QCD & Monte Carlo Event Generators

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PART I: Basics of QCD

PART II: Monte Carlo & Perturbative QCD

PART III: Monte Carlo & Non–Perturbative QCD

PART IV: Towards Precision Simulations

ROUND I: QCD BASICS

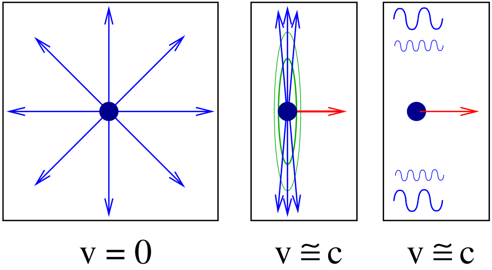
QCD BASICS

UNDERLYING IDEAS

Contents

An electromagnetic analogy

- consider a charge Z moving at constant velocity v



- at $v = 0$: radial E field only
- at $v = c$: B field emerges: $\vec{E} \perp \vec{B}$, $\vec{B} \perp \vec{v}$, $\vec{E} \perp \vec{v}$,
  energy flow $\sim$ Poynting vector $\vec{S} \sim \vec{E} \times \vec{B}$, $\parallel \vec{v}$
- approximate classical fields by "equivalent quanta": photons

- spectrum of photons:

(in dependence on energy ω and transverse distance $b_\perp$)

$$dn_\gamma = \frac{Z^2\alpha}{\pi}\cdot\frac{d\omega}{\omega}\cdot\frac{db_\perp^2}{b_\perp^2} \xrightarrow{\text{electron}(Z=1)} \frac{\alpha}{\pi}\cdot\frac{d\omega}{\omega}\cdot\frac{db_\perp^2}{b_\perp^2}$$

- Fourier transform to transverse momenta $k_\perp$:

$$dn_\gamma = \frac{\alpha}{\pi}\cdot\frac{d\omega}{\omega}\cdot\frac{dk_\perp^2}{k_\perp^2}$$

note: divergences for $k_\perp \to 0$ (collinear) and $\omega \to 0$ (soft)

- therefore: Fock state for lepton = superposition (coherent):

$$|e\rangle_{\text{phys}} = |e\rangle + |e\gamma\rangle + |e\gamma\gamma\rangle + |e\gamma\gamma\gamma\rangle + \ldots$$

photon fluctuations will “recombine”

- lifetime of electron–photon fluctuations: $e(P) \to e(p) + \gamma(k)$
- first estimate: use uncertainty relation and Lorentz time dilation
  - $P^2 = (p+k)^2 = M^2_{\rm virt}$ the virtual mass of the incident electron
  - life time = life time in rest frame · time dilation

$$\tau \sim \frac{1}{M_{\rm virt}} \cdot \frac{E}{M_{\rm virt}} = \frac{E}{(p+k)^2} \sim \frac{E}{2Ek(1-\cos\theta)} \approx \frac{k}{k^2 \sin^2\theta/2} \approx \frac{\omega}{k_\perp^2}$$

- second estimate: use uncertainty relation and assume only photon off-shell
  - energy balance of photon

$$P^2 = 2p\cdot k + k^2\,, \quad \text{therefore}\;\; k^2 \approx -k_\perp^2 \approx -2p\cdot k < 0\,.$$

  - assume photon momentum to be $k^\mu = (\omega, \vec{k}_\perp, k_\parallel)$,
    shift in energy for photon going on-shell: $\delta\omega \sim k_\perp^2/\omega$, therefore

$$\tau_\gamma \sim \frac{1}{\delta\omega} \approx \frac{\omega}{k_\perp^2} \approx \frac{\omega}{\omega^2 \sin^2\theta} \approx \frac{1}{\omega\theta^2}$$

- lifetime larger with smaller transverse momentum

(i.e. with larger transverse distance)

DGLAP equations for QED

(Dokshitser–Gribov–Lipatov–Altarelli–Parisi Equations)

- define probability to find electron or photon in electron:

$$\begin{aligned} \text{at LO in } \alpha(\text{noemission}): \;\; \ell(x,\, k_\perp^2) &= \delta(1-x) \\ \text{and} \;\; \gamma(x,\, k_\perp^2) &= 0 \end{aligned}$$

(introduced $x =$ energy fraction w.r.t. physical state)

- including emissions:
    - probabilities change
    - energy fraction ξ of lepton parton w.r.t. the physical lepton object reduced by some fraction $z = x/\xi$
    - reminder: differential of photon number w.r.t. $k_\perp^2$:

$$\mathrm{d}n_\gamma \;=\; \frac{\alpha}{\pi}\frac{\mathrm{d}k_\perp^2}{k_\perp^2}\frac{\mathrm{d}\omega}{\omega} \;\leftrightarrow\; \frac{\mathrm{d}n_\gamma}{\mathrm{d}\log k_\perp^2} \;=\; \frac{\alpha}{\pi}\frac{\mathrm{d}x}{x}$$

- evolution equations (trivialised)

$$\frac{\mathrm{d}\ell(x,\,k_\perp^2)}{\mathrm{d}\log k_\perp^2} = \frac{\alpha(k_\perp^2)}{2\pi}\int\limits_x^1 \frac{\mathrm{d}\xi}{\xi}\mathcal{P}_{\ell\ell}\left(\frac{x}{\xi},\,\alpha(k_\perp^2)\right)\ell(\xi,\,k_\perp^2)$$

$$\frac{\mathrm{d}\gamma(x,\,k_\perp^2)}{\mathrm{d}\log k_\perp^2} = \frac{\alpha(k_\perp^2)}{2\pi}\int\limits_x^1 \frac{\mathrm{d}\xi}{\xi}\mathcal{P}_{\gamma\ell}\left(\frac{x}{\xi},\,\alpha(k_\perp^2)\right)\ell(\xi,\,k_\perp^2)\,.$$

- $k_\perp^2$ plays the role of "resolution parameter"
- the $\mathcal{P}_{ab}(z)$ are the splitting functions, encoding quantum mechanics of the "splitting cross section", for example (at LO)

$$\mathcal{P}_{\ell\ell}(z) = \left(\frac{1+z^2}{1-z}\right)_+ + \frac{3}{2}\delta(1-z)$$

- if $\gamma\to\ell\bar{\ell}$ splittings included, have to add entries/splitting functions into evolution equations above

QED Initial and Final State Radiation

- physical interpretation:
  equivalent quanta = quantum manifestation of accompanying fields
- in absence of interaction: recombination enforced by coherence
- but: hard interaction possibly "kicks out" quantum
  $\longrightarrow$ coherence broken
  $\longrightarrow$ equivalent (virtual) quanta become real
  $\longrightarrow$ emission pattern unravels
- alternative idea:
  initial state radiation of photons off incident electron

- consider final state radiation in $\gamma^* \to \ell\bar{\ell}$
  (electron velocities/momenta labelled as v and v'/p and p')
- classical electromagnetic spectrum from radiation function:

(this is from Jackson or any other reasonable book on ED)

$$\frac{\mathrm{d}^2 I}{\mathrm{d}\omega \mathrm{d}\Omega} = \frac{e^2}{4\pi^2}\left|\vec{\epsilon}^{\,*}\cdot\left(\frac{\vec{v}}{1-\vec{v}\cdot\vec{n}}-\frac{\vec{v}'}{1-\vec{v}'\cdot\vec{n}}\right)\right|^2,$$

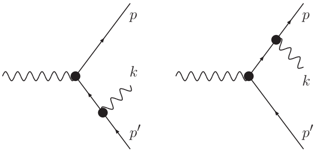
with ϵ the polarisation vector and $\vec{n}(\Omega)$ the direction of the radiation

- recast with four–momenta, equivalent photon spectrum:

$$\begin{aligned}\mathrm{d}N &= \frac{\mathrm{d}^3 k}{(2\pi)^3 2k_0}\,\frac{\alpha}{\pi}\left|\epsilon^*_\mu\left(\frac{p^\mu}{p\cdot k}-\frac{p'^\mu}{p'\cdot k}\right)\right|^2 \\ &= \frac{\mathrm{d}^3 k}{(2\pi)^3 2k_0}\,\frac{\alpha}{\pi}\left|\mathrm{W}_{pp';k}\right|^2\end{aligned}$$

with the eikonal $\mathrm{W}_{pp';k}$

- repeat exercise in QFT, Feynman diagrams:



$$\begin{aligned}\mathcal{M}_{X\to e^+e^-\gamma} &= e\bar{u}(p)\left[\Gamma\,\frac{\not{p}'-\not{k}}{(p'-k)^2}\,\gamma^\mu-\gamma^\mu\,\frac{\not{p}+\not{k}}{(p+k)^2}\,\Gamma\right]u(p')\epsilon^*_\mu(k)\\ \stackrel{\text{soft}}{\longrightarrow}\quad e\epsilon^*_\mu(k)\left[\frac{p^\mu}{p\cdot k}-\frac{p'^\mu}{p'\cdot k}\right]\bar{u}(p')\Gamma u(p) &= e\mathcal{M}_{X\to e^+e^-\gamma}\cdot\mathrm{W}_{pp';k}\end{aligned}$$

- manifestation of Low's theorem:
  soft radiation independent of spin ($\to$ classical)

(radiation decoposes into soft, classical part with logs – i.e. dominant – and hard collinear part)

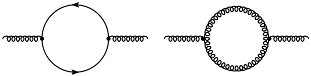
- quantum effect due to loops: couplings change with scale
- "running" for QCD
  "crawling" for QED
- running driven by β–function

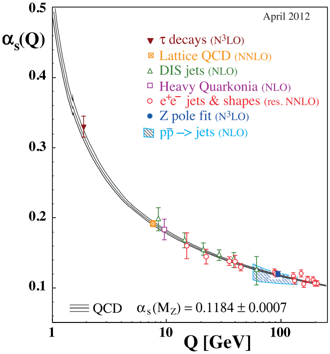
$$\beta(\alpha_s) = \mu_R^2 \frac{\partial \alpha_s(\mu_R^2)}{\partial \mu_R^2} = \frac{\beta_0}{4\pi}\alpha_s^2 + \frac{\beta_1}{(4\pi)^2}\alpha_s^3 + \dots$$

with

$$\beta_0 = \frac{11}{3}C_A - \frac{4}{3}T_R n_f$$

$$\beta_1 = \frac{34}{3}C_A^2 - \frac{20}{3}C_A T_R n_f - 4\,C_F T_R n_f$$





- Casimir operators in the fundamental and adjoint representation:

$$C_F = \frac{N_c^2-1}{2N_c} \quad \text{and} \quad C_A = N_c$$

  with $N_c = 3$ colours and $T_R = 1/2$.
- n_f = the number of (quark) flavours
- the Casimirs correspond to quark and gluon colour charges
- explicit expression for strong coupling

$$\alpha_s(\mu_R^2) \equiv \frac{g_s^2(\mu_R^2)}{4\pi} = \frac{1}{\frac{\beta_0}{4\pi}\log\frac{\mu_R^2}{\Lambda_{\rm QCD}{}^2}}$$

  with $\Lambda_{\rm QCD}$ the Landau pole of QCD, $\Lambda_{\rm QCD} \approx 250{\rm MeV}$.

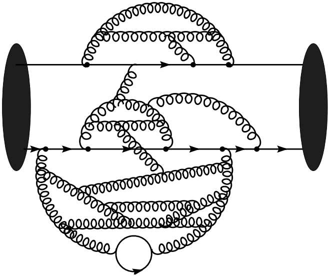
- result of running strong coupling:
  infrared slavery & asymptotic freedom
  (two sides of the same coin)
- at low scales: bound states only, no free QCD quanta,
  confinement
- at high scales: free QCD quanta
  perturbative expansion in α_s
- trivial particle spectrum:

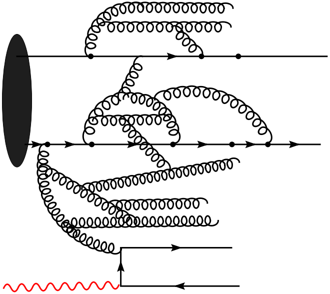
$$\mathrm{d}n_g^{q,g} = C_{q,g} \cdot \frac{\alpha_s(k_\perp^2)}{\pi} \cdot \frac{\mathrm{d}\omega}{\omega} \cdot \frac{\mathrm{d}k_\perp^2}{k_\perp^2}$$

but now the gluons carry charge!
$\Longrightarrow$ massive proliferation of soft particles

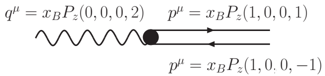
(from gluons emitted by gluons and from α_s at small $k_\perp$)

- picture of hard interaction: coherence broken by interaction
- hadron does not recombine but desintegrates





- probing the deep structure of hadrons with photon:
  deep–inelastic scattering (DIS)
- condition for DIS on protons $Q \gg m_P \approx 1/R_P$ for scale Q
  (typically the virtual mass of the photonic probe)
- using Breit frame:



$$\begin{aligned} P^\mu &= (P_0, \vec{0}, P_z) \text{ with } P_z = \sqrt{P_0^2 - m_P^2} \approx P_0 \\ q^\mu &= (0, \vec{0}, -q_z) \text{ with } Q^2 = -q^2 = q_z^2 \,. \end{aligned}$$

- introducing Bjorken variable ("Bjorken–x")

$$x_B = -\frac{q^2}{2P\cdot q} = \frac{q_z^2}{2P_z q_z}$$

- interaction time between photon and proton from longitudinal wavelength of photon $\tau_{\rm int} \sim \lambda_z \sim 1/q_z$
- parton wavelength must be as large as photon wavelength for them to "see" each other: $\Longrightarrow \quad p_z = q_z$
- lifetime of parton $\tau_{\rm life} \sim p_z/p_\perp^2 \;\; \geq \;\; \tau_{\rm int}$

  must be larger than interaction time!

  let the parton live before it interacts!

- therefore: $p_z \geq p_\perp$ for interactionto happen
- if this holds: collision of two quasi-free particles

  collinear factorisation

  into hard process and **independent** proton $\to$ parton transitions

- then cross section proportional to probabilities to find partons in proton, given by the parton distribution functions $f_{q/p}$

$$\sigma_{\rm DIS} \;\sim\; \sum_q e_q^2 f_{q/p}(x,\, Q^2)\,,$$

Hadrons in initial state: DGLAP equations of QCD

- similar to QED case:
  define probabilities (at LO) to find a parton q – quark or gluon – in hadron h at energy fraction x and resolution parameter/scale Q:
  parton distribution function (PDF) $f_{q/h}(x,\, Q^2)$
- scale-evolution of PDFs: DGLAP equations

$$\frac{\partial}{\partial \log Q^2}\left(\begin{array}{c} f_{q/h}(x,\, Q^2) \\ f_{g/h}(x,\, Q^2) \end{array}\right)$$
$$= \frac{\alpha_s(Q^2)}{2\pi}\int\limits_x^1 \frac{\mathrm{d}z}{z}\left(\begin{array}{cc} \mathcal{P}_{qq}\left(\frac{x}{z}\right) & \mathcal{P}_{qg}\left(\frac{x}{z}\right) \\ \mathcal{P}_{gq}\left(\frac{x}{z}\right) & \mathcal{P}_{gg}\left(\frac{x}{z}\right) \end{array}\right)\left(\begin{array}{c} f_{q/h}(z,\, Q^2) \\ f_{g/h}(z,\, Q^2) \end{array}\right),$$

- QCD splitting functions:

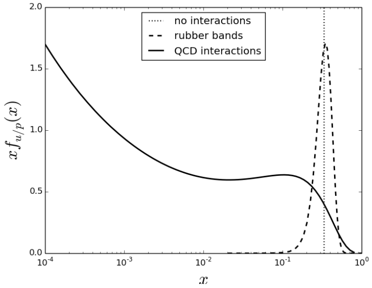
$$\begin{aligned}
\mathcal{P}^{(1)}_{qq}(x) &= C_F\left[\frac{1+x^2}{(1-x)_+}+\frac{3}{2}\delta(1-x)\right] = \left[P^{(1)}_{qq}(x)\right]_+ + \gamma^{(1)}_q\delta(1-x)\\
\mathcal{P}^{(1)}_{qg}(x) &= T_R\left[x^2+(1-x)^2\right] = P^{(1)}_{qg}(x)\\
\mathcal{P}^{(1)}_{gq}(x) &= C_F\left[\frac{1+(1-x)^2}{x}\right] = P^{(1)}_{gq}(x)\\
\mathcal{P}^{(1)}_{gg}(x) &= 2C_A\left[\frac{x}{(1-x)_+}+\frac{1-x}{x}+x(1-x)\right]\\
&\quad +\frac{11C_A-4n_fT_R}{6}\delta(1-x) = \left[P^{(1)}_{gg}(x)\right]_+ + \gamma^{(1)}_g\delta(1-x)\,.
\end{aligned}$$

- remark: IR regularisation by $+$–prescription & terms $\sim\delta(1-x)$ from physical conditions on splitting functions

