Lecture 7: Percolation

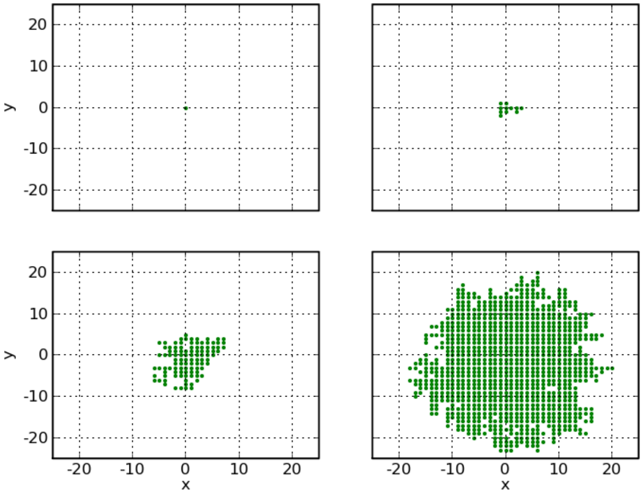
Cluster growth models

- Last lecture: Random walkers and diffusion
- Related process: Cluster growth
  - Start from a small seed
  - Add cluster sites according to different rules
  - Larger structures emerge
- Examples: Growth of tumors, snowflakes, etc..
- In the following: two different models/operational descriptions of the modalities of cluster growth, main difference in “how do new sites dock to the cluster”
- N.B.: only two-dimensional clusters on a square lattice

Eden clusters: Tumors

- Start with a seed at $x, y = (0, 0)$.
- Any unoccupied nearest neighbour eligible for added site - pick one of these "perimeter sites" at random.
- Repeat until cluster is finished (predefined size, number of sites included or similar)
- Note: As the cluster grows, perimeter sites could also be "inside" the cluster, surrounded by four occupied sites at the nearest neighbours.
- Resulting cluster roughly circular (spherical), with a somewhat "fuzzy" edge and, eventually, some holes in it.
- Model also known as "cancer model".

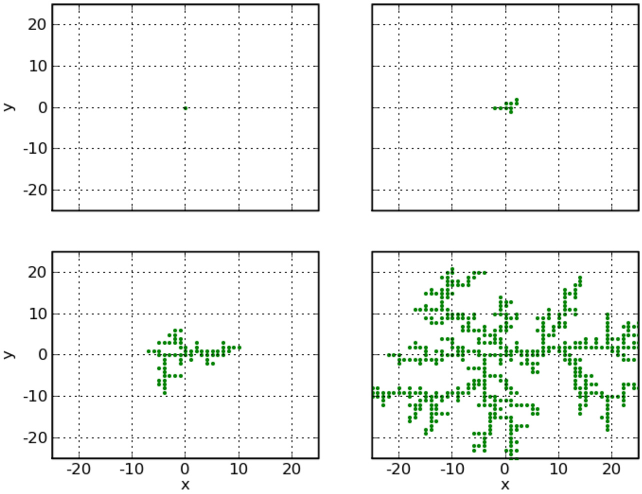
An example Eden cluster



DLA clusters: Snowflakes

- Diffusion-limited aggregation (DLA) adds sites from the outside:
  - Again, start with a seed at $(0,0)$
  - Initialise a random walker at a large enough distance and let it walk. As soon as it hits a perimeter site, it sticks.
  - For efficient implementation: discard random walkers moving too far away or "direct" the walk toward the cluster.
  - Resulting cluster much more "airy":
    A fluffy object of irregular shape with large holes in it.

An example DLA cluster



Fractal dimensions: An operational definition

- Quantify the difference between Eden and DLA clusters $\Longrightarrow$ need a quantitative measure for fluffiness.
- Basic idea: How to measure the dimension of an object (at first sight a silly idea - both clusters are 2D)
- Start with a disk in xy-plane; mass m depends on radius r through $m(r) = \sigma\pi r^2$, if σ is the area density.
- In contrast, a straight line has length-dependent mass $m(r) = 2\lambda r$, if $l = 2r$ and λ is the length density.
- Masses scale as r^2 and r^1, respectively, giving another effective measure of dimension through the scaling.
- Therefore, for a more general definition, try $m(r) \sim r^{d_f}$
- The effective dimension d_f known as fractal dimension.

