Lecture 3: Harmonic Motion

The mathematical pendulum

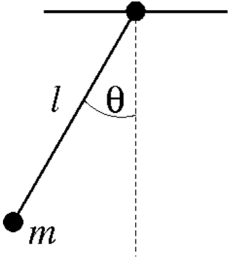
- Simple system: mass m at a rigid massless rod of length l, deflection angle θ as relevant coordinate.
- Gravitational force:

$$F_\theta = -mg \sin\theta \approx -mg\theta$$

in the small angle approximation.

- Equation of motion from Newton's law:

$$\begin{aligned} m\ddot{r} &= ml\ddot{\theta} = -mg\theta \\ \Longrightarrow \ddot{\theta} &= -\frac{g}{l}\theta\,. \end{aligned}$$



Analytic solution

- Analytic solution straightforward: $\theta(t) = \theta_0 \sin(\Omega t + \phi)$.
- Angular eigen-frequency Ω: $\Omega^2 = g/l$.
- Maximal amplitude: θ_0
  $\rightarrow$ will serve as initial condition: $\theta(0) = \theta_0$
- ϕ from initial velocity (will be set to 0)
- Check energy conservation:

$$E = E_{\mathrm{kin}} + E_{\mathrm{pot}} = \frac{ml^2}{2}\omega^2(t) + \frac{mgl}{2}\theta^2(t)$$

Using that $\omega = \mathrm{d}\theta/\mathrm{d}t$ and the solution above:

$$E = \frac{mgl}{2}\theta_0^2 = \mathrm{const.}\,.$$

Numerical solution: Euler method

- Differential eqn's using the angle θ and angular velocity ω:

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega \quad \text{and} \quad \frac{\mathrm{d}\omega}{\mathrm{d}t} = -\frac{g}{l}\theta \,.$$

- Discretisation with Δt:

$$\theta_{i+1} = \theta_i + \omega_i \Delta t \quad \text{and} \quad \omega_{i+1} = \omega_i - \frac{g}{l}\theta_i \Delta t$$

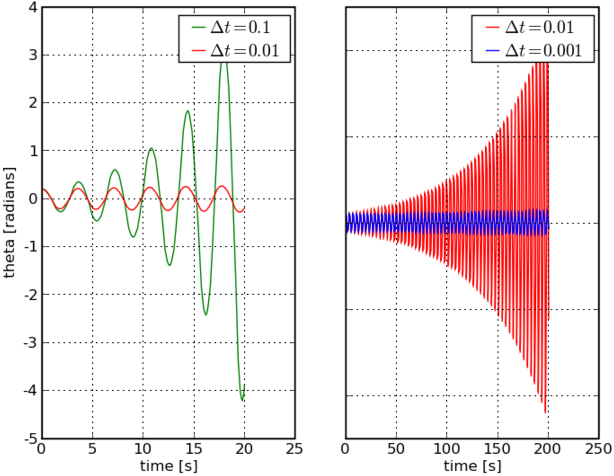
- Problematic results (see next slide):

  Euler method does not respect energy conservation!

Results with the Euler method

- Amplitude increases with time, independent of Δt

(just need to run long enough . . .)



The source of the instability

- Increasing amplitude $\Longleftrightarrow$ increasing energy.
- Check energy conservation. Write energy for step $i+1$ and express through coordinates at i:

$$
\begin{aligned}
E_{i+1} &= \frac{ml^2}{2}\left[\omega_{i+1}^2 + \frac{g}{l}\theta_{i+1}^2\right] \\
&= \frac{ml^2}{2}\left[\left(\omega_i - \frac{g}{l}\theta_i \Delta t\right)^2 + \frac{g}{l}\left(\theta_i + \omega_i \Delta t\right)^2\right] \\
&= E_i + \frac{mgl}{2}\left(\frac{g}{l}\theta_i^2 + \omega_i^2\right)(\Delta t)^2 \,.
\end{aligned}
$$

- Energy non-conservation exists independent of step-size.
- Euler method not good for harmonic motion!

Lessons

- Euler method not good for harmonic motion.
- Okay, fine, but why was it good before? Did we not have also energy non-conservation there?
  Answer: Yes, already there (cannon ball).
  Euler/Runge-Kutta violate energy conservation, but not a problem (small effect).