(flavour conservation for $q\to qg$ and momentum conservation for $g\to gg,\,q\bar{q}$)

Parton distribution functions

- simple idea: proton $= |uud\rangle$
  (valence quarks) only
- naively: no interactions
  $\longrightarrow f_{u,d/p} \sim \delta\left(x - \frac{1}{3}\right)$
- elastic interactions
  $\longrightarrow$ Gaussian smearing
- strong interactions:
  develop "sea" = soft partons
  will depend on resolution scale
  remember: $\mathrm{d}n \propto \log\omega \log k_\perp^2$



- in fact, due to $g \to gg$, sea increaes much faster,

$$f_{\mathrm{sea}/p}(x,\ Q^2) \sim x^{-\lambda}\,, \quad \lambda \approx 1\,.$$

- however, sum rules put constraints,
  example: momentum sum rule

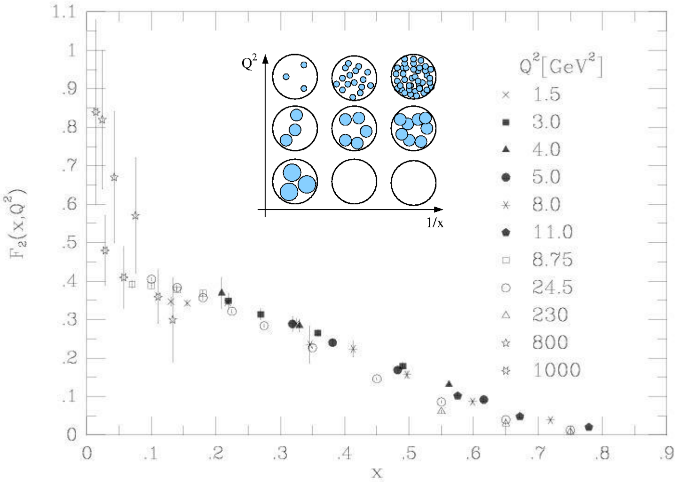
$$\int\limits_0^1 \mathrm{d}x\, x \sum_i f_{i/h}(x,\, \mu^2) = 1 \;\; \forall \mu^2 \text{ and for all hadrons } h\,,$$

- also flavour sum rules, for protons:

$$\int\limits_0^1 \mathrm{d}x \left[f_{u/p}(x,\, \mu^2) - f_{\bar{u}/p}(x,\, \mu^2)\right] = 2$$

$$\int\limits_0^1 \mathrm{d}x \left[f_{d/p}(x,\, \mu^2) - f_{\bar{d}/p}(x,\, \mu^2)\right] = 1$$

$$\int\limits_0^1 \mathrm{d}x \left[f_{q/p}(x,\, \mu^2) - f_{\bar{q}/p}(x,\, \mu^2)\right] = 0 \;\text{ for } q \in \{s,\, c,\, b\}\,,$$


$F_2(x,Q^2)$
x
Q^2
1/x
Q^2[GeV^2]
1.5
3.0
4.0
5.0
8.0
11.0
8.75
24.5
230
800
1000
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1
1.1

Hadrons in the final state

- consider QCD final state radiation
- pattern for $q \to qg$ similar to $\ell \to \ell\gamma$ in QED:

$$
\begin{aligned}
\mathrm{d}w^{q\to qg} &= \frac{\alpha_s(k_\perp^2)}{2\pi}\, C_F\, \frac{\mathrm{d}k_\perp^2}{k_\perp^2}\, \frac{\mathrm{d}\omega}{\omega}\, \left[1+\left(1-\frac{\omega}{E}\right)^2\right] \\
&\stackrel{\omega=E(1-z)}{=} \frac{\alpha_s(k_\perp^2)}{2\pi}\, C_F\, \frac{\mathrm{d}k_\perp^2}{k_\perp^2}\, \mathrm{d}z\, \frac{1+z^2}{1-z} = \frac{\alpha_s(k_\perp^2)}{2\pi}\, C_F\, \frac{\mathrm{d}k_\perp^2}{k_\perp^2}\, \mathrm{d}z\, P_{qg}^{(1)}(z)\,.
\end{aligned}
$$

- divergent structures for:

$z \to 1$ (soft divergence)	$\longleftrightarrow$	infrared/soft logarithms
$k_\perp^2 \to 0$ (collinear/amss divergence)	$\longleftrightarrow$	collinear logarithms

- cut regularise with cut-off $k_{\perp,\mathrm{min}} \sim 1\mathrm{GeV} > \Lambda_{\mathrm{QCD}}$

- find two perturbative regimes:
  - a regime of jet production, where $k_\perp \sim k_\| \sim \omega \gg k_{\perp,\mathrm{min}}$ and emission probabilities scale like $w \sim \alpha_s(k_\perp) \ll 1$; and
  - a regime of jet evolution, where $k_{\perp,\mathrm{min}} \leq k_\perp \ll k_\| \leq \omega$ and therefore emission probabilities scale like $w \sim \alpha_s(k_\perp) \log^2 k_\perp^2 \gtrsim 1$.
- in jet production:
  standard fixed–order perturbation theory
- in jet evolution regime,
  perturbative parameter not α_s any more
  but rather towers of $\exp\left[\alpha_s \log k_\perp^2 \log k_\|\right]$
- induces counting of leading logarithms (LL), $\alpha_s L^{2n}$,
  next-to leading logarithms (NLL), $\alpha_s L^{2n-1}$, etc.

- consider spatio-temporal structure in classical QED case
  - assume a charge comes into existence at $t = 0$ ith $v = 0$: in its rest frame radial $\vec{E}$ spreads out in sphere $r' \leq t'$
  - assume charge moves with $v \to 1$ and Lorentz factor E/m: then in lab frame field at radial distance $r_\perp$ will arrive at $t = \gamma t' = E r_\perp / m$
- translate to classical QCD:
  - light quarks with constituent mass $m \approx \Lambda_{\rm QCD} \approx 1/R$ or $m = m_Q$ for heavy quarks

(assume here typical hadron radius R)

  - identify $r_\perp$ with typical hadronic size R
  - then: hadronization time

$$t^{(\rm had)} \approx \begin{cases} ER^2 & \text{for light quarks} \\ \dfrac{ER}{m_Q} & \text{for heavy quarks.} \end{cases}$$

- repeat exercise in quantum mechanics
- confining forces assoicated with gluons with $k \approx k_\perp \approx k_\parallel \approx 1/R \approx m$ in hadronic rest frame
- they emerge at (as before with Lorentz factor E/m)

$$t^{(\mathrm{had})} \approx \frac{k_\parallel}{k_\perp^2} \approx \frac{ER}{m} \approx k_\parallel R^2 \,,$$

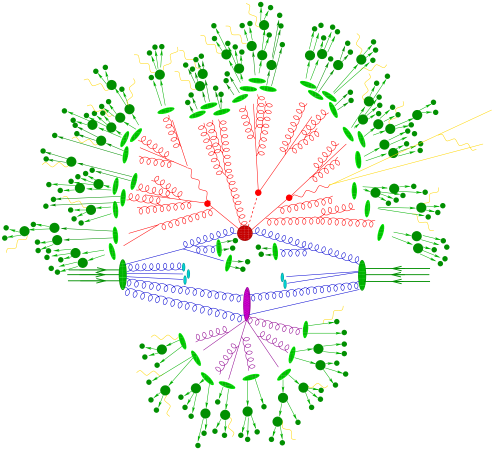
- demand hadronization time $\geq$ formation time:

$$t^{(\mathrm{form})} \approx \frac{k_\parallel}{k_\perp^2} \leq k_\parallel R^2 \approx t^{(\mathrm{had})}$$

- therefore $k_\perp \geq 1/R = \mathcal{O}\left(\mathrm{few}\,\Lambda_{\mathrm{QCD}}\right)$

- therefore: breakdown of perturbative picture at scales/transverse momenta $\mathcal{O}(\text{few}\,\Lambda_{\text{QCD}})$
- "gluers" replace gluons
- transition to bound states (phase transition)
- no first–principle understanding: $\Longrightarrow$ models

Summary



Detector Event

Hadron-hadron collision

Hard

Parton-parton collision

hardest interaction: the "*signal process*"
description: fixed-order perturbation theory (Chapter 3)
important input: PDFs, factorisation theorem (Chapter 6)

Initial- & final-state radiation

emission of secondary particles
description: resummation or parton shower (Chapter 5)
important: matching to fixed order & logarithmic precision
note: final state radiation is perturbative part of "*fragmentation*"

Underlying event

multiple parton-parton interactions
description: $2 \to 2$ parton-parton scatterings in perturbation theory
or other QCD-inspired models (Chapter 7)
treated only in full simulations, comes with further parton showering

Hadronisation

transition of partons to primordial hadrons
description: fragmentation functions in calculations &
non-perturbative models in simulations (Chapter 7)

Hadron decays

decay cascades of the primordial hadrons
description: data, effective theories, symmetries, models (Chapter 7)

Soft

Pile-up

Multiple hadron-hadron collisions, typically soft.
Description: Models, often QCD-inspired (Chapter 7)

QCD BASICS

QCD @ FIXED ORDER

Contents

Master Formula for Cross Sections at Hadron Colliders

$$\sigma_{2\to n} = \sum_{a,b}\int_0^1 \mathrm{d}x_a \mathrm{d}x_b\, f_{a/h_1}(x_a,\mu_F)\, f_{b/h_2}(x_b,\mu_F)\, \hat{\sigma}_{ab\to n}(\mu_F,\mu_R)$$

$$= \sum_{a,b}\int_0^1 \overbrace{\mathrm{d}x_a \mathrm{d}x_b}^{\text{IS phase space}}\; \overbrace{f_{a/h_1}(x_a,\mu_F) f_{b/h_2}(x_b,\mu_F)}^{\text{parton distribution functions}}$$

$$\times \underbrace{\frac{1}{2\hat{s}}}_{\text{incoming flux}} \times \underbrace{\int \mathrm{d}\Phi_n}_{\text{FS phase space}}\; \underbrace{|\mathcal{M}_{ab\to n}|^2(\Phi_n;\mu_F,\mu_R)}_{\text{amplitude squared}}.$$

- inspect N–particle final state phase space integral

$$\mathrm{d}\Phi_N = \left[\prod_{i=1}^{N} \overbrace{\frac{\mathrm{d}^4 p_i}{(2\pi)^4}}^{\text{Lorentz-invariant volume}} \overbrace{(2\pi)\delta(p_i^2 - m_i^2)}^{\text{particle on-shell}} \overbrace{\Theta(E_i)}^{\text{with positive energy}}\right]$$

$$\times \underbrace{(2\pi)^4\delta^4\left(x_a P_{h_1} + x_b P_{h_2} - \sum_{i=1}^{N} p_i\right)}_{\text{total 4-momentum conservation}}$$

- note: incoming parton momenta are given by Bjorken-x times hadron momenta:

$$p_{a,b} = x_{a,b} P_{h_{1,2}}$$

Example: W Production at Leading Order

- test case for the development of ideas: W production
- matrix element/amplitude for real W production:

$$\mathcal{M}_{u\bar{d}\to W^+} = -\frac{iV_{ud}g_W\delta_{ij}}{\sqrt{2}}\bar{d}_i(p_2)\gamma^\mu\frac{1-\gamma_5}{2}u_j(p_1)\epsilon_\mu^{(W)},$$

- CKM matrix element V_{ud}, weak coupling g_W (from Fermi constant), subscripts i, j indicate quark colours
- squared amplitude (summing/averaging over FS/IS internal d.o.f.)

internal d.o.f. = polarisations, colours, ...

$$\begin{aligned}\bar{\sum}|\mathcal{M}|^2 &= \frac{3}{4\cdot 9}\frac{|V_{ud}|^2g_W^2}{2}\mathrm{Tr}\left[\not{p}_2\gamma^\mu\not{p}_1\gamma^\nu\frac{1-\gamma_5}{2}\right]\left[-g_{\mu\nu}+\frac{Q_\mu Q_\nu}{m_W^2}\right]\\ &= \frac{|V_{ud}|^2g_W^2}{12}Q^2 = \frac{|V_{ud}|^2g_W^2}{12}m_W^2\,.\end{aligned}$$

- add in one–particle final state phase space

$$\begin{aligned}\hat{\sigma}^{(\mathrm{LO})}_{u\bar{d}\to W^+} &= \frac{1}{2\hat{s}}\int\frac{\mathrm{d}^4 p_W}{(2\pi)^4}(2\pi)^4\delta^4(p_u+p_{\bar{d}}-p_W)(2\pi)\delta(p_W^2-m_W^2)|\mathcal{M}|^2\\ &= \frac{\pi\delta(\hat{s}-m_W^2)}{\hat{s}}|\mathcal{M}|^2 = \frac{\pi\delta(\hat{s}-m_W^2)}{\hat{s}}\,\frac{g_W^2|V_{ud}|^2 m_W^2}{12}\,.\end{aligned}$$

- therefore, hadronic cross section:

$$\sigma^{(\mathrm{LO})} = \int\limits_0^1 \mathrm{d}x_u\mathrm{d}x_{\bar{d}}\sum_{u,\bar{d}} f_{u/h_1}(x_u,\,\mu_F)\,f_{\bar{d}/h_2}(x_{\bar{d}},\,\mu_F)\,\hat{\sigma}^{(\mathrm{LO})}_{u\bar{d}\to W^+}$$

- with the integral over the energy fractions rewritten as

$$\mathrm{d}x_u \mathrm{d}x_{\bar{d}} = \frac{\mathrm{d}\hat{s}}{s}\, \mathrm{d}y_W \,.$$

from c.m.-energy squared $\hat{s}$ and rapidity y

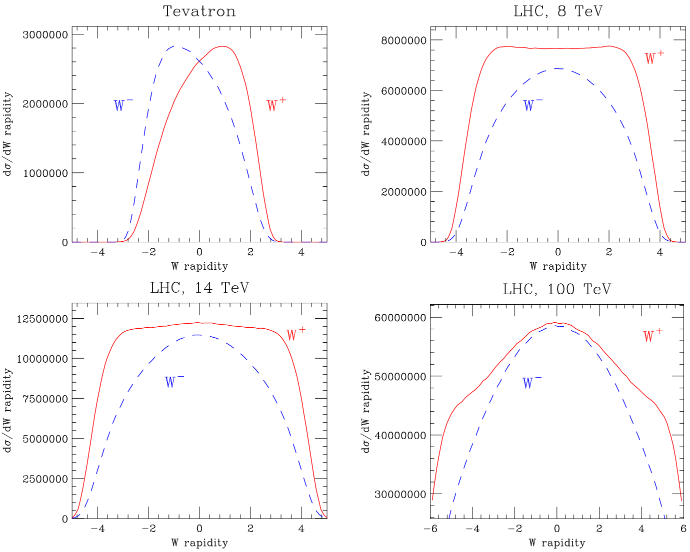
$$\hat{s} \;=\; x_u x_{\bar{d}} s \quad \text{and} \quad y_W \;=\; y_{\mathrm{c.m.}} \;=\; \frac{1}{2}\log\frac{x_u}{x_{\bar{d}}}\,.$$

- limits on rapidity from $x_u \;=\; m_W^2/(s x_{\bar{d}}) \geq m_W^2/s$ since $x_{\bar{d}} \leq 1$:

$$|y_W| \;\leq\; y_{\max} \;=\; \frac{1}{2}\log\frac{s}{m_W^2}\,.$$

- therefore cross section given by

$$\sigma^{(\mathrm{LO})} \;=\; \frac{\pi g_W^2\, |V_{ud}|^2}{12s} \int\limits_{-y_{\max}}^{y_{\max}} \mathrm{d}y_W \sum_{u,\bar{d}} f_{u/h_1}(x_u,\,\mu_F)\, f_{\bar{d}/h_2}(x_{\bar{d}},\,\mu_F)\,.$$


Tevatron
LHC, 8 TeV
LHC, 14 TeV
LHC, 100 TeV
dσ/dW rapidity
W rapidity
W⁻
W⁺
3000000
2000000
1000000
0
8000000
6000000
4000000
2000000
12500000
10000000
7500000
5000000
2500000
60000000
50000000
40000000
30000000
−6
−4
−2
0
2
4
6

- including decays $W^+\to\ell^+\nu_\ell$:

$$\begin{aligned}\overline{\sum}|\mathcal{M}_{u\bar{d}\to\ell^+\nu_\ell}|^2 &= \frac{3}{9\cdot 4}\,\frac{|V_{ud}|^2 g_W^4}{4}\,\frac{\left(g_{\mu\nu}-\frac{Q_\mu Q_\nu}{m_W^2}\right)\left(g_{\rho\sigma}-\frac{Q_\rho Q_\sigma}{m_W^2}\right)}{(Q^2-m_W^2)^2+m_W^2\Gamma_W^2}\\ &\quad\times \mathrm{Tr}\left[\not{p}_{\nu_\ell}\gamma^\nu\not{p}_{\bar{\ell}}\gamma^\sigma\frac{1-\gamma_5}{2}\right]\mathrm{Tr}\left[\not{p}_{\bar{d}}\gamma^\mu\not{p}_u\gamma^\rho\frac{1-\gamma_5}{2}\right]\\ &= \frac{|V_{ud}|^2 g_W^4}{12}\,\frac{\hat{t}^2}{(Q^2-m_W^2)^2+m_W^2\Gamma_W^2}\end{aligned}$$

