it walk. As soon as it hits a perimeter site, it sticks.
  - For efficient implementation: discard random walkers moving too far away or “direct” the walk toward the cluster.
  - Resulting cluster much more “airy”:
    A fluffy object of irregular shape with large holes in it.
An example DLA cluster
Fractal dimensions: An operational definition

- Quantify the difference between Eden and DLA clusters → need a quantitative measure for fluffiness.
- Basic idea: How to measure the dimension of an object (at first sight a silly idea - both clusters are 2D)
- Start with a disk in $xy$-plane; mass $m$ depends on radius $r$ through $m(r) = \sigma \pi r^2$, if $\sigma$ is the area density.
- In contrast, a straight line has length-dependent mass $m(r) = 2\lambda r$, if $l = 2r$ and $\lambda$ is the length density.
- Masses scale as $r^2$ and $r^1$, respectively, giving another effective measure of dimension through the scaling.
- Therefore, for a more general definition, try $m(r) \sim r^{d_f}$
- The effective dimension $d_f$ known as fractal dimension.
Fractal dimensions: An operational definition (cont’d)

▶ Using $m(r) \sim r^{d_f}$ for a definition of fractal dimension:
  
  - $d_f = 2$ for a (massive) circle
  - $d_f = 1$ for a (massive) line

▶ In general: $\ln m(r) \sim d_f \ln r$.

▶ Can be applied to our clusters:
  
  - $d_f \approx 1.99$ for an Eden cluster
  - $d_f \approx 1.65$ for a DLA cluster
Fractal dimension of clusters

Mass vs. radius: Eden cluster

Mass vs. radius: DLA cluster
Percolation

- Percolation as a physical phenomenon:
  for example the motion of ground water through soil, oil
  oozing through a porous rock, or burning of a forest.
- Random processes where cells with a finite size within an
  area or volume are filled or activated
- An interesting field of research, a large number of
  applications
- Many surprising results: phase transitions etc..
- Simple example: Lattice of finite size, the sites are filled
  randomly, with probability $p$ (see below).
Simple results

- Lattice of dimension $40 \times 40$. 
Analysis of model above

- Introduce concept of a cluster: connected sites (nearest neighbours).
- At $p = 0.2$ most clusters consist of 1-2 sites.
- At $p = 0.4$ cluster size increased, typically 8-10 sites.
- At $p = 0.8$ more or less one large structure
- Most interesting at $p = 0.6$:
  - Typically at this value the first spanning cluster emerges: connecting all four edges of the lattice.
  - Emergence of a spanning cluster indicates percolation.
  - Usually cluster decays when removing individual sites.
  - Stated differently: Occupancy of single sites determines average cluster size $\Rightarrow$ a phase transition
More on percolating clusters

- Analyse emergence of percolating cluster as function of $p$.

- **Transition between two regimes** (no percolating cluster, or existence of such a cluster) is **sharp** and depends on size of lattice $d$.

- In the limit of $d \to \infty$ the **critical density** for appearance of a percolating cluster is $p_c \approx 0.593$.

  $\implies$ a posteriori explanation, why $p = 0.6$ was interesting.

- How to calculate the critical value? Need to check the configuration of lattice sites.

- **Simple algorithm**: Successively choose sites at random, stop when percolating cluster appears. Repeat a couple of times and sample over the respective densities. Tricky bit: Check for spanning cluster (solution see below).
Cluster labelling - pseudo-code

Main routine:

1. Begin with an empty lattice, all sites labelled by ’0’

2. Pick a site at random, label it with ’1’
   (its the first site hence the first cluster - label clusters consecutively)

3. Repeat step 2 until a spanning cluster has emerged.
   Pick a site at random. Check for occupied neighbours:
   ▶ If no, open a new cluster with new integer label.
   ▶ If yes, call it a bridging site and deal with it.

4. If a spanning cluster has emerged keep track of \( p_c \),
given by the fraction of occupied sites.
Cluster labelling - pseudo-code (cont’d)

Bridging sites:

1. For every bridging site BS, check number of occupied neighbours:

2. One occupied neighbour: BS inherits number of neighbouring cluster.

3. Two or more occupied neighbour:
   ▶ Find minimum of neighbour numbers.
   ▶ BS inherits this number
   ▶ All clusters belonging to this site are relabelled

⇒ merged cluster has unique number

Check for spanning cluster:

1. For each cluster keep track of sites at each edge - need four bools, switched to “true” on the flight.
Percolation in 2 D

- Lattice with dimension $L$, sampled over $50L$ runs.
- Statistical fluctuations, linear fit in agreement with 0.593.
Phase transition in percolation

- Study the behaviour near the percolation threshold $p_c$
- Second order phase transition (first derivative jumps). Examples for this: transitions between the paramagnetic and ferromagnetic phases of a metal or melting of materials
- Here: transition between macroscopically connected and disconnected phases.
- Typical for phase transitions: Singular behaviour of some observables, often described by power laws.
- Obvious observable here: fraction of spanning clusters.
Phase transition: Results

- Results for $25 \times 25$ lattices

![Graphs showing phase transition results](image-url)
Phase transition in percolation

- Number of percolating clusters and their relative occupancy drops very steeply at around 0.6.
- Write this as $F = F_0(p - p_0)^\beta$ and $N = N_0(p - p_0)^\gamma$
- $\beta, \gamma$ known as critical exponents (more in lecture 8)
- Guess: $\frac{d\{F, N\}}{dp} \to \infty$ for $p \to p_0$.
- For infinitely large lattices, finite size effects are unimportant, and $p_0 \to p_c$ as $d \to \infty$.
- Also for $d \to \infty$: for all two-dimensional lattices $\beta = 5/36$.
- Also: $F \to 0$ for $d \to \infty$
  $\iff$ percolating cluster has infinite size but zero volume - a fractal!
Summary

- More on simulation:
  - Another set of simple random processes (closely related):
    - Cluster growth models
    - Percolation
  - A new way of classifying objects: fractal dimension
  - First exposure to a phase transition
- Interesting feature:
  - Simplistic tools may model complicated physics
  - Typical problem: Map results from toy model to reality.