  > Remember the trajectory of the cannon ball: For larger step-size higher peak in trajectory than for smaller step-size (with roughly the same range).
  >
  > But we could reach stability by making Δt small.

- For harmonic motion, we want to simulate many cycles.
  $\Longrightarrow$ small effects accumulate here.
- There is no single perfect method for all problems.

Improving the Euler method: Euler-Cromer

- Obvious alternative to Euler method: Runge-Kutta methods.
- Here, we try a small change in the algorithm: Instead of

$$\omega_{i+1} = \omega_i - \frac{g}{l}\theta_i \Delta t \quad \text{and} \quad \theta_{i+1} = \theta_i + \omega_i \Delta t$$

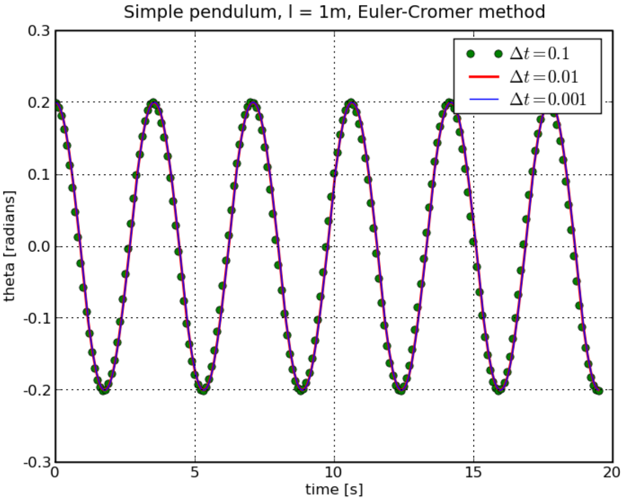
  we use

$$\omega_{i+1} = \omega_i - \frac{g}{l}\theta_i \Delta t \quad \text{and} \quad \theta_{i+1} = \theta_i + \omega_{i+1} \Delta t$$

- Simple calculation: This ensures energy conservation!

Results with Euler-Cromer

- Amplitude stable with time, independent of Δt!



Adding dissipation

- To add dissipation, we take a linear term:

$$\ddot{\theta} = -q\dot{\theta} - \Omega\theta \,.$$

- Can be solved analyticly (see textbooks for methods). There are three regimes:
  1. Under-damped regime: amplitude decays exponentially.

$$\theta(t) = \theta_0 \exp\left(-\frac{qt}{2}\right) \sin\left(\sqrt{\Omega^2 - q^2/4} \cdot t + \phi\right)$$

  2. Over-damped regime: no oscillations

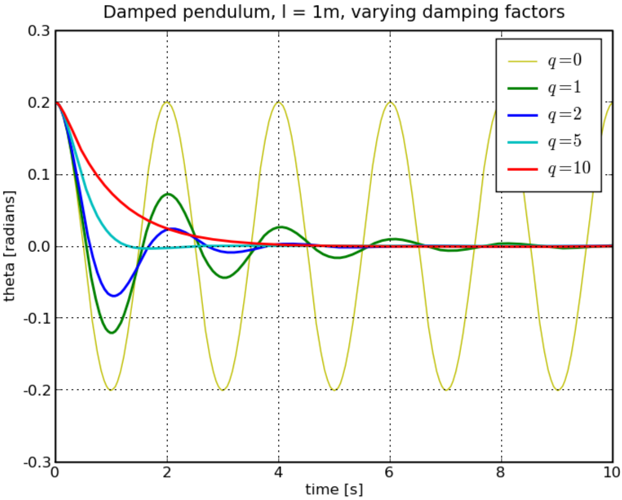
$$\theta(t) = \theta_0 \exp\left[-\left(\frac{q}{2} + \sqrt{q^2/4 - \Omega^2}\right) \cdot t\right]$$