- used Mandelstam variables

$$\hat{s} = Q^2 = (p_u+p_{\bar{d}})^2\,,\quad \hat{t} = (p_u-p_{\ell^+})^2\,,\quad \text{and}\quad \hat{u} = (p_u-p_\nu)^2$$

- for phase space integration, rewrite $\hat{t}$ with angle θ^* in c.m.-system:

$$\hat{t} = -2p_u\cdot p_\ell \xrightarrow{\text{cms}} -\frac{\hat{s}}{2}(1-\cos\theta^*)\,,$$

- evaluate final state phase space integral

$$\int \mathrm{d}\Phi_{\ell\nu} = \int \frac{\mathrm{d}^4 p_\ell}{(2\pi)^4}(2\pi)\delta(p_\ell^2)\,\frac{\mathrm{d}^4 p_\nu}{(2\pi)^4}(2\pi)\delta(p_\nu^2)(2\pi)^4\delta^4(p_u+p_d-p_\ell-p_\nu)$$
$$= \frac{1}{32\pi^2}\int \mathrm{d}^2\Omega_\ell^*\,,$$

- therefore

$$\hat{\sigma}^{(LO)} = \frac{1}{2\hat{s}}\int \mathrm{d}\Phi|\mathcal{M}|^2$$
$$= \frac{g_W^4\,|V_{ud}|^2}{12\cdot 2\hat{s}}\int\limits_{-1}^{1}\frac{2\pi\mathrm{d}\cos\theta^*}{4\cdot 32\pi^2}\,\frac{\hat{s}^2(1-\cos\theta^*)^2}{\left[(\hat{s}-m_W^2)^2+m_W^2\Gamma_W^2\right]}$$
$$= \frac{g_W^4\,|V_{ud}|^2}{576\pi}\,\frac{\hat{s}}{(\hat{s}-m_W^2)^2+m_W^2\Gamma_W^2}\,,$$

- include initial state (PDFs):

$$\sigma^{(\mathrm{LO})}_{h_1 h_2\to\nu_\ell\bar\ell} = \frac{g_W^4\,\left|V_{ud}\right|^2}{576\pi}\int \mathrm{d}y_W\,\mathrm{d}\hat s\left[\frac{1}{\left[\hat s - m_W^2\right]^2 + m_W^2\Gamma_W^2}\right.$$

$$\left.\times\sum_{u,\bar d} x_u f_{u/h_1}(x_u,\,\mu_F)\, x_{\bar d} f_{\bar d/h_2}(x_{\bar d},\,\mu_F)\right]\,,$$

- defining lepton rapidity w.r.t. parton c.m.-system, $\hat{y}_\ell$ and relate to lab frame through

$$\begin{aligned}\hat{y}_\ell &= y_\ell - y_W\\ \log\cot\frac{\theta^*}{2} &= \frac{1}{2}\log\frac{1+\cos\theta^*}{1-\cos\theta^*}\end{aligned}$$

- this allows to write differentially w.r.t. the lepton kinematics:

$$\frac{\mathrm{d}\sigma_{h_1h_2\to\bar{\ell}\nu_\ell}}{\mathrm{d}y_{\bar{\ell}}} = \int \mathrm{d}x_u\mathrm{d}x_{\bar{d}} f_{u/h_1}(x_u,\,\mu_F) f_{\bar{d}/h_2}(x_{\bar{d}},\,\mu_F)\frac{\sin^2\theta^*\mathrm{d}\hat{\sigma}_{u\bar{d}\to\bar{\ell}\nu_\ell}}{\mathrm{d}\cos\theta^*}\,.$$

- θ^* is angle of lepton w.r.t. u quark
- remember: $\mathrm{d}\hat{\sigma}^{(LO)} \sim (1-\cos\theta^*)^2$

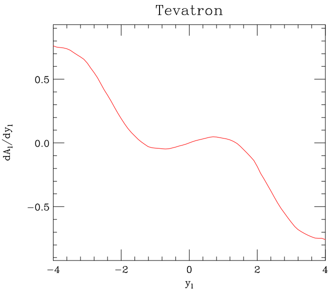
vanishes if $\bar{\ell}/\ell$ points into direction of $u/\bar{d}$,
and is maximal for $\bar{\ell}/\ell$ in direction of $\bar{d}/u$

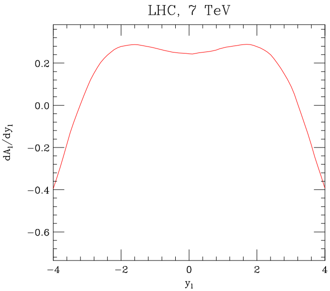
- W asymmetry not directly measurable
  $\Longrightarrow$ must construct lepton observable
- construct lepton asymmetry:

$$\mathcal{A}_\ell = \frac{\left(\frac{\mathrm{d}\sigma_{\ell^+}}{\mathrm{d}y_{\ell^+}}\right) - \left(\frac{\mathrm{d}\sigma_{\ell^-}}{\mathrm{d}y_{\ell^-}}\right)}{\left(\frac{\mathrm{d}\sigma_{\ell^+}}{\mathrm{d}y_{\ell^+}}\right) + \left(\frac{\mathrm{d}\sigma_{\ell^-}}{\mathrm{d}y_{\ell^-}}\right)},$$

- ℓ^- and ℓ^+ denote negatively and positively charged lepton

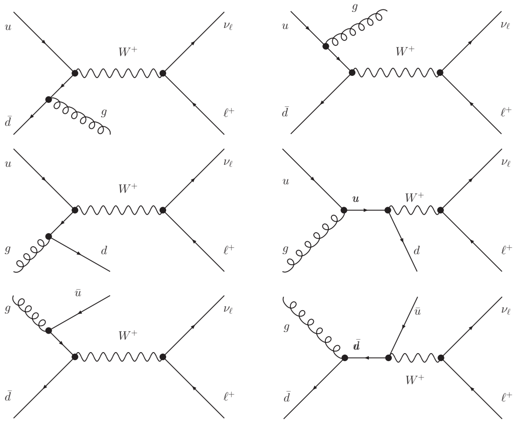
- lepton asymmetry at Tevatron and LHC
- $A > 0$: more positively than negatively charged leptons





Example: $W + j$ Production at Leading Order

- add additional quark or gluon to final state; Feynman diagrams:



- amplitudes (ignoring W-decays):

$$\begin{aligned}
\mathcal{M}_{u\bar{d}\to gW^+} &= \frac{ig_s g_W V_{ud}}{\sqrt{2}}\;\bar{v}_{d,i}\left[\gamma_\nu\; T^a_{ij}\frac{\not{p}_{\bar{d}}-\not{p}_g}{(p_{\bar{d}}-p_g)^2}\gamma_\mu L\right.\\
&\qquad\left.+\gamma_\mu L\frac{\not{p}_u-\not{p}_g}{(p_u-p_g)^2}\gamma_\nu\; T^a_{ij}\right]u_{u,j}\epsilon^\mu_W\epsilon^{\nu,a}_g\,,\\
\mathcal{M}_{ug\to dW^+} &= \frac{ig_s g_W V_{ud}}{\sqrt{2}}\;\bar{u}_{d,i}\left[\gamma_\nu\; T^a_{ij}\frac{\not{p}_g-\not{p}_d}{(p_g-p_d)^2}\gamma_\mu L\right.\\
&\qquad\left.+\gamma_\mu L\frac{\not{p}_u+\not{p}_g}{(p_u+p_g)^2}\gamma_\nu\; T^a_{ij}\right]u_{u,j}\epsilon^\mu_W\epsilon^{*\nu,a}_g\\
\mathcal{M}_{\bar{d}g\to\bar{u}W^+} &= \frac{ig_s g_W V_{ud}}{\sqrt{2}}\;\bar{v}_{d,i}\left[\gamma_\mu L\frac{\not{p}_g-\not{p}_{\bar{u}}}{(p_g-p_{\bar{u}})^2}\gamma_\nu\; T^a_{ij}\right.\\
&\qquad\left.+\gamma_\nu\; T^a_{ij}\frac{\not{p}_g+\not{p}_{\bar{d}}}{(p_g+p_{\bar{d}})^2}\gamma_\mu L\right]v_{u,j}\epsilon^\mu_W\epsilon^{*\nu,a}_g
\end{aligned}$$

- squaring them and using Mandelstam variables:

$$|\mathcal{M}|^2_{u\bar{d}\to gW^+} = \frac{4\pi\alpha_s C_F\, g_W^2 |V_{ud}|^2}{12}\, \frac{\hat{t}^2+\hat{u}^2+2m_W^2\hat{s}}{\hat{t}\hat{u}}$$

and

$$|\mathcal{M}|^2_{ug\to dW^+} = |\mathcal{M}|^2_{\bar{d}g\to \bar{u}W^+} = \frac{4\pi\alpha_s T_R\, g_W^2 |V_{ud}|^2}{12}\, \frac{\hat{s}^2+\hat{u}^2+2m_W^2\hat{t}}{-\hat{s}\hat{u}}\,.$$

- used identity below for compactness ($ab\to 12$):

$$\hat{s}+\hat{t}+\hat{u} \;=\; m_a^2+m_b^2+m_1^2+m_2^2\,,$$

- rewrite as LO matrix element times "radiation part":

$$
\begin{aligned}
|\mathcal{M}|^2_{u\bar{d}\to gW^+} &= \frac{\hat{\sigma}^{(\mathrm{LO})}_{u\bar{d}\to W^+}}{\pi}\cdot(4\pi\alpha_s C_F)\,\frac{\hat{t}^2+\hat{u}^2+2m_W^2\hat{s}}{\hat{t}\hat{u}}\\
|\mathcal{M}|^2_{ug\to dW^+} &= \frac{\hat{\sigma}^{(\mathrm{LO})}_{u\bar{d}\to W^+}}{\pi}\cdot(4\pi\alpha_s T_R)\,\frac{\hat{s}^2+\hat{u}^2+2m_W^2\hat{t}}{-\hat{s}\hat{u}}\,.
\end{aligned}
$$

- identifying, with c.m.angles θ^*, and for $u\bar{d}\to gW^+$

$$
\begin{aligned}
\hat{t} &= -2p_u p_g = -2E_u E_g(1-\cos\theta^*_{ug})\\
\hat{u} &= -2p_{\bar{d}} p_g = -2E_{\bar{d}} E_g(1-\cos\theta^*_{dg})\,.
\end{aligned}
$$

shows two possible divergences:
soft ($E_g\to 0$) and collinear ($\theta^*_g\to 0$)

- must regularise with cuts (jet algorithm)

Aside: Jet finders

- QCD final state radiation and hadronisation:
  hard partons turn into collimated spray of hadrons: jets
- need a quantitative definition: a jet algorithm
- most simple: cones – (transverse) energy in cone with fixed radius R
- typically direction defined by "seed hadron", $\vec{n}_{\rm jet} = \vec{p}_{\rm seed}/|\vec{p}_{\rm seed}|$, yields pseudorapidity $\eta_{\rm jet}$ and azimuthal angle $\phi_{\rm jet}$
- distance ΔR of hadrons w.r.t. direction:

$$\Delta R_{h,\,{\rm jet}} = \sqrt{(\eta_h - \eta_{\rm jet})^2 + (\phi_h - \phi_{\rm jet})^2}$$

- jets defined by demanding

$$E_{\rm tot} = \sum_{h\in{\rm cone}} E_h = \sum_{h,\Delta R_{h,\,{\rm jet}}<R} E_h > E_{\rm min}$$

- "seeded" cone algorithms not infrared-safe

  (soft and/or collinear radiation can change number of jets)

- modern jet algorithms are iterative:
  - in each step, go over all pairs of particles i & j
  - calculate distances:

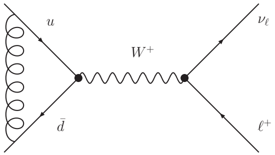
$$\begin{aligned} d_{ij} &= \min\{p_{\perp,i}^{2p},\, p_{\perp,j}^{2p}\}\,\frac{\Delta R_{i,j}^2}{R^2} \\ d_{i,\mathrm{beam}} &= p_{\perp,i}^2 \end{aligned}$$

p	$=$	1	k_T-algorithm
p	$=$	0	Cambridge–Aachen
p	$=$	-1	anti-k_T-algorithm

  - cluster partons with smallest d_{ij}: $(ij) = i + j$
    or if smallest d_i, parton is absorbed into beam
- can be applied to partons and particles
- standard @ LHC: anti-k_T with $R = 0.4$ (ATLAS)/0.5 (CMS)

Example: W Production at Next-To-Leading Order

- diagrams for $W + j$ above can act as
  - **LO** contributions to $W + j$, or as
  - real corrections at **NLO** to **inclusive** W **production**
- must add virtual corrections (loops)



$$\mathcal{M}^{(1)}_{u\bar{d}\to W^+} = \frac{g_W}{\sqrt{2}} g_s^2\, \bar{v}_i(\bar{d}) \left\{ \mu^{4-D} \int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{g^{\nu\rho}\delta^{ab}}{k^2} \times \left[\gamma_\nu\, T^a_{ik} \frac{\not{p}_d + \not{k}}{(p_d+k)^2} \gamma^{\mu L} \frac{\not{p}_u - \not{k}}{(p_u-k)^2} \gamma_\rho\, T^b_{kj} \right] \right\} u_j(u)\, \epsilon_\mu(W^+)$$

- UV and IR divergent

- matrix element above to be multiplied with LO matrix element:

$$\hat{\sigma}^{(\mathrm{NLO})} \sim \left|\mathcal{M}^{(0)} + \mathcal{M}^{(1)}\right|^2_{\mathcal{O}(\alpha_s)} = \left|\mathcal{M}^{(0)}\right|^2 + 2\left|\mathcal{M}^{(0*)}\mathcal{M}^{(1)}\right| + \mathcal{O}\left(\alpha_s^2\right)$$

- technology:
  - analytically continue to $D = 4 - 2\varepsilon$ dimensions
  - evaluate numerator (traces, contractions, etc.)
  - use Feynman trick(s) to map onto master integrals of the form

$$\mu^{4-D} \int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{k^{2m}}{(k^2 + p^2)^n}$$

  - evaluate integrals over Feynman parameters

- therefore, result for relevant contribution

$$2\left|\mathcal{M}^{(1*)}_{u\bar{d}\to W^+}\mathcal{M}^{(0)}_{u\bar{d}\to W^+}\right| =$$
$$\left|\mathcal{M}^{(0)}_{u\bar{d}\to W^+}\right|^2 \frac{\alpha_s}{2\pi} C_F \left(\frac{\mu^2}{Q^2}\right)^\varepsilon c_\Gamma \left(-\frac{2}{\varepsilon^2}-\frac{3}{\varepsilon}-8+\pi^2\right)$$

with $Q^2 = (p_u + p_d)^2 = m_W^2$ and $c_\Gamma = 4\pi^\varepsilon/\Gamma(1-\varepsilon)$

- note: finite term "8" is scheme-dependent

(scheme for evaluation of Dirac algebra/ numerator in D dimensions)

- note: UV divergences already renormalised

(this is explicit, when self–energies/wave function renormalisation is added, gauge invariance at work!)

