The Euler method

The following discussion of the Euler method considers the case of an ordinary differential equation of order one in one dimension,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x,\, t)\,. \tag{Euler Method.1}$$

A generalisation to an arbitrary number of dimensions n is straightforward, and the emerging algorithm will be shown at the end.

Basic idea

The idea behind the Euler method consists of using the Taylor expansion of $x(t)$ around t with interval Δt,

$$x(t+\Delta t) = x(t) + \frac{\mathrm{d}x(t)}{\mathrm{d}t}\,\Delta t + \frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2}\,\frac{(\Delta t)^2}{2} + \cdots + \frac{\mathrm{d}^nx(t)}{\mathrm{d}t^n}\,\frac{(\Delta t)^n}{n} + \ldots\,. \tag{Euler Method.2}$$

Assuming that the function and its derivative is sufficiently smooth and that Δt is sufficiently small, $x(t)$ can be propagated to $x(t+\Delta t)$ with sufficiently high precision. Ignoring terms of order $calO[(\Delta t)^2]$ or higher, thus

$$x(t_{i+1}) \approx x(t_i) + \frac{\mathrm{d}x(t_i)}{\mathrm{d}t}\,\Delta t = x(t_i) + f(x,\, t)\Delta t\,, \tag{Euler Method.3}$$

where Eq. () has been used.

Error estimate

Comparing this equation, Eq. (), with the Taylor expansion shows that the Euler method essentially constitutes a local first order approximation in each step. If, starting from some starting time t_{ini} the solution at a later time t_{fin} is to be calculated, then the Euler step must be repeated

$$n_{\mathrm{steps}} = \frac{|t_{\mathrm{fin}} - t_{\mathrm{ini}}|}{\Delta t} \tag{Euler Method.4}$$

times. Since in each individual step the error is of order $(\Delta t)^2$, the total error of this method scales linearly with Δt. Therefore, employing the proper terminology, the Euler method globally is said to be a zeroth order approximation. Thus, in order to reduce the error at t_{fin}, you would have to reduce Δt, but the compuational cost would rise linearly with the numerical accuracy.

It should also be noted here that choosing the time steps too large can lead to unstable or meaningless results.

Implementation

The algorithm for the Euler method basically reads:

1. Initialise $x(t_{\mathrm{ini}}) = x_{\mathrm{ini}}$.
2. Use Eq. () to propagate from $x(t_i)$ to $x(t_{i+1})$, where $t_{i+1} = t_i + \Delta t$.
3. Repeat step 2, until $t_{i+1} \geq t_{\mathrm{end}}$.