  3. Critically damped regime: Pendulum "crawls" to 0.

$$\theta(t) = (\theta_0 + Ct) \exp\left(-\frac{qt}{2}\right) \,.$$

Results

- Amplitude decreases with time



Adding a driving force

- Sinusoidal force, with amplitude F_D and frequency Ω_D:

$$\ddot{\theta} = -q\dot{\theta} - \Omega\theta + F_D \sin(\Omega_D t)\,.$$

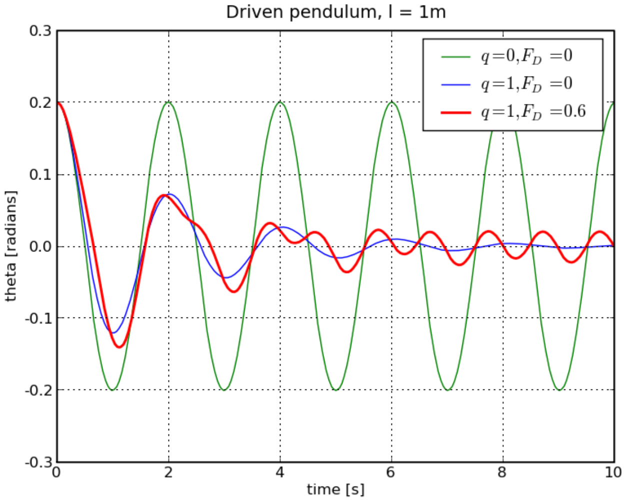
- Effect: Driving force pumps energy into the system, its frequency competes with the eigen-frequency of the pendulum and takes over. Can be solved analyticly. After "swinging in": $\theta(t) = \theta_{\max} \sin(\Omega_D t + \phi)$. The maximal amplitude is given by

$$\theta_{\max} = \frac{F_D}{\sqrt{(\Omega^2 - \Omega_D^2)^2 + (q\Omega_D)^2}}\,.$$

- This leads to an interesting situation when q is small and $\Omega_D \to \Omega$, called resonance.

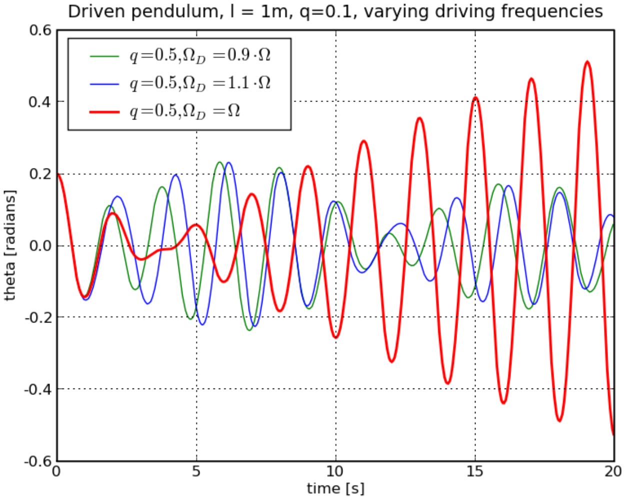
Results

- Driving force with amplitude $\Omega_D = 2\Omega$ takes over!



Emerging resonance

- Driving force with amplitude $\Omega_D \to \Omega$ resonates $\longrightarrow$ amplitude increases (catastrophic w/o dissipation)!



Adding non-linearity

- With a driving force, the pendulum does not necessarily remain in a region of small amplitudes for the oscillations. $\Longrightarrow$ So far, we discussed the mathematical pendulum, i.e. approximation of small angles, replacing $\sin\theta \to \theta$.
- For the description of a realistic pendulum, we reinstate the non-linearity, and we will use $\sin\theta$.
- This will have interesting consequences:
  - In the non-driven, non-dissipative pendulum, the eigen-frequency is not constant any more, but depends on the initial amplitude.
  - Adding a driving force opens the road to chaos.

Summary

- Harmonic motion a very important phenomenon in physics, worthwhile to study in great detail. We chose a pendulum here.
- The Euler method fails to describe harmonic motion properly, due to non-conservation of energy. The Euler-Cromer method rectifies the situation.
- Adding dissipation and driving force adds new-phenomena: dampening and resonances.
- Adding non-linearity paves the road towards deterministic chaos, the subject of next lecture.
- In the homework assignment you'll be asked to implement a full simulation of the pendulum in the Euler-Cromer method, including dissipation, driving force, and non-linearity.