Fractal dimensions: An operational definition (cont'd)

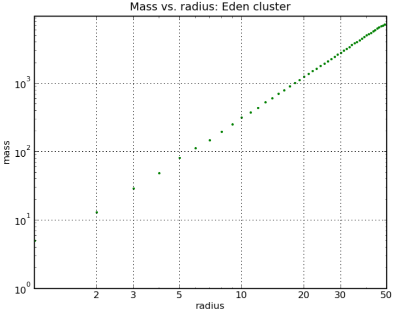
- Using $m(r) \sim r^{d_f}$ for a definition of fractal dimension:

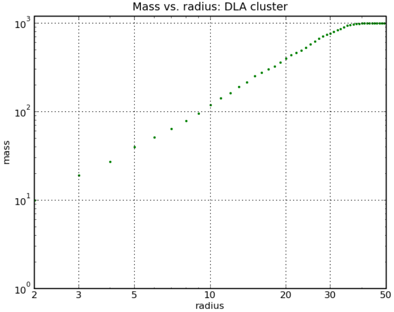
$$\begin{aligned} d_f &= 2 \quad \text{for a (massive) circle} \\ d_f &= 1 \quad \text{for a (massive) line} \end{aligned}$$

- In general: $\ln m(r) \sim d_f \ln r$.
- Can be applied to our clusters:

$$\begin{aligned} d_f &\approx 1.99 \quad \text{for an Eden cluster} \\ d_f &\approx 1.65 \quad \text{for a DLA cluster} \end{aligned}$$

Fractal dimension of clusters



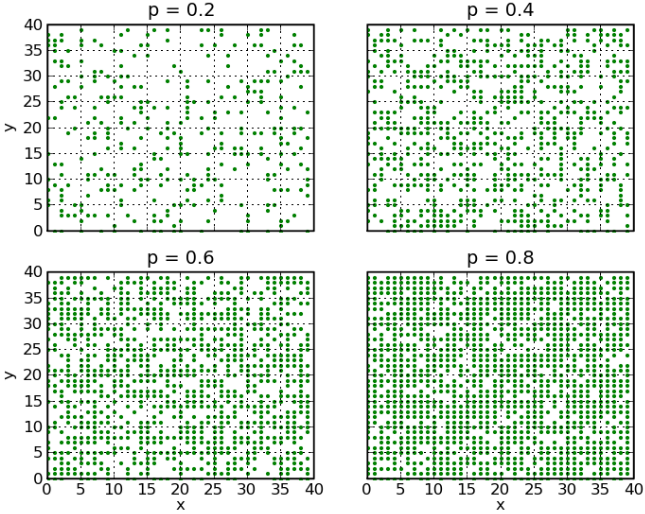


Percolation

- Percolation as a physical phenomenon:
  for example the motion of ground water through soil, oil oozing through a porous rock, or burning of a forest.
- Random processes where cells with a finite size within an area or volume are filled or activated
- An interesting field of research, a large number of applications
- Many surprising results: phase transitions etc..
- Simple example: Lattice of finite size, the sites are filled randomly, with probability p (see below).

Simple results

- Lattice of dimension 40×40.



Analysis of model above

- Introduce concept of a cluster: connected sites (nearest neighbours).
- At $p = 0.2$ most clusters consist of 1-2 sites.
- At $p = 0.4$ cluster size increased, typically 8-10 sites.
- At $p = 0.8$ more or less one large structure
- Most interesting at $p = 0.6$:
  - Typically at this value the first spanning cluster emerges: connecting all four edges of the lattice.
  - Emergence of a spanning cluster indicates percolation.
  - Usually cluster decays when removing individual sites.
  - Stated differently: Occupancy of single sites determines average cluster size $\Longrightarrow$ a phase transition

More on percolating clusters

- Analyse emergence of percolating cluster as function of p.
- Transition between two regimes (no percolating cluster, or existence of such a cluster) is sharp and depends on size of lattice d.
- In the limit of $d \to \infty$ the critical density for appearance of a percolating cluster is $p_c \approx 0.593$.
  $\Longrightarrow$ a posteriori explanation, why $p = 0.6$ was interesting.
- How to calculate the critical value? Need to check the configuration of lattice sites.
- Simple algorithm: Successively choose sites at random, stop when percolating cluster appears. Repeat a couple of times and sample over the respective densities. Tricky bit: Check for spanning cluster (solution see below).

Cluster labelling - pseudo-code

Main routine:

1. Begin with an empty lattice, all sites labelled by '0'
2. Pick a site at random, label it with '1'
   (its the first site hence the first cluster - label clusters consecutively)
3. Repeat step 2 until a spanning cluster has emerged.
   Pick a site at random. Check for occupied neighbours:
   - If no, open a new cluster with new integer label.
   - If yes, call it a bridging site and deal with it.
4. If a spanning cluster has emerged keep track of p_c, given by the fraction of occupied sites.