- IR divergences in virtual part = poles in $1/\varepsilon$
- but also: IR divergences in real part, if $p_\perp \to 0$

(or, equivalently, if $\hat{t} \to 0$ or $\hat{u} \to 0$)

- Kinoshita–Lee–Nauenberg theorem guarantees that
  IR divergences cancel for physically sensible observables
- examples for such observables:
  total cross section σ, differential cross sections $\mathrm{d}\sigma/\mathrm{d}\mathcal{O}$ for observables $\mathcal{O}$ based on physical particles (jets, pions, ...)
- counter-examples:
  number of emitted gluons, or their individual transverse momenta

- direct integration of two–body phase space in D dimensions:

$$\mu^{4-D}\int\frac{\mathrm{d}^D k}{(2\pi)^D}\left|\mathcal{M}^{(0)}_{u\bar{d}\to W^+g}\right|^2 = \left|\mathcal{M}^{(0)}_{u\bar{d}\to W^+}\right|^2\frac{\alpha_s}{2\pi}C_F\left(\frac{\mu^2}{Q^2}\right)^{\varepsilon}c_\Gamma$$

$$\times\left[\overbrace{\left(\frac{2}{\varepsilon^2}+\frac{3}{\varepsilon}+\frac{\pi^2}{3}\right)\delta(1-z)}^{\text{cancelled with virtual}}\right.$$

$$+\left(\frac{4}{1-x}\log\frac{(1-z)^2}{z}\right)_+ - 2(1+z)\log\frac{(1-z)^2}{z}$$

$$\left.\underbrace{-\frac{2}{\varepsilon}\frac{\mathcal{P}^{(1)}_{qq}(z)}{C_F}}_{\text{absorbed into PDF}}\right]$$

- total result for gluon emission:

$$\begin{aligned}\hat{\sigma}^{(\mathrm{NLO})}_{u\bar{d}\to W^+} &= \hat{\sigma}^{(\mathrm{LO})}_{u\bar{d}\to W^+}\left\{1+\frac{\alpha_s(\mu_R)}{2\pi}C_F\left[\left(\frac{4\pi^2}{3}-8\right)\delta(1-z)\right.\right.\\ &\quad +\left(\frac{4}{1-z}\log\frac{(1-z)^2}{z}\right)_+ -2(1+z)\log\frac{(1-z)^2}{z}\\ &\quad \left.\left.-2\frac{\mathcal{P}^{(1)}_{qq}(z)}{C_F}\log\frac{\mu_F^2}{Q^2}\right]\right\}\end{aligned}$$

- add in gluon initial states:

$$\begin{aligned}\frac{\hat{\sigma}^{(\mathrm{NLO})}_{ug\to dW^+}}{\hat{\sigma}^{(\mathrm{LO})}_{u\bar{d}\to W^+}} &= \frac{\alpha_s(\mu_R)}{2\pi}T_R\left\{-\frac{\mathcal{P}^{(1)}_{qg}(z)}{T_R}\log\frac{\mu_F^2}{m_W^2}\right.\\ &\quad \left.+\left[z^2+(1-z)^2\right]\log\frac{(1-z)^2}{z}+\frac{1}{2}(1-z)(1+7z)\right\}\end{aligned}$$

- in general, cannot perform phase space integral for real emission contribution $\mathcal{R}_N$ analytically
- must resort to Monte Carlo integration for N–particle Born phase space $\Phi_{\mathcal{B}}$ and $(N+1)$–particle real emission phase space $\Phi_{\mathcal{R}}$

(note: have absorbed PDFs etc. into phase space)

- but: cannot MC integrate in D dimensions
- solution: identify singularities in real contribution
- subtract them from real contribution with term $\mathcal{S}$
- have to add it back to virtual $\mathcal{V}$

- therefore structure of NLO calculation for N-body production

$$\begin{aligned}
\mathrm{d}\sigma &= \mathrm{d}\Phi_{\mathcal{B}}\mathcal{B}_N(\Phi_{\mathcal{B}}) + \mathrm{d}\Phi_{\mathcal{B}}\mathcal{V}_N(\Phi_{\mathcal{B}}) + \mathrm{d}\Phi_{\mathcal{R}}\mathcal{R}_N(\Phi_{\mathcal{R}}) \\
&= \mathrm{d}\Phi_{\mathcal{B}}\left(\mathcal{B}_N + \mathcal{V}_N + \mathcal{I}_N^{(\mathcal{S})}\right) + \mathrm{d}\Phi_{\mathcal{R}}\left(\mathcal{R}_N - \mathcal{S}_N\right)
\end{aligned}$$

- assumed here that $\Phi_{\mathcal{R}} = \Phi_{\mathcal{B}} \otimes \Phi_1$

$$\int \mathrm{d}\Phi_1 \mathcal{S}_N(\Phi_{\mathcal{B}} \otimes \Phi_1) = \mathcal{I}_N^{(\mathcal{S})}(\Phi_{\mathcal{B}})$$

- nice feature: universal IR behaviour of gauge theories
$\longrightarrow$ can write process independent subtraction kernels as

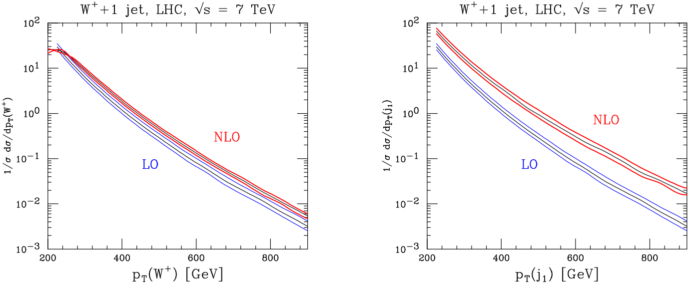
$$\begin{aligned}
\mathcal{S}_N(\Phi_{\mathcal{B}} \otimes \Phi_1) &= \mathcal{B}_N(\Phi_{\mathcal{B}}) \otimes \mathcal{S}_1(\Phi_{\mathcal{B}} \otimes \Phi_1) \\
\mathcal{I}_N^{(\mathcal{S})}(\Phi_{\mathcal{B}} \otimes \Phi_1) &= \mathcal{B}_N(\Phi_{\mathcal{B}}) \otimes \mathcal{I}_1^{(\mathcal{S})}(\Phi_{\mathcal{B}})
\end{aligned}$$

with universal $\mathcal{S}_1(\Phi_{\mathcal{B}} \otimes \Phi_1)$ and $\mathcal{I}_1^{(\mathcal{S})}(\Phi_{\mathcal{B}})$

Subtleties: scaling, giant K-factors and all that

- which order is which? can be tricky ...
- higher–order reduce scale dependence
  however: effect still may be large!
  $\Longrightarrow$ how to chose the "right" scale?
- generic idea (borrowed from later):
  analyse process in terms of "parton splittings"
  - choose renormalisation scale as geometric mean of scales
  - choose factorisation scale from largest deflection of colour flow
- K-factor as ratio to LO cross section
  of course, just a number for total cross section
  but not necessarily flat for distributions!
- occurrence of "giant" K-factors when new channels open

- example: giant K–factor in $W + j$



- $p_\perp^{(W)}$ relatively stable, $1^{\rm st}$ jet not
- hints at large impact of 2–parton configurations

Summary

- calculated W production
  in different channels (inclusive, W+jet)
  and at different accuracies (LO, NLO)
- a word on perturbative orders:
  - simply: leading order (LO) is the first order with non-trivial results
  - examples:

    $\longrightarrow$ inclusive W production cross section & rapidity spectrum

    $\longrightarrow$ leptonic forward–backward charge asymmetry at LO, NLO

    all already at $\mathcal{O}\left(\alpha_s^0\right)$

    calculation at $\mathcal{O}\left(\alpha_s^1\right)$ yields $1^{\rm st}$ quantum correction

    $\longrightarrow$ $p_\perp$ spectrum of W from $W+$ parton at LO

    no distribution at $\mathcal{O}\left(\alpha_s^0\right)$ ...

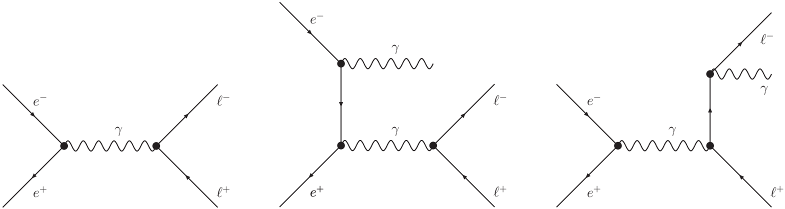
QCD BASICS

QCD TO ALL ORDERS

Contents

A QED example

- consider $e^- e^+ \to \mu^- \mu^+$ and $p_\perp$ spectrum of $\mu^- \mu^+$ pair
- Feynman diagrams:



- ignore emission of FS leptons – can concentrate on $e^- e^+ \to \gamma^* \gamma$ with virtual mass Q

- matrix element squared for $e^-e^+ \to \gamma^*\gamma$ with photon virtual mass Q

$$|\mathcal{M}|^2_{e^-e^+\to\gamma\gamma^*} = 32\pi^2\alpha^2\left[\frac{\hat{t}}{\hat{u}}+\frac{\hat{u}}{\hat{t}}+\frac{2Q^2(Q^2-\hat{t}-\hat{u})}{\hat{t}\hat{u}}\right],$$

- LO cross section for $e^-e^+ \to \gamma^*$: $\hat{\sigma}_0 = 4\pi\alpha^2/(3\hat{s})$, therefore

$$\frac{\mathrm{d}\hat{\sigma}_R}{\mathrm{d}\hat{t}\mathrm{d}Q^2} = \frac{\alpha}{2\pi}\frac{\hat{\sigma}_0}{Q^2}\left[\frac{\hat{t}}{\hat{u}}+\frac{\hat{u}}{\hat{t}}+\frac{2Q^2(Q^2-\hat{t}-\hat{u})}{\hat{t}\hat{u}}\right]$$

- note symmetry under $\hat{t} \longleftrightarrow \hat{u}$
- remember Mandelstam identity, here $\hat{s}+\hat{t}+\hat{u} = Q^2$

$p_\perp$ spectrum in single– and all–emission approximation

- introduce $Q_\perp$ the transverse momentum of the virtual photon and assume low-$Q_\perp$ limit, $Q^2 \gg Q_\perp^2 \to 0$
- symmetry under $\hat{t} \longleftrightarrow \hat{u}$, consider $\hat{t} \to 0$ only:

$$Q_\perp^2 \approx -\hat{t} \to 0 \quad \text{and} \quad \hat{u} \stackrel{\hat{t}\to 0}{\longrightarrow} Q^2 - \hat{s}$$

- integrate over all masses Q^2, with $4m_\ell^2 \le Q^2 \le \hat{s} - 2\sqrt{\hat{s}}\,Q_\perp$ results in (approximatively)

$$\begin{aligned}\frac{\mathrm{d}\hat{\sigma}_R}{\mathrm{d}Q_\perp^2} &= \hat{\sigma}_0\,\frac{\alpha}{\pi}\,\frac{1}{Q_\perp^2}\left[\log\frac{\hat{s}}{Q_\perp^2} + \mathcal{O}(1)\right]\\ \longrightarrow \quad \mathrm{d}\hat{\sigma}_R &\approx \hat{\sigma}_0\,\frac{\alpha}{\pi}\,\frac{\mathrm{d}Q_\perp^2}{Q_\perp^2}\,\log\frac{\hat{s}}{Q_\perp^2}\end{aligned}$$

(this is a double-logarithmic spectrum → DL approximation, DLLA)

- calculate cross section for $Q_\perp > Q_{\perp,\rm min}$:

$$\mathrm{d}\hat{\sigma}_R(>Q_{\perp,\rm min}) \approx \hat{\sigma}_0\,\frac{\alpha}{\pi}\int\limits_{Q^2_{\perp,\rm min}}^{\hat{s}}\frac{\mathrm{d}Q^2_\perp}{Q^2_\perp}\,\log\frac{\hat{s}}{Q^2_\perp} \approx \hat{\sigma}_0\,\frac{\alpha}{2\pi}\,\log^2\frac{\hat{s}}{Q^2_{\perp,\rm min}}$$

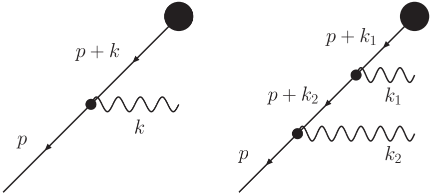
- but total cross section for γ^* production finite:

$$\begin{aligned}\sigma_{e^-e^+\to\gamma^*} &= \hat{\sigma}_0\left(1+\mathcal{O}(\alpha)\cdot\mathrm{non-logarithmic}\right)\\ &= \int\limits_0^{Q^2_{\perp,\rm min}}\mathrm{d}Q^2_\perp\,\frac{\mathrm{d}\hat{\sigma}_R+\mathrm{d}\hat{\sigma}_V}{\mathrm{d}Q^2_\perp}+\int\limits_{Q^2_{\perp,\rm min}}^{\hat{s}}\mathrm{d}Q^2_\perp\,\frac{\mathrm{d}\hat{\sigma}_R}{\mathrm{d}Q^2_\perp}\end{aligned}$$

- this implies that

$$\int\limits_0^{Q^2_{\perp,\rm min}}\mathrm{d}Q^2_\perp\,\frac{\mathrm{d}\hat{\sigma}_R+\mathrm{d}\hat{\sigma}_V}{\mathrm{d}Q^2_\perp} \approx \hat{\sigma}_0\left(1-\frac{\alpha}{2\pi}\,\log^2\frac{\hat{s}}{Q^2_{\perp,\rm min}}\right) = \hat{\sigma}_0\left(1+\Sigma^{(1)}\right)$$

- consider many emissions



- expression for single emission in soft limit

$$e\,\epsilon_\mu(k)\,\bar{u}(p)\gamma^\mu\frac{\not p+\not k}{(p+k)^2}\,\dots\approx e\,\bar{u}(p)\frac{p\cdot\epsilon}{p\cdot k}\,\dots,$$

becomes for n emissions

$$\frac{e^n}{n!}\,\frac{p\cdot\epsilon_1}{p\cdot k_1}\,\frac{p\cdot\epsilon_2}{p\cdot k_2}\dots\frac{p\cdot\epsilon_n}{p\cdot k_n}\,\dots.$$

- therefore, n–photon contribution $\Sigma^{(n)}$

$$\Sigma^{(n)}(p_\perp^2) = \frac{1}{n!}\left[\frac{\alpha}{\pi}\int\limits_0^{p_\perp^2} \mathrm{d}\log Q_\perp^2 \log\frac{\hat{s}}{Q_\perp^2}\right]^n = \frac{1}{n!}\left[-\frac{\alpha}{2\pi}\log^2\frac{\hat{s}}{p_\perp^2}\right]^n,$$

- summing all orders:

$$\Sigma(Q_\perp^2) = \sum_{n=0}^{\infty}\frac{1}{n!}\left[-\frac{\alpha}{2\pi}\log^2\frac{\hat{s}}{Q_\perp^2}\right]^n = \exp\left[-\frac{\alpha}{2\pi}\log^2\frac{\hat{s}}{Q_\perp^2}\right]$$

- normalised differential cross section with respect to the transverse momentum squared in DLLA

$$\frac{1}{\hat\sigma_0}\frac{\mathrm{d}\hat\sigma}{\mathrm{d}Q_\perp^2} = -\frac{\mathrm{d}}{\mathrm{d}Q_\perp^2}\Sigma(Q_\perp^2) = \frac{\alpha}{\pi}\frac{1}{Q_\perp^2}\log\frac{\hat s}{Q_\perp^2}\exp\left[-\frac{\alpha}{2\pi}\log^2\frac{\hat s}{Q_\perp^2}\right].$$

- Σ is the Sudakov form factor

(a no–emission probability, we will see much more of it!)

Impact parameter space

- problem in reasoning so far:
  every emitted photon gives a transverse momentum $k_\perp$, total transverse momentum comes as the sum over all of them:

$$\vec{Q}_\perp = -\sum_i \vec{k}_{\perp,i}$$

- rewrite as δ function and Fourier transform:

$$\delta^2\left(\vec{Q}_\perp + \sum_{i=0}^{n} \vec{k}_{\perp,i}\right) = \frac{1}{2\pi}\int \mathrm{d}^2 b_\perp \exp\left[i\vec{b}_\perp \cdot \left(\vec{Q}_\perp + \sum_{i=0}^{n} \vec{k}_{\perp,i}\right)\right].$$

- rewrite single emission terms in impact parameter space:

$$\nu(k_{\perp,i}) = \frac{\alpha}{\pi}\frac{1}{k_{\perp,i}^2}\log\frac{\hat{s}}{k_{\perp,i}^2}$$
$$\longleftrightarrow \frac{1}{2\pi}\int \mathrm{d}^2 k_{\perp,i}\exp\left[-i\vec{b}_{\perp,i}\cdot\vec{k}_{\perp,i}\right]\nu(k_{\perp,i}) = \nu(b_{\perp,i})$$

- summing over all emissions

$$\frac{\mathrm{d}\hat{\sigma}^{(\mathrm{all})}}{\mathrm{d}Q_\perp^2} = \frac{\hat{\sigma}_0}{\pi}\int \mathrm{d}b_\perp^2\,\mathrm{J}_0(b_\perp Q_\perp)\exp\left[-\frac{1}{2\pi}\frac{\mathrm{d}k_\perp^2}{k_\perp^2}\frac{\alpha(k_\perp^2)}{\pi}\log\frac{\hat{s}}{k_{\perp,i}^2}\right]$$

after δ–function has been taken care of

Q_T resummation in QCD

- apply the same reasoning on $p_\perp^{(W)}$:

$$\frac{\mathrm{d}\sigma_{AB\to W^+}}{\mathrm{d}y\mathrm{d}Q_\perp^2} = \sum_{ij} \frac{\hat{\sigma}^{(LO)}_{ij\to W^+}}{2\pi} \left\{ \int \frac{\mathrm{d}b_\perp^2}{\pi} \left[\mathrm{J}_0(\vec{b}_\perp \cdot \vec{Q}_\perp)\, \tilde{W}_{ij}(b;\, Q,\, x_A,\, x_B) \right] \right\}$$

with

$$\hat{\sigma}^{(LO)}_{ij\to W^+} = \frac{4\pi^2\alpha |V_{ij}|^2}{12\sin^2\theta_W m_W^2}$$

- function $\tilde{W}_{ij}(b;\, Q,\, x_A,\, x_B)$ incorporates resummation with: Sudakov form factor, PDFs, potential higher orders
- may have to supplement (hard) real emission corrections

- at LL accuracy (and with $k_{\perp,\mathrm{min}} = 1/b_\perp$)

$$\tilde{W}_{ij}(b;\, Q,\, x_A,\, x_B) \;=\; f_{i/A}\left(x_A,\, \frac{1}{b_\perp}\right)\, f_{j/B}\left(x_B,\, \frac{1}{b_\perp}\right)\, \exp\left[-\int\limits_{1/b_\perp^2}^{Q^2} \frac{\mathrm{d}k_\perp^2}{k_\perp^2}\left(A(k_\perp^2)\log\frac{Q^2}{k_\perp^2}\right)\right]$$

(remember $x_{A,B} = M_X/\sqrt{s}\, e^{\pm y}$)

- must regularise large-$b_\perp$/low-$k_\perp$ regime:
  add non–perturbative dampening function (such as $\exp(-b_\perp^2)$)

(will ignore this added term)

- full beauty result:

$$\frac{\mathrm{d}\sigma_{AB\to W^+}}{\mathrm{d}y\mathrm{d}Q_\perp^2} = \sum_{ij} \frac{\hat{\sigma}^{(LO)}_{ij\to W^+}}{2\pi} \left\{ \int \frac{\mathrm{d}b_\perp^2}{\pi} \left[\mathrm{J}_0(\vec{b}_\perp \cdot \vec{Q}_\perp)\, \tilde{W}_{ij}(b;\, Q,\, x_A,\, x_B) \right] \right.$$
$$\left. + Y_{ij\to W^+}(Q_\perp;\, Q,\, x_A,\, x_B) \right\}$$

- function Y as differrence of full real correction and real emission parts coming from $\tilde{W}_{ij}$ ("matching")

- anatomy of resummation part

$$
\begin{aligned}
&\tilde{W}_{ij}(b;\, Q,\, x_A,\, x_B) \\
&= \sum_{ab} \left\{ \int\limits_{x_A}^{1} \frac{\mathrm{d}\xi_A}{\xi_A} \int\limits_{x_B}^{1} \frac{\mathrm{d}\xi_B}{\xi_B} \left[f_{a/A}\left(\xi_A,\, \frac{1}{b_\perp}\right) f_{b/B}\left(\xi_B,\, \frac{1}{b_\perp}\right) \right.\right. \\
&\qquad \times \overbrace{C_{ia}\left(\frac{x_A}{\xi_A},\, b_\perp;\, \mu\right) C_{jb}\left(\frac{x_B}{\xi_B},\, b_\perp;\, \mu\right)}^{\text{corrections to factorisation}} \overbrace{H_{ab}\left(\frac{x_A}{\xi_A},\, \frac{x_B}{\xi_B};\, \mu\right)}^{\text{genuine loop corrections}} \\
&\qquad \left.\left. \times \underbrace{\exp\left[-\int\limits_{1/b_\perp^2}^{Q^2} \frac{\mathrm{d}k_\perp^2}{k_\perp^2} \left(A(k_\perp^2) \log\frac{Q^2}{k_\perp^2} + B(k_\perp^2) \right) \right]}_{\text{Sudakov form factor}} \right] \right\}
\end{aligned}
$$

- terms A, B, C, and H all with expansion in α_s, e.g.

$$A(\mu) = \sum_N \left(\frac{\alpha_s(\mu)}{2\pi}\right)^N A^{(N)} \ldots$$

- process-independent are the $A^{(i)}$ and

$$\begin{aligned}
A^{(1)}_{q,g} &= 2C_{F,A} \\
A^{(2)}_{q,g} &= 2C_{F,A}K = 2C_{q,g}\left[C_A\left(\frac{67}{18}-\frac{\pi^2}{6}\right)-\frac{10}{9}T_R n_f\right] \\
B^{(1)}_{q,g} &= -3C_F\,, \quad -2\beta_0 \\
C^{(0)}_{ia}\left(z,\,\frac{1}{b_\perp},\,\mu_F^2\right) &= \delta_{ia}\,\delta(1-z) \\
C^{(1)}_{ia}\left(z,\,\frac{1}{b_\perp},\,\mu_F^2\right) &= P^{(1)}_{ia}\log\frac{b_0^2}{b_\perp^2\mu_F^2} - P^{\epsilon}_{ia}(z) + \delta_{ia}\delta(1-z)C_a\frac{\pi^2}{6}\,.
\end{aligned}$$

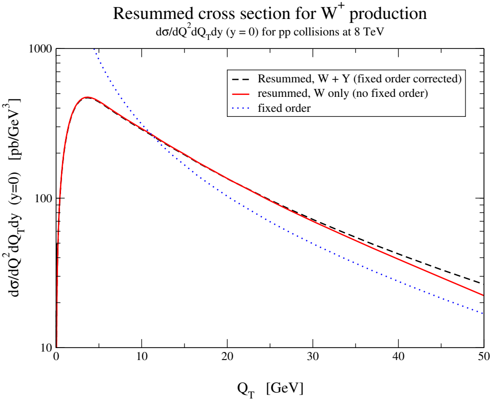
- logarithmic accuracy:

accuracy	terms included
LL	$A^{(1)}$
NLL	$A^{(2)}$, $B^{(1)}$, $C^{(1)}$
NNLL	$A^{(3)}$, $B^{(2)}$, $C^{(2)}$

- note: log–counting differs by community

(I adopt counting of Collins–Soper–Sterman and Catani–deFlorian–Grazzini)

- example result: interplay of fixed order and resummation



- note: parton shower will act similar to Q_T resummation

Threshold resummation in QCD

- another source of large logarithms: production thresholds
- produce heavy system (singlet) with mass Q, add multiple soft gluon emissions to "hit" threshold Q from above, $\sim 1/(1-z)_+$ from splitting function
- integration over splitting yields logarithms of the form

$$\alpha_s^n \left[\frac{\log^k(1-z)}{1-z}\right]_+ .$$

- interesting limit: large logs from "squeezed" phase space, gluon emissions do not change kinematics of system

- log–enhancement of cross section encoded in $W_{ij}^{(\mathrm{thres})}$:

$$\frac{\mathrm{d}\sigma_{AB\to X}}{\mathrm{d}Q^2} = \int\limits_0^1 \mathrm{d}\tau \int\limits_0^1 \mathrm{d}x_i \mathrm{d}x_j\, f_{i/A}(x_i,\,\mu_F) f_{j/B}(x_j,\,\mu_F)\,\delta\left(\tau - \frac{Q^2}{Sx_ix_j}\right) \cdot W_{ij}^{(\mathrm{thres})}(\tau,\,Q,\,\mu_R,\,\mu_F)\,,$$

- relevant limit $\tau = Q^2/\hat{s} \to 1$
- usually discussed after Mellin transform, therefore only qualitatively
  - important: no phase space for hard collinear emissions,
  - therefore completely driven by soft eikonal, identical A's!!!!

$$\mathbf{M}_N\left[W_{ij}^{(\mathrm{thres})}\right]^{(1)} = \frac{4C_F}{2\pi}\int\limits_0^1 \mathrm{d}z\,\frac{z^N-1}{1-z}\int\limits_{Q^2}^{(1-z)Q^2}\frac{\mathrm{d}q^2}{q^2}\,\alpha_s((1-z)q^2)\,,$$

  - subleading terms start at $\mathcal{O}\left(\alpha_s^2\right)$
  - same choice of scale, $k_\perp^2 = (1-z)q^2$

Resummed jet rates in e^+e^-–annihilations

- use Durham jet measure ($k_\perp$-type):

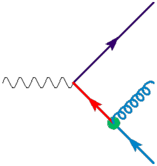
$$k^2_{\perp,ij} = 2\min(E_i^2,\, E_j^2)(1-\cos\theta_{ij}) > Q^2_{\rm jet}$$

- jets given by $k_\perp \geq Q_{\rm jet}$
- remember probabilistic interpretation of Sudakov form factor

$$\begin{aligned}\mathcal{P}_{\rm no\ emission} &= \Delta_{q,g}(Q,Q_0)\\ &= \exp\left[-\int\limits_{Q_0^2}^{Q^2}\frac{{\rm d}q_\perp^2}{q_\perp^2}\frac{C_{F,A}\alpha_s(q_\perp^2)}{2\pi}\left(A\log\frac{Q^2}{q_\perp^2}+B\right)\right]\end{aligned}$$

- jet rates from no–emission probabilities

- evaluate 2– and 3–jet rate
- for example: 2–jet rate given by probs of no emission above $Q_{\rm jet}$ for either quark



$$\begin{aligned}\mathfrak{R}_2(Q_{\rm jet}) &= \left[\Delta_q(E^2_{\rm c.m.}, Q^2_{\rm jet})\right]^2\\ \mathfrak{R}_3(Q_{\rm jet}) &= 2\Delta_q(E^2_{\rm c.m.}, Q^2_{\rm jet})\\ &\cdot \int \frac{dq^2_\perp}{q^2_\perp}\left[\Gamma_q(q^2_\perp)\Delta_q(E^2_{\rm c.m.}, q^2_\perp)\Delta_q(q^2_\perp, Q^2_{\rm jet})\Delta_g(q^2_\perp, Q^2_{\rm jet})\right]\end{aligned}$$

- used "integrated splitting function"

$$\Gamma_{q,g}(q^2_\perp) = \frac{C_{F,A}\alpha_s(q^2)}{2\pi}\left(A\log\frac{Q^2}{q^2} + B\right)$$

- different resummation techniques for different situations
- common denominator: going to soft/collinear limits of parton emission phase space
- replacing perturbative parameter α_s with $\alpha_s L^2$ or similar
- important object: Sudakov form factor

(identified as no–emission probability)

- will put perturbative technology into Monte Carlo tomorrow

ROUND II: GOING MONTE CARLO

GOING MONTE CARLO

GENERAL MONTE CARLO

Contents

Prelude: Selecting from a distribution

- a typical Monte Carlo/simulation problem:
  - wanted: random numbers $x \in [x_{\rm min},\, x_{\rm max}]$, distributed according to (probability) density $f(x)$, i.e.

$$\mathcal{P}(x \in [x', x' + {\rm d}x']) = f(x'){\rm d}x'$$

  - but: only "usual" random numbers $\#$ available: "flat" in $[0, 1]$
- exact solution:
  - must know integral F of density f and its inverse F^{-1}
  - x given by

$$\int\limits_{x_{\rm min}}^{x} {\rm d}x' f(x') = \# \int\limits_{x_{\rm min}}^{x_{\rm max}} {\rm d}x' f(x')$$

    and therefore

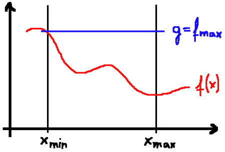
$$x = F^{-1}\left[F(x_{\rm min}) + \#\left(F(x_{\rm max}) - F(x_{\rm min})\right)\right]$$

- case above is very untypical – integral sometimes known, inverse practically never
- need a work-around solution: "Hit-or-miss"
  (solution, if exact case does not work.)
- construct good "over-estimator" $g(x)$ (G and G^{-1} known):

$$g(x) > f(x) \;\; \forall x \in [x_{\rm min},\, x_{\rm max}]$$

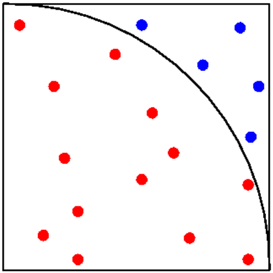
- exact algorithm to select a trial-x according to overestimator g
- accept x with probability $f(x)/g(x)$
- obvious fall-back choice for $g(x)$:

$$g(x) = {\rm Max}_{[x_{\rm min},\, x_{\rm max}]}\{f(x)\}.$$



Monte Carlo integration

- underlying idea: determination of π with random number generator



$$\frac{\text{Hits}}{\text{Misses} + \text{Hits}} \longrightarrow \frac{\pi}{4}$$

Throw random points (x,y), with x, y in [0,1]

For hits: $(x^2 + y^2) < r^2 = 1$

- MC integration: estimate integral by N probes

$$I_f^{(a,b)} = \int_a^b \mathrm{d}x f(x)$$

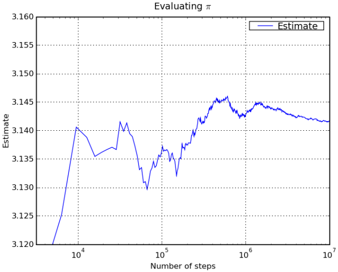
$$\longrightarrow \langle I_f^{(a,b)} \rangle = \frac{b-a}{N} \sum_{i=1}^{N} f(x_i) = \langle f \rangle_{a,b}$$

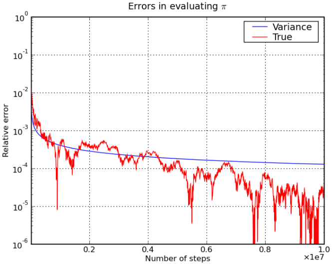
where x_i homogeneously distributed in $[a, b]$

- error estimate from statistical sample $\Longrightarrow$ standard deviation

$$\langle E_f^{(a,b)}(N) \rangle = \sigma = \left[\frac{\langle f^2 \rangle_{a,b} - \langle f \rangle_{a,b}^2}{N} \right]^{1/2}$$

- independent of the number of integration dimensions!
$\Longrightarrow$ method of choice for high-dimensional integrals.


Evaluating π
Estimate
Estimate
Number of steps
3.160
3.155
3.150
3.145
3.140
3.135
3.130
3.125
3.120
10^4
10^5
10^6
10^7

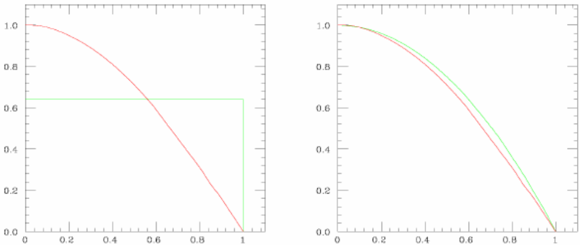

Errors in evaluating π
Variance
True
Relative error
Number of steps
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
0.2
0.4
0.6
0.8
1.0
×1e7

Monte Carlo integration: refinements

- want to minimise number of potentially expensive function calls
  $\Longrightarrow$ need to improve convergence of MC integration.
- first basic idea: sample in regions, where f largest

  ($\Longrightarrow$ corresponds to a Jacobian transformation of integral)
- alternative algorithm: minimise error by "smoothing" integrand ("importance sampling")
  - assume a function $g(x)$ similar to $f(x)$.
  - $f(x)/g(x)$ smooth $\Longrightarrow \langle E(f/g)\rangle$ small
  - must sample according to $\mathrm{d}x\, g(x)$ rather than $\mathrm{d}x$:
    $g(x)$ plays role of probability distribution; we know already how to deal with this!
- works, if $f(x)$ is well-known, but hard to generalise.

- importance sampling
- consider $f(x) = \cos\frac{\pi x}{2}$ and $g(x) = 1 - x^2$:



$$I = \int_0^1 dx\,\cos\frac{\pi}{2}x = 0.637 \pm 0.308/\sqrt{N}$$

$$I = \int_0^1 dx(1-x^2)\,\frac{\cos\frac{\pi}{2}x}{1-x^2} = \int d\rho\,\frac{\cos\frac{\pi}{2}x}{1-x^2}[x(\rho)] = 0.637 \pm 0.032/\sqrt{N}$$

- yet another idea: decompose integral in M sub-integrals

$$\begin{aligned}\langle I(f)\rangle &= \sum_{j=1}^{M}\langle I_j(f)\rangle\\ \langle E(f)\rangle^2 &= \sum_{j=1}^{M}\langle E_j(f)\rangle^2\end{aligned}$$

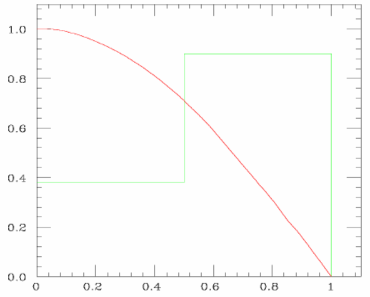
- overall variance smallest, if "equally distributed".

($\Longrightarrow$ sample, where the fluctuations are.)

("stratified sampling")

- algorithm:
  - divide interval in bins (variable bin-size or weight);
  - adjust such that variance identical in all bins.

- stratified sampling
- consider $f(x) = \cos\frac{\pi x}{2}$ and $g(x) = 1 - x^2$:



$$\langle I \rangle = 0.637 \pm 0.147/\sqrt{N}$$

- a hybrid of stratified and importance sampling:
  replace independent bins of stratified sampling with independent functions of importance sampling
- "bins" – with weight α_i – of "eigenfunctions" – $g_i(x)$:
  $\Longrightarrow \quad g(\vec{x}) = \sum_{i=1}^{N} \alpha_i g_i(\vec{x})$.
- in particle physics, this is the method of choice for parton level event generation:
  - translate each Feynman diagram into one or more channels
  - optimise interplay of channels, cuts, etc. through weights α_i
  - optional: add VEGAs to "best" channels

Traditional MC simulation

- a classical example: two-dimensional Ising model:

(spins s_i fixed on 2-D lattice with nearest neighbour interactions.)

$$\mathcal{H} = -J \sum_{\langle ij \rangle} s_i s_j$$

- evaluation of observable $\mathcal{O}$ by summing over all micro states $\phi_{\{i\}}$, given as spin ensembles (similar to path integral in QFT.)

$$\langle \mathcal{O} \rangle = \int \mathcal{D}\phi_{\{i\}} \,\mathrm{Tr}\, \left\{ \mathcal{O}(\phi_{\{i\}}) \exp\left[-\frac{\mathcal{H}(\phi_{\{i\}})}{k_B T} \right] \right\}$$

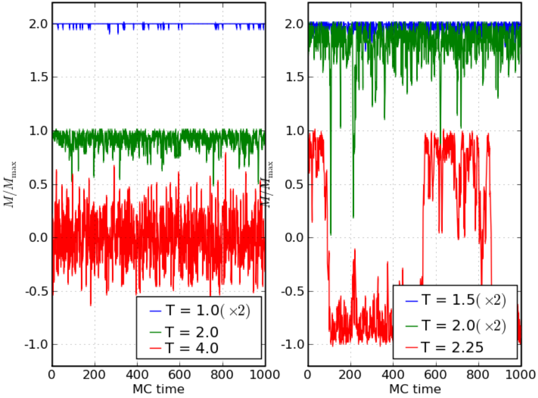
- typical problem in such calculations (integrations!): phase space too large $\Longrightarrow$ need to sample.

- Metropolis algorithm simulates the canonical ensemble, summing/integrating over micro-states with MC method.
- necessary ingredient: interactions among spins in probabilistic language (this will come back to us!)
- algorithm:
    - go over the spins,
    - check whether they flip:
        - compare $\mathcal{P}_{\mathrm{flip}}$ with random number
        - $\mathcal{P}_{\mathrm{flip}}$ from energies of the two micro-states (before and after flip) and Boltzmann factors
    - repeat to equilibrate.
    - evaluate observables directly during run &take thermal average (average over many steps).

- why does this work? detailled balance!
  - consider one spin flip, connecting micro-states 1 and 2.
  - rate of transitions given by the transition probabilities $\mathcal{W}$
  - if $E_1 > E_2$ then $\mathcal{W}_{1\to 2} = 1$ and $\mathcal{W}_{2\to 1} = \exp\left(-\frac{E_1 - E_2}{k_B T}\right)$
  - in thermal equilibrium, both transitions equally often:
    $$\mathcal{P}_2 \mathcal{W}_{2\to 1} = \mathcal{P}_1 \mathcal{W}_{1\to 2}$$
    takes into account that the respective states are occupied according to their Boltzmann factors.

    $(\mathcal{P}_i \sim \exp(-E_i/k_B T))$
- in principle, all systems in thermal equilibrium can be studied with Metropolis - just need to write transition probabilities in accordance with detailed balance, as above $\Longrightarrow$ general simulation strategy in thermodynamics.

- example results on a 10×10 lattice



GOING MONTE CARLO

PARTON LEVEL

Contents

Calculating matrix elements efficiently

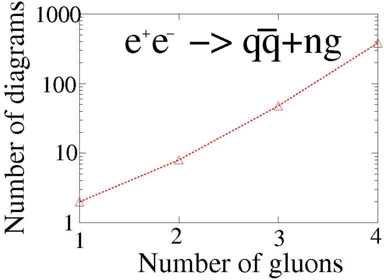
- stating the problem(s):
  - multi-particle final states for signals & backgrounds.
  - need to evaluate $\mathrm{d}\sigma_N$:

$$\int\limits_{\text{cuts}} \left[\prod_{i=1}^{N} \frac{\mathrm{d}^3 q_i}{(2\pi)^3 2E_i}\right] \delta^4\left(p_1 + p_2 - \sum_i q_i\right) \left|\mathcal{M}_{p_1 p_2 \to N}\right|^2 .$$

  - problem 1: factorial growth of number of amplitudes.
  - problem 2: complicated phase-space structure.
  - solutions: numerical methods.

- example for factorial growth: $e^+e^- \to q\bar{q} + ng$

n	$\#_{\rm diags}$
0	1
1	2
2	8
3	48
4	384



- obvious: traditional textbook methods (squaring, completeness relations, traces) fail
  $\Longrightarrow$ result in proliferation of terms ($\mathcal{M}_i\mathcal{M}_j^*$)
- better ideas of efficient ME calculation:
  $\Longrightarrow$ realise: amplitudes just are complex numbers,
  $\Longrightarrow$ add them before squaring!
- remember: spinors, gamma matrices have explicit form
  could be evaluated numerically (brute force)
  but: Rough method, lack of elegance, CPU-expensive
- can do better with smart basis for spinors (see next slide)
- this is still on the base of traditional Feynman diagrams!

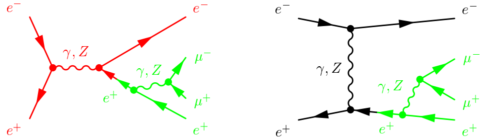
- helicity method:
  - introduce basic helicity spinors (needs to "gauge"-vectors)
  - write everything as spinor products, e.g.
    $$\bar{u}(p_1,\, h_1)u(p_2,\, h_2) = \text{complex numbers.}$$
  - have completeness relations such as

$$(\not{p} + m) \Longrightarrow \frac{1}{2}\sum_h \left[\left(1 + \frac{m^2}{p^2}\right) \bar{u}(p,\, h)u(p,\, h) \right.$$
$$\left. + \quad \left(1 - \frac{m^2}{p^2}\right) \bar{v}(p,\, h)v(p,\, h) \right]$$

  - there are other genuine expressions ...
  - translate Feynman diagrams into "helicity amplitudes": complex-valued functions of momenta & helicities.
  - spin-correlations etc. nearly come for free.

- taming the factorial growth in the helicity method
  - by reusing pieces: calculate only once!
  - factoring out: reduce number of multiplications!

  can be implemented as a-posteriori manipulations of amplitudes.



- better method: recursion relations (recycling built in).
  best candidate so far: off-shell recursions

(Dyson-Schwinger, Berends-Giele etc.)

- improvement: off-shell recursion relations
- general idea: recursively construct generalised currents $\mathcal{J}_\alpha(\pi)$ for a set π of external particles on their mass shell plus one internal one

$$
\begin{aligned}
\mathcal{J}_\alpha(\pi) \; &= P_\alpha(\pi) \left\{ \sum_{\mathcal{P}_2(\pi)} \sum_{\mathcal{V}_\alpha^{\alpha_1\alpha_2}} \left[S(\pi_1,\, \pi_2)\, \mathcal{V}_\alpha^{\alpha_1\alpha_2} \mathcal{J}_{\alpha_1}(\pi_1) \mathcal{J}_{\alpha_2}(\pi_2) \right] \right. \\
&\left. \qquad + \sum_{\mathcal{P}_3(\pi)} \sum_{\mathcal{V}_\alpha^{\alpha_1\alpha_2\alpha_3}} \left[S(\pi_1,\, \pi_2,\, \pi_3)\, \mathcal{V}_\alpha^{\alpha_1\alpha_2\alpha_3} \mathcal{J}_{\alpha_1}(\pi_1) \mathcal{J}_{\alpha_2}(\pi_2) \mathcal{J}_{\alpha_3}(\pi_3) \right] \right\}
\end{aligned}
$$

  - $P_\alpha(\pi)$ denotes the propagator denominator
  - $S(\pi_1,\, \pi_2)$ and $S(\pi_1,\, \pi_2,\, \pi_3)$ for symmetry factors
  - $\mathcal{V}_\alpha$ for three– and four–particle vertices
  - go over all permutations of external particles π

- recursion relations particularly powerful due to massive recycling as integral part of structure $\longrightarrow$ bookkeeping problem only
- there are sub–classes of particularly simple amplitudes:
  maximally helicity violating (MHV) amplitudes
- all-gluon amplitudes with helicities given

(Parke–Taylor/Berends–Giele amplitudes)

$$\begin{aligned}
\mathcal{A}(1^+, 2^+, \dots, i^-, \dots, j^-, \dots\dots n^+) &= ig_s^{n-2}\,\frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\dots\langle (n-1)n\rangle\langle n1\rangle} \\
\mathcal{A}(1^-, 2^-, \dots, i^+, \dots, j^+, \dots\dots n^-) &= ig_s^{n-2}\,\frac{[ij]^4}{[12][23]\dots[(n-1)n][n1]}\,.
\end{aligned}$$

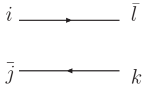
- terms $[ij]$ etc. are products of two–component left– and right–handed Weyl spinors (particularly simple)
- note: all-sign identical amplitudes vanish due to the conservation of angular momentum

- in principle also factorial growth with number of colours
- sampling over colours improves situation.

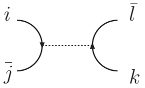
(but still, e.g. naively $\simeq (n-1)!$ permutations/colour-ordering for n external gluons).

- improved scheme: colour dressing

$$T^a_{i\bar{j}}T^a_{k\bar{l}} = \delta_{i\bar{l}}\delta_{k\bar{j}} - \frac{1}{N_c}\,\delta_{i\bar{j}}\delta_{k\bar{l}} \longleftrightarrow$$



$-\dfrac{1}{N_c}$



- works very well with Berends-Giele recursions

Final State	BG CO	BG CD	BCF CO	BCF CD	CSW CO	CSW CD
$2g$	0.24	0.28	0.28	0.33	0.31	0.26
$3g$	0.45	0.48	0.42	0.51	0.57	0.55
$4g$	1.20	1.04	0.84	1.32	1.63	1.75
$5g$	3.78	2.69	2.59	7.26	5.95	5.96
$6g$	14.2	7.19	11.9	59.1	27.8	30.6
$7g$	58.5	23.7	73.6	646	146	195
$8g$	276	82.1	597	8690	919	1890
$9g$	1450	270	5900	127000	6310	29700
$10g$	7960	864	64000	-	48900	-

Time $[s]$ for the evaluation of 10^4 phase space points, sampled over helicities & colour.

Phase spacing for professionals

("Amateurs study strategy, professionals study logistics")

- democratic, process-blind integration methods:
  - Rambo/Mambo: Flat & isotropic

    R.Kleiss, W.J.Stirling & S.D.Ellis, Comput. Phys. Commun. **40** (1986) 359;
  - HAAG/Sarge: Follows QCD antenna pattern

    A.van Hameren & C.G.Papadopoulos, Eur. Phys. J. C **25** (2002) 563.
- multi-channeling: each Feynman diagram related to a phase space mapping (= "channel"), optimise their relative weights

  R.Kleiss & R.Pittau, Comput. Phys. Commun. **83** (1994) 141.
- main problem: practical only up to $\mathcal{O}(10\mathrm{k})$ channels.
- some improvement by building phase space mappings recursively: more channels feasible, efficiency drops a bit.

- quality measure for integration performance: unweighting efficiency
- want to generate events "as in nature".
- basic idea: use hit-or-miss method;
  - generate $\vec{x}$ with integration method,
  - compare actual $f(\vec{x})$ with maximal value during sampling $\Longrightarrow$ "Unweighted events".
- comments:
  - unweighting efficiency, $w_{\mathrm{eff}} = \langle f(\vec{x}_j)/f_{\max}\rangle$ = number of trials for each event.
  - expect $\log_{10} w_{\mathrm{eff}} \approx 3-5$ for good integration of multi-particle final states at tree-level.
  - maybe acceptable to use $f_{\max,\mathrm{eff}} = Kf_{\max}$ with $K < 1$.
    problem: what to do with events where $f(\vec{x}_j)/f_{\max,\mathrm{eff}} > 1$?
    answer: Add $\mathrm{int}[f(\vec{x}_j)/f_{\max,\mathrm{eff}}] = k$ events and perform hit-or-miss on $f(\vec{x}_j)/f_{\max,\mathrm{eff}} - k$.

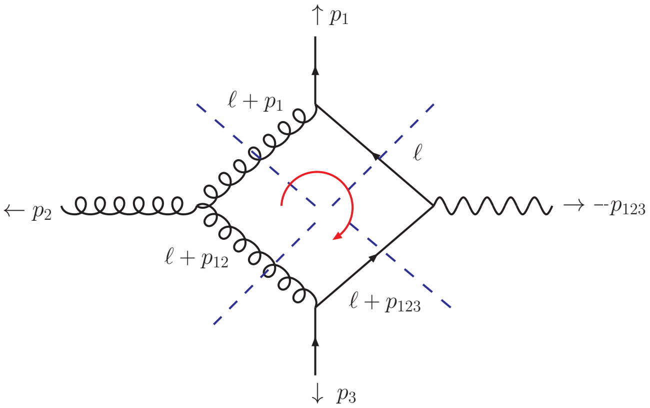
Including higher order corrections

- traditional bottleneck of higher–order calculations: virtual parts
- algorithm before about 2005:
  - Passarino–Veltman reduction of tensors in numerator (replace $2p \cdot k = (p+k)^2 - p^2 - k^2$)
  - reduce to scalar master integrals of the form

$$\int \frac{\mathrm{d}^D k}{[(p_1+k)^2(p_2+k)^2\dots]}$$

  - further reduce to integrals with up to four propagators only (but careful: introduces instabilities through "Gram determinants")

- about 2005: begin of "NLO revolution"
- basic idea: reduce to master integrals numerically by cutting



- coefficients of master integrals emerge as solutions of linear equations

Cancellation of infrared divergences

- need a mechanism to cancel IR divergences for higher multiplicities in final states
- toy model in one dimension:
  - $|\mathcal{M}^R_{m+1}|^2 = \frac{1}{x}R(x)$, where x=gluon energy or similar.
  - $|\mathcal{M}^V_m|^2 = \frac{1}{\epsilon}V$, regularised in $d = 4 - 2\epsilon$ dimensions.
  - cross section in d dimensions with jet measure F^J:

    $$\sigma = \int_0^1 \frac{\mathrm{d}x}{x^{1+\epsilon}} R(x) F^J_1(x) + \frac{1}{\epsilon} V F^J_0$$
  - infrared safety of jet measure: $F^J_1(0) = F^J_0$

    $\Longrightarrow$ "a soft/collinear parton has no effect."

    (tricky issue - without it, no reliable NLO calculation!)
  - KLN theorem: $R(0) = V$.

- rewrite

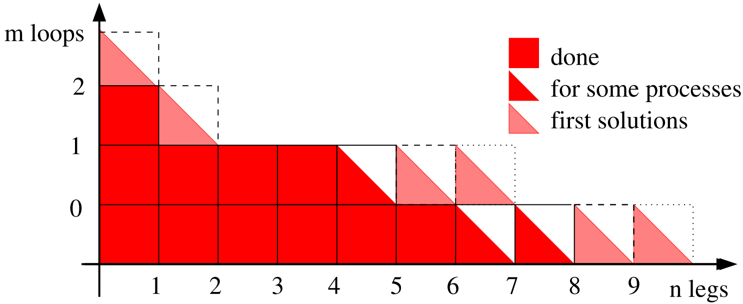
$$\sigma = \int\limits_0^1 \frac{\mathrm{d}x}{x^{1+\epsilon}} R(x) F_1^J(x) - \int\limits_0^1 \frac{\mathrm{d}x}{x^{1+\epsilon}} V F_0^J + \int\limits_0^1 \frac{\mathrm{d}x}{x^{1+\epsilon}} V F_0^J + \frac{1}{\epsilon} V F_0^J$$

$$= \int\limits_0^1 \frac{\mathrm{d}x}{x^{1+\epsilon}} \left(R(x) F_1^J(x) - V F_0^J\right) + \mathcal{O}(1) V F_0^J\,.$$

- two separately finite integrals, with no large numbers to be added/subtracted.
- subtraction terms are universal (analytical bit can be calculated once and for all).

Availability of exact calculations (hadron colliders)

- fixed order matrix elements ("parton level") are exact to a given perturbative order. (and often quite a pain!)
- important to understand limitations:
  only tree-level and one-loop level fully automated, beyond: prototyping



Survey of existing parton-level tools @ tree–level

	Models	$2 \to n$	Ampl.	Integ.	public?	lang.
ALPGEN	SM	$n = 8$	rec.	Multi	yes	Fortran
AMEGIC++	SM,MSSM,ADD	$n = 6$	hel.	Multi	yes	C++
COMIX	SM	$n = 8$	rec.	Multi	yes	C++
COMPHEP	SM,MSSM	$n = 4$	trace	1Channel	yes	C
HELAC	SM	$n = 8$	rec.	Multi	yes	Fortran
MADEVENT	SM,MSSM,UED	$n = 6$	hel.	Multi	yes	Python/Fortran
WHIZARD	SM,MSSM,LH	$n = 8$	rec.	Multi	yes	O'Caml

Survey of existing parton-level tools @ NLO

	type	technology dependencies on other codes
LoopTools	integrals	
OneLoop	integrals	
QCDLoop	integrals	
Collier	reduction	
CutTools	reduction	OPP
FormCalc	reduction	PV
Ninja	reduction	Laurent expansion
Samurai	reduction	
BlackHat	library (amplitudes)	OPP (unitarity)
McFm	library (full calculation)	PV & OPP
MJet	library (amplitudes)	OPP
GoSam	generator (amplitudes)	OPP Samurai +Ninja +
MadLoop	generator (full calculation)	OL+OPP CutTools +
OpenLoops	generator (amplitudes)	OL+OPP Collier +CutTools +
Helac-NLO	generator (full calculation)	OPP CutTools +

GOING MONTE CARLO

PARTON SHOWERS – THE BASICS

Contents

An analogy: Radioactive decays

- consider radioactive decay of an unstable isotope with half-life τ.

(and ignore factors of ln 2.)

- "survival" probability after time t is given by

$$\mathcal{S}(t) = \mathcal{P}_{\rm nodec}(t) = \exp[-t/\tau]$$

(note "unitarity relation": $\mathcal{P}_{\rm dec}(t) = 1 - \mathcal{P}_{\rm nodec}(t)$.)

- probability for an isotope to decay at time t:

$$\frac{{\rm d}\mathcal{P}_{\rm dec}(t)}{{\rm d}t} = -\frac{{\rm d}\mathcal{P}_{\rm nodec}(t)}{{\rm d}t} = \frac{1}{\tau}\exp(-t/\tau)$$

- now: connect half-life with width $\Gamma = 1/\tau$.
- probability for the isotope to decay at any fixed time t determined by Γ.

- spice things up now: add time–dependence, $\Gamma = \Gamma(t')$
- rewrite

$$\Gamma t \longrightarrow \int\limits_0^t \mathrm{d}t'\Gamma$$

- decay-probability at a given time t is given by

$$\frac{\mathrm{d}\mathcal{P}_{\mathrm{dec}}(t)}{\mathrm{d}t} = \Gamma(t)\,\exp\left[-\int\limits_0^t \mathrm{d}t'\Gamma(t')\right] = \Gamma(t)\,\mathcal{P}_{\mathrm{nodec}}(t)$$

(unitarity strikes again: $\mathrm{d}\mathcal{P}_{\mathrm{dec}}(t)/\mathrm{d}t = -\mathrm{d}\mathcal{P}_{\mathrm{nodec}}(t)/\mathrm{d}t$.)

- interpretation of l.h.s.:
  - first term is for the actual decay to happen.
  - second term is to ensure that no decay before t
    $\Longrightarrow$ conservation of probabilities.
    the exponential is - of course - called the Sudakov form factor.

The pattern of QCD radiation

- a detour: Altarelli-Parisi equation, once more
- AP describes the scaling behaviour of the parton distribution function

(which depends on Bjorken-parameter and scale Q^2)

$$\frac{\mathrm{d}q(x,\ Q^2)}{\mathrm{d}\ln Q^2} = \int\limits_x^1 \frac{\mathrm{d}y}{y}\left[\alpha_s(Q^2)P_q(x/y)\right]q(y,Q^2)$$

- term in square brackets determines the probability that the parton emits another parton at scale Q^2 and Bjorken-parameter y

(after the splitting, $x \to yx + (1-y)x$.)

- driving term: Splitting function $P_q(x)$
important property: universal, process independent

- differential cross section for gluon emission in $e^+e^- \to$ jets

$$\frac{\mathrm{d}\sigma_{ee\to 3j}}{\mathrm{d}x_1\mathrm{d}x_2} = \sigma_{ee\to 2j}\frac{C_F\alpha_s}{\pi}\frac{x_1^2+x_2^2}{(1-x_1)(1-x_2)}$$

singular for $x_{1,2}\to 1$.

- rewrite with opening angle θ_{qg} and gluon energy fraction $x_3 = 2E_g/E_{\mathrm{c.m.}}$:

$$\frac{\mathrm{d}\sigma_{ee\to 3j}}{\mathrm{d}\cos\theta_{qg}\mathrm{d}x_3} = \sigma_{ee\to 2j}\frac{C_F\alpha_s}{\pi}\left[\frac{2}{\sin^2\theta_{qg}}\frac{1+(1-x_3)^2}{x_3} - x_3\right]$$

singular for $x_3\to 0$ ("soft"), $\sin\theta_{qg}\to 0$ ("collinear").

- re-express collinear singularities

$$\begin{aligned}\frac{2\mathrm{d}\cos\theta_{qg}}{\sin^2\theta_{qg}} &= \frac{\mathrm{d}\cos\theta_{qg}}{1-\cos\theta_{qg}} + \frac{\mathrm{d}\cos\theta_{qg}}{1+\cos\theta_{qg}}\\ &= \frac{\mathrm{d}\cos\theta_{qg}}{1-\cos\theta_{qg}} + \frac{\mathrm{d}\cos\theta_{\bar{q}g}}{1-\cos\theta_{\bar{q}g}} \approx \frac{\mathrm{d}\theta^2_{qg}}{\theta^2_{qg}} + \frac{\mathrm{d}\theta^2_{\bar{q}g}}{\theta^2_{\bar{q}g}}\end{aligned}$$

- independent evolution of two jets (q and $\bar{q}$)

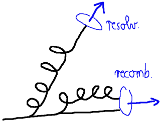
$$\mathrm{d}\sigma_{ee\to 3j} \approx \sigma_{ee\to 2j} \sum_{j\in\{q,\bar{q}\}} \frac{C_F\alpha_s}{2\pi}\frac{\mathrm{d}\theta^2_{jg}}{\theta^2_{jg}} P(z)\,,$$

- note: same form for any $t \propto \theta^2$:
- transverse momentum $k_\perp^2 \approx z^2(1-z)^2 E^2\theta^2$
- invariant mass $q^2 \approx z(1-z)E^2\theta^2$

$$\frac{\mathrm{d}\theta^2}{\theta^2} \approx \frac{\mathrm{d}k_\perp^2}{k_\perp^2} \approx \frac{\mathrm{d}q^2}{q^2}$$

- parametrisation-independent observation:
  (logarithmically) divergent expression for $t \to 0$.
- practical solution: cut-off Q_0^2.
  $\Longrightarrow$ divergence will manifest itself as $\log Q_0^2$.
- similar for $P(z)$: divergence for $z \to 0$ cured by cut-off.

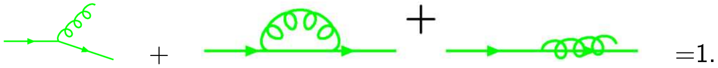
- what is a parton?
  collinear pair/soft parton recombine!
- introduce resolution criterion $k_\perp > Q_0$.



- combine virtual contributions with unresolvable emissions:
  cancels infrared divergences $\Longrightarrow$ finite at $\mathcal{O}(\alpha_s)$

  (Kinoshita-Lee-Nauenberg, Bloch-Nordsieck theorems)

- unitarity: probabilities add up to one
  $\mathcal{P}(\text{resolved}) + \mathcal{P}(\text{unresolved}) = 1.$



- the Sudakov form factor, once more
- differential probability for emission between q^2 and $q^2 + \mathrm{d}q^2$:

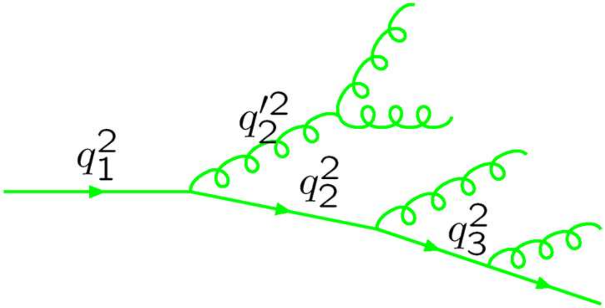
$$\mathrm{d}\mathcal{P} = \frac{\alpha_s}{2\pi}\frac{\mathrm{d}q^2}{q^2}\int\limits_{z_{\rm min}}^{z_{\rm max}} \mathrm{d}z P(z) =: \mathrm{d}q^2\,\Gamma(q^2)$$

- from radioactive example: evolution equation for Δ

$$-\frac{\mathrm{d}\Delta(Q^2,\,q^2)}{\mathrm{d}q^2} = \Delta(Q^2,\,q^2)\frac{\mathrm{d}\mathcal{P}}{\mathrm{d}q^2} = \Delta(Q^2,\,q^2)\Gamma(q^2)$$

$$\Longrightarrow\ \Delta(Q^2,\,q^2) = \exp\left[-\int\limits_{q^2}^{Q^2}\mathrm{d}k^2\Gamma(k^2)\right]$$

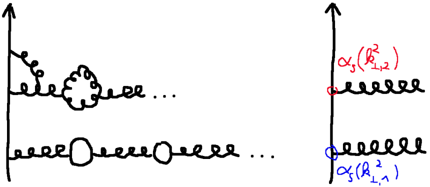
- maximal logs if emissions ordered
- impacts on radiation pattern: in each emission t becomes smaller



$$q_1^2 > q_2^2 > q_3^2,\ q_1^2 > {q_2'}^2$$

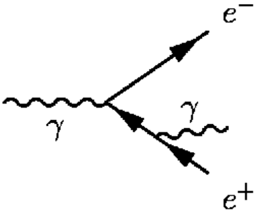
Quantum improvements

- improvement: inclusion of various quantum effects
- trivial: effect of summing up higher orders (loops) $\alpha_s \to \alpha_s(k_\perp^2)$



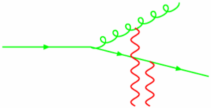
- much faster parton proliferation, especially for small $k_\perp^2$.
- avoid Landau pole: $k_\perp^2 > Q_0^2 \gg \Lambda^2_{\rm QCD} \Longrightarrow Q_0^2 =$ physical parameter.

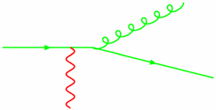
- soft limit for single emission also universal
- problem: soft gluons come from all over (not collinear!) quantum interference? still independent evolution?
- answer: not quite independent.
- consider case in QED



- assume photon into e^+e^- at θ_{ee} and photon off electron at θ
  photon momentum denoted as k
- energy imbalance at vertex: $k_\perp^\gamma \sim k_\| \theta$, hence $\Delta E \sim k_\perp^2/k_\| \sim k_\| \theta^2$.
- formation time for photon emission: $\Delta t \sim 1/\Delta E \sim k_\|/k_\perp^2 \sim 1/(k_\| \theta^2)$.
- ee-separation: $\Delta b \sim \theta_{ee} \Delta t$
- must be larger than transverse wavelength of photon:
  $\theta_{ee}/(k_\| \theta^2) > 1/k_\perp = 1/(k_\| \theta)$
- thus: $\theta_{ee} > \theta$ must be satified for photon to form
- angular ordering as manifestation of quantum coherence

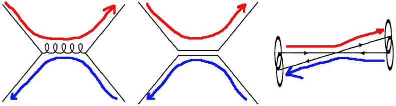
- pictorially:

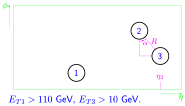


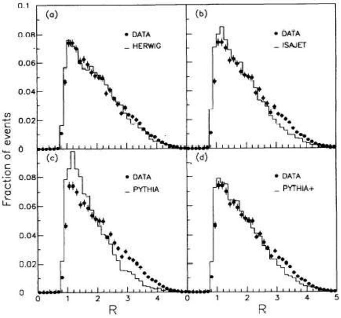


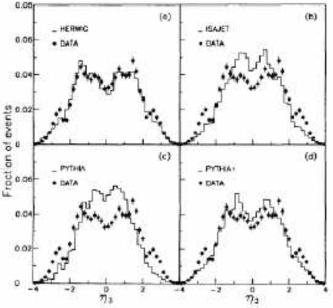
gluons at large angle from combined colour charge!

- experimental manifestation:
  ΔR of 2nd & 3rd jet in multi-jet events in pp-collisions









ROUND III: SIMULATING SOFT QCD

SIMULATING SOFT QCD

HADRONISATION

Contents

QCD radiation, once more

- remember QCD emission pattern

$$\mathrm{d}w^{q\to qg} = \frac{\alpha_s(k_\perp^2)}{2\pi}\, C_F\, \frac{\mathrm{d}k_\perp^2}{k_\perp^2}\, \frac{\mathrm{d}\omega}{\omega}\, \left[1 + \left(1 - \frac{\omega}{E}\right)\right] .$$

- spectrum cut-off at small transverse momenta and energies by onset of hadronization, at scales $R \approx 1\,\mathrm{fm}/\Lambda_{\mathrm{QCD}}$
- two (extreme) classes of emissions: gluons and gluers
  determined by relation of formation and hadronization times

- gluers formed at times R, with momenta $k_{\parallel} \sim k_{\perp} \sim \omega \gtrsim 1/R$
- assuming that hadrons follow partons,

$$\mathrm{d}N_{\rm (hadrons)} \sim \int\limits_{k_\perp > 1/R}^{Q} \frac{\mathrm{d}k_\perp^2}{k_\perp^2} \frac{C_F\,\alpha_s(k_\perp^2)}{2\pi} \left[1 + \left(1 - \frac{\omega}{E}\right)\right] \frac{\mathrm{d}\omega}{\omega}$$
$$\sim \frac{C_F\alpha_s(1/R^2)}{\pi} \log(Q^2R^2)\,\frac{\mathrm{d}\omega}{\omega}$$

or - relating their energyn with that of the gluers -

$$\mathrm{d}N_{\rm (hadrons)}/\mathrm{d}\log\epsilon = \mathrm{const.}\,,$$

a plateau in log of energy (or in rapidity)

- impact of additional radiation
- new partons must separate before they can hadronize independently
- therefore, one more time

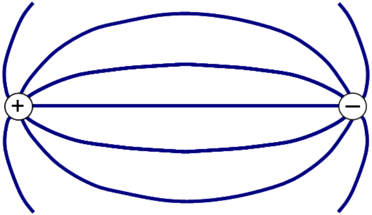
$$\begin{aligned} t^{\rm form} &\sim \frac{k_\|}{k_\perp^2} & & \\ t^{\rm sep} &\sim R\theta &\sim& \; t^{\rm form}\,(Rk_\perp) \\ t^{\rm had} &\sim k_\| R^2 &\sim& \; t^{\rm form}\,(Rk_\perp)^2\,. \end{aligned}$$

- for gluers $Rk_\perp \approx 1$: all times the same
- naively; new & more hadrons following new partons
- but: colour coherence
  primary and secondary partons not separated enough in
  $$1/R \lesssim \omega_{\rm (hadron)} \lesssim 1/(R\theta)$$
  and therefore no independent radiation

Hadronisation: General thoughts

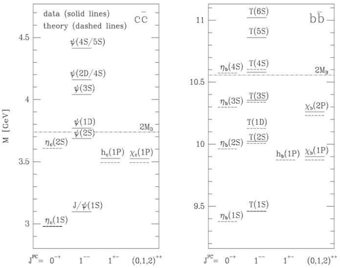
- confinement the striking feature of low–scale sotrng interactions
- transition from partons to their bound states, the hadrons
- the Meissner effect in QCD

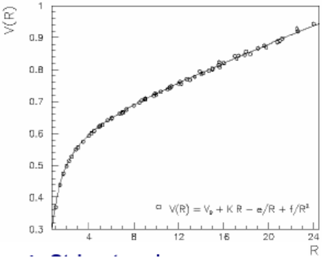
QED:



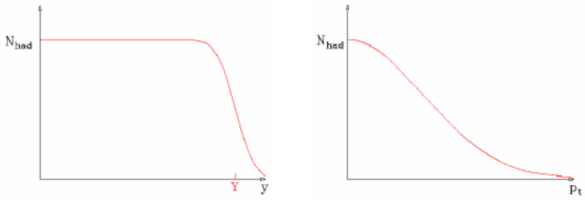
QCD:

- linear QCD potential in Quarkonia – like a string





- combine some experimental facts into a naive parameterisation
- in $e^+e^- \to$ hadrons: exponentially decreasing $p_\perp$, flat plateau in y for hadrons



- try "smearing": $\rho(p_\perp^2) \sim \exp(-p_\perp^2/\sigma^2)$

- use parameterisation to "guesstimate" hadronisation effects:

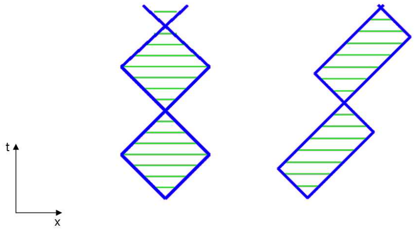
$$
\begin{aligned}
E &= \int_0^Y \mathrm{d}y\mathrm{d}p_\perp^2\,\rho(p_\perp^2)p_\perp \cosh y = \lambda \sinh Y \\
P &= \int_0^Y \mathrm{d}y\mathrm{d}p_\perp^2\,\rho(p_\perp^2)p_\perp \sinh y = \lambda(\cosh Y - 1) \approx E - \lambda \\
\lambda &= \int \mathrm{d}p_\perp^2\,\rho(p_\perp^2)p_\perp = \langle p_\perp \rangle\,.
\end{aligned}
$$

- estimate $\lambda \sim 1/R_{\rm had} \approx m_{\rm had}$, with $m_{\rm had}$ 0.1-1 GeV.
- effect: jet acquire non-perturbative mass $\sim 2\lambda E$ ($\mathcal{O}(10\mathrm{GeV})$ for jets with energy $\mathcal{O}(100\mathrm{GeV})$).

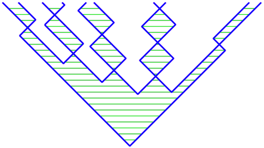
- similar parametrization underlying Feynman-Field model for independent fragmentation
- recursively fragment $q \to q' +$ had, where
  - transverse momentum from (fitted) Gaussian;
  - longitudinal momentum arbitrary (hence from measurements);
  - flavour from symmetry arguments + measurements.
- problems: frame dependent, "last quark", infrared safety, no direct link to perturbation theory, ...

The string model

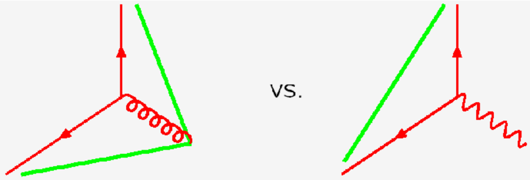
- a simple model of mesons: yoyo strings
  - light quarks ($m_q = 0$) connected by string, form a meson
  - area law: $m^2_{\rm had} \propto$ area of string motion
  - L=0 mesons only have 'yo-yo' modes:



- turn this into hadronisation model $e^+e^- \to q\bar{q}$ as test case
- ignore gluon radiation: $q\bar{q}$ move away from each other, act as point-like source of string
- intense chromomagnetic field within string:
  more $q\bar{q}$ pairs created by tunnelling and string break-up
- analogy with QED (Schwinger mechanism):
  $\mathrm{d}\mathcal{P} \sim \mathrm{d}x\mathrm{d}t \exp\left(-\pi m_q^2/\kappa\right)$, $\kappa =$ "string tension".



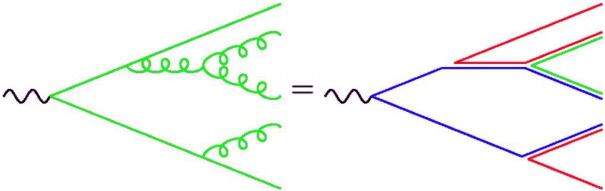
- string model = well motivated model, constraints on fragmentation (Lorentz-invariance, left-right symmetry, ...)
- how to deal with gluons?
- interpret them as kinks on the string $\Longrightarrow$ the string effect



- infrared-safe, advantage: smooth matching with PS.

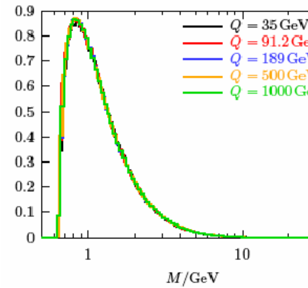
The cluster model

- underlying idea: preconfinement/LPHD
    - typically, neighbouring colours will end in same hadron
    - hadron flows follow parton flows $\longrightarrow$ don't produce any hadrons at places where you don't have partons
    - works well in large–N_c limit with planar graphs
- follow evolution of colour in parton showers



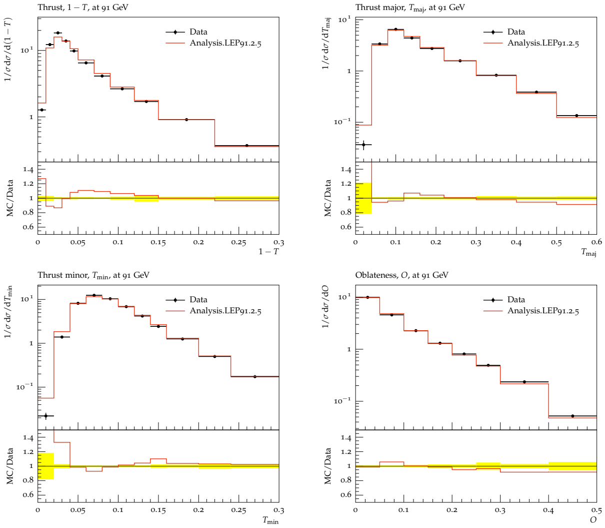
- paradigm of cluster model: clusters as continuum of hadron resonances
- trace colour through shower in $N_c \to \infty$ limit
- force decay of gluons into $q\bar{q}$ or $\bar{d}d$ pairs, form colour singlets from neighbouring colours, usually close in phase space
- mass of singlets: peaked at low scales $\approx Q_0^2$
- decay heavy clusters into lighter ones or into hadrons (here, many improvements to ensure leading hadron spectrum hard enough, overall effect: cluster model becomes more string-like)
- if light enough, clusters will decay into hadrons
- naively: spin information washed out, decay determined through phase space only $\to$ heavy hadrons suppressed (baryon/strangeness suppression)

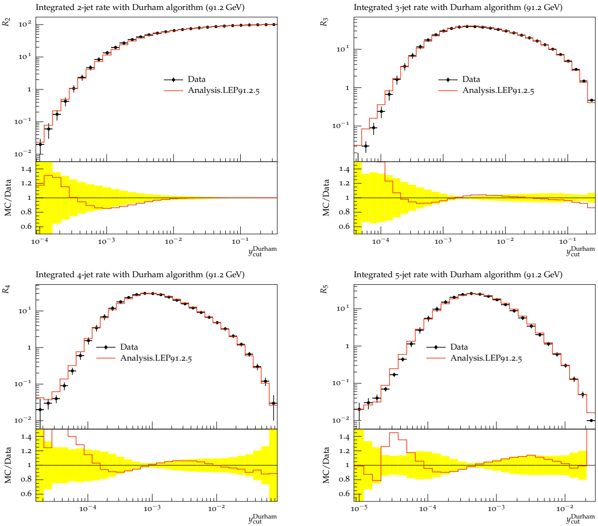
- self–similarity of parton shower will end with roughly the same **local** distribution of partons, with roughly the same invairant mass for colour singlets
- adjacent pairs colour connected, form colourless (white) clusters.
- clusters ("≈ excited hadrons) decay into hadrons

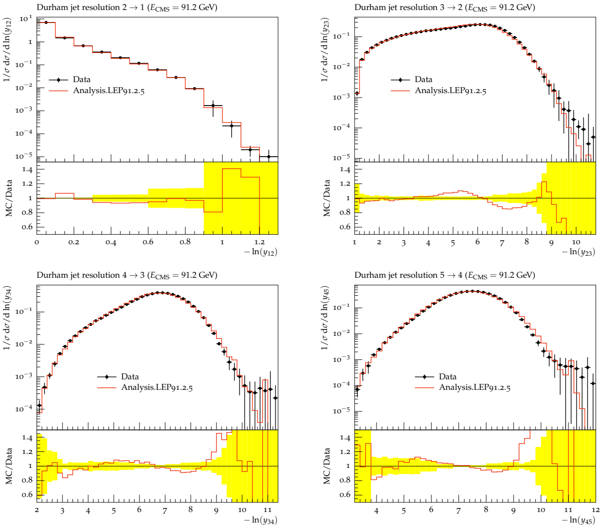


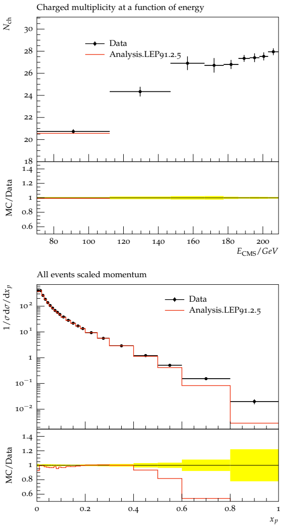
Observables

- in the following a selection of data from the LEP collaboration relevant for the tuning of hadronisation models
- all compared with an actual tune of SHERPA
- typically, PYTHIA does as good (or somwetimes slightly better)
- so, this is the level we talk about these days, agreement of 5% or better over large ranges of observables and scales

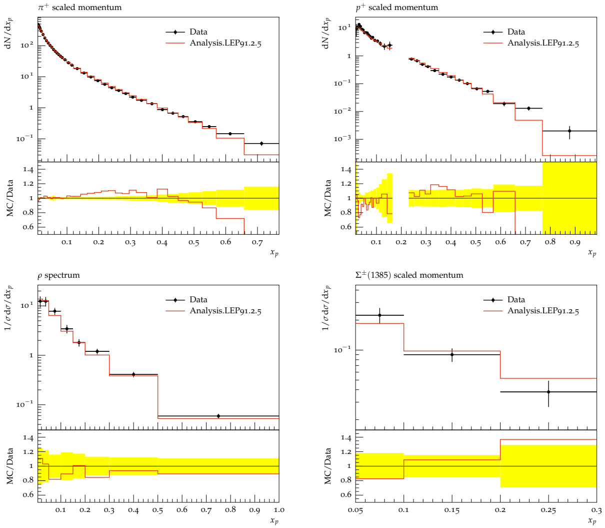

Thrust, $1-T$, at 91 GeV
Thrust major, T_{maj}, at 91 GeV
Thrust minor, T_{min}, at 91 GeV
Oblateness, O, at 91 GeV
Data
Analysis.LEP91.2.5
$1/\sigma\, d\sigma/d(1-T)$
$1/\sigma\, d\sigma/dT_{\text{maj}}$
$1/\sigma\, d\sigma/dT_{\text{min}}$
$1/\sigma\, d\sigma/dO$
MC/Data
$1-T$
T_{maj}
T_{min}
O

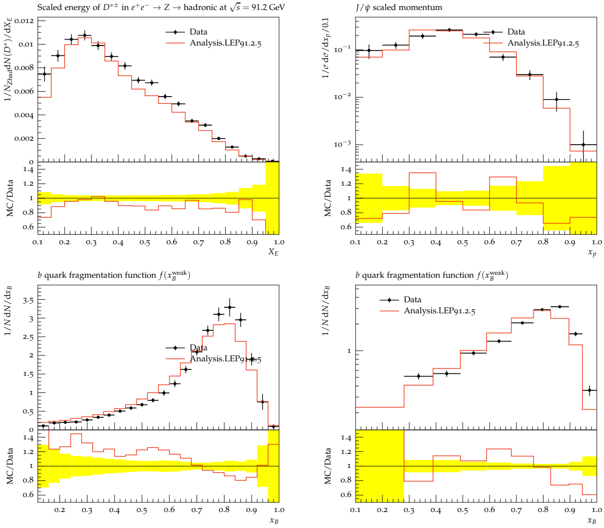

Integrated 2-jet rate with Durham algorithm (91.2 GeV)
R_2
Data
Analysis.LEP91.2.5
MC/Data
$y_{\mathrm{cut}}^{\mathrm{Durham}}$
Integrated 3-jet rate with Durham algorithm (91.2 GeV)
R_3
Data
Analysis.LEP91.2.5
MC/Data
$y_{\mathrm{cut}}^{\mathrm{Durham}}$
Integrated 4-jet rate with Durham algorithm (91.2 GeV)
R_4
Data
Analysis.LEP91.2.5
MC/Data
$y_{\mathrm{cut}}^{\mathrm{Durham}}$
Integrated 5-jet rate with Durham algorithm (91.2 GeV)
R_5
Data
Analysis.LEP91.2.5
MC/Data
$y_{\mathrm{cut}}^{\mathrm{Durham}}$


Durham jet resolution 2 → 1 (E_{CMS} = 91.2 GeV)
Data
Analysis.LEP91.2.5
$1/\sigma\, d\sigma/d\ln(y_{12})$
MC/Data
$-\ln(y_{12})$
Durham jet resolution 3 → 2 (E_{CMS} = 91.2 GeV)
Data
Analysis.LEP91.2.5
$1/\sigma\, d\sigma/d\ln(y_{23})$
MC/Data
$-\ln(y_{23})$
Durham jet resolution 4 → 3 (E_{CMS} = 91.2 GeV)
Data
Analysis.LEP91.2.5
$1/\sigma\, d\sigma/d\ln(y_{34})$
MC/Data
$-\ln(y_{34})$
Durham jet resolution 5 → 4 (E_{CMS} = 91.2 GeV)
Data
Analysis.LEP91.2.5
$1/\sigma\, d\sigma/d\ln(y_{45})$
MC/Data
$-\ln(y_{45})$


Charged multiplicity at a function of energy
N_{ch}
Data
Analysis.LEP91.2.5
MC/Data
E_{CMS}/GeV
Charged multiplicity distribution
$1/N\, dN/dN_{ch}$
Data
Analysis.LEP91.2.5
MC/Data
N_{ch}
All events scaled momentum
$1/\sigma\, d\sigma/dx_p$
Data
Analysis.LEP91.2.5
MC/Data
x_p
Log of scaled momentum, $\log(1/x_p)$
$N\, d\sigma/d\log(1/x_p)$
Data
Analysis.LEP91.2.5
MC/Data
$\log(1/x_p)$

Observables




Scaled energy of $D^{*\pm}$ in $e^+e^- \to Z \to$ hadronic at $\sqrt{s} = 91.2$ GeV
$1/N_{Z\mathrm{had}} \mathrm{d}N(D^*)/\mathrm{d}X_E$
Data
Analysis.LEP91.2.5
MC/Data
X_E
J/ψ scaled momentum
$1/\sigma\, \mathrm{d}\sigma/\mathrm{d}x_p/0.1$
Data
Analysis.LEP91.2.5
MC/Data
x_p
b quark fragmentation function $f(x_B^{\mathrm{weak}})$
$1/N\, \mathrm{d}N/\mathrm{d}x_B$
Data
Analysis.LEP91.2.5
MC/Data
x_B
b quark fragmentation function $f(x_B^{\mathrm{weak}})$
$1/N\, \mathrm{d}N/\mathrm{d}x_B$
Data
Analysis.LEP91.2.5
MC/Data
x_B

- practicalities of hadronisation models: parameters
    - kinematics of string or cluster decay: 2-5 parameters
    - must "pop" quark or diquark flavours in string or cluster decay — cannot be completely democratic or driven by masses alone
      $\longrightarrow$ suppression factors for strangeness, diquarks 2-10 parameters
    - transition to hadrons, cannot be democratic over multiplets
      $\longrightarrow$ adjustment factors for vectors/tensors etc. 2-6 parameters
- tuned to LEP data, overall agreement satisfying
- validity for hadron data not quite clear

  (beam remnant fragmentation not in LEP.)
- there are some issues with inclusive strangeness/baryon production

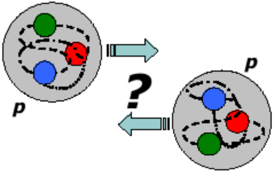
SIMULATING SOFT QCD

UNDERLYING EVENT

Contents

Multiple parton scattering

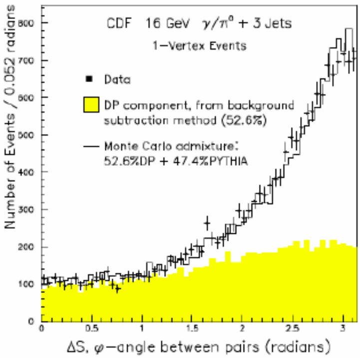
- hadrons = extended objects!
- no guarantee for one scattering only.
- running of α_S
  $\Longrightarrow$ preference for soft scattering.



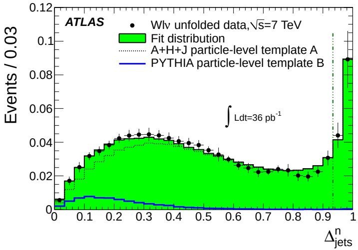
- first experimental evidence for double–parton scattering: events with $\gamma + 3$ jets:
  - cone jets, $R = 0.7$, $E_T > 5$ GeV; $|\eta_j| < 1.3$;
  - "clean sample": two softest jets with $E_T < 7$ GeV;
- cross section for DPS

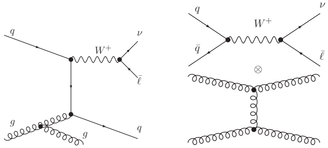
$$\sigma_{\rm DPS} = \frac{\sigma_{\gamma j}\sigma_{jj}}{\sigma_{\rm eff}}$$

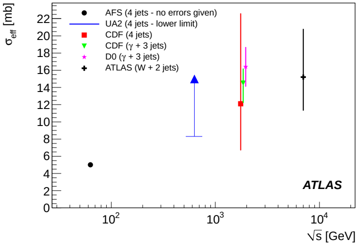
$\sigma_{\rm eff} \approx 14 \pm 4$ mb.