Cluster labelling - pseudo-code (cont'd)

Bridging sites:

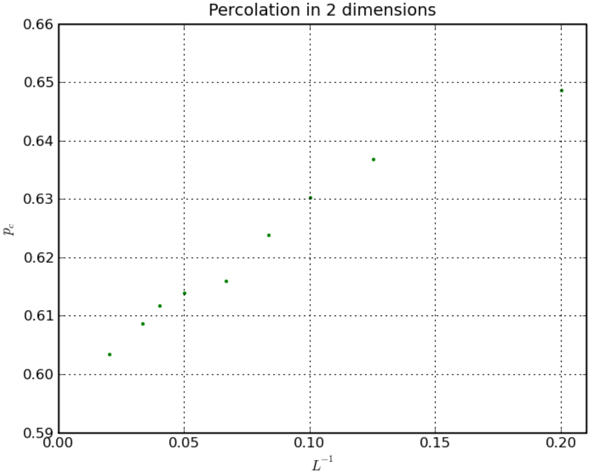
1. For every bridging site BS, check number of occupied neighbours:
2. One occupied neighbour: BS inherits number of neighbouring cluster.
3. Two or more occupied neighbour:
   - Find minimum of neighbour numbers.
   - BS inherits this number
   - All clusters belonging to this site are relabelled $\Longrightarrow$merged cluster has unique number

Check for spanning cluster:

1. For each cluster keep track of sites at each edge - need four bools, switched to "true" on the flight.

Percolation in 2 D

- Lattice with dimension L, sampled over $50L$ runs.
- Statistical fluctuations, linear fit in agreement with 0.593.

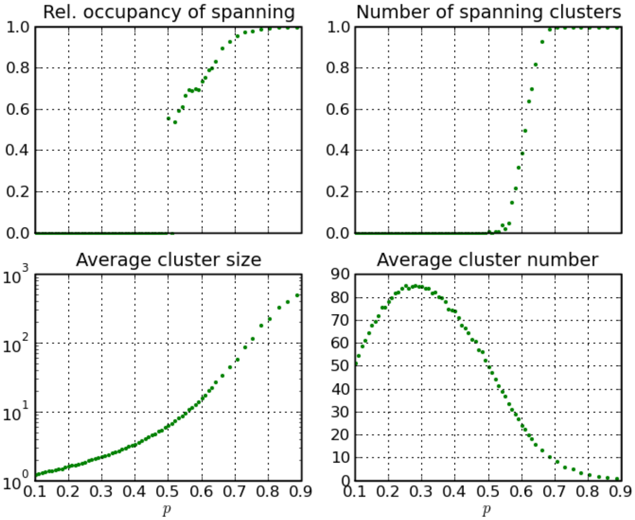


Phase transition in percolation

- Study the behaviour near the percolation threshold p_c
- Second order phase transition (first derivative jumps). Examples for this: transitions between the paramagnetic and ferromagnetic phases of a metal or melting of materials
- Here: transition between macroscopically connected and disconnected phases.
- Typical for phase transitions: Singular behaviour of some observables, often described by power laws.
- Obvious observable here: fraction of spanning clusters.

Phase transition: Results

- Results for 25×25 lattices



Phase transition in percolation

- Number of percolating clusters and their relative occupancy drops very steeply at around 0.6.
- Write this as $F = F_0(p - p_0)^\beta$ and $N = N_0(p - p_0)^\gamma$
- β, γ known as critical exponents (more in lecture 8)
- Guess: $\mathrm{d}\{F, N\}/\mathrm{d}p \to \infty$ for $p \to p_0$.
- For infinitely large lattices, finite size effects are unimportant, and $p_0 \to p_c$ as $d \to \infty$.
- Also for $d \to \infty$: for **all** two-dimensional lattices
$$\beta = 5/36.$$
- Also: $F \to 0$ for $d \to \infty$
$\Longleftrightarrow$ percolating cluster has infinite size but zero volume - a fractal!

Summary

- More on simulation:
  Another set of simple random processes (closely related):
  - Cluster growth models
  - Percolation
- A new way of classifying objects: fractal dimension
- First exposure to a phase transition
- Interesting feature:
  - Simplistic tools may model complicated physics
  - Typical problem: Map results from toy model to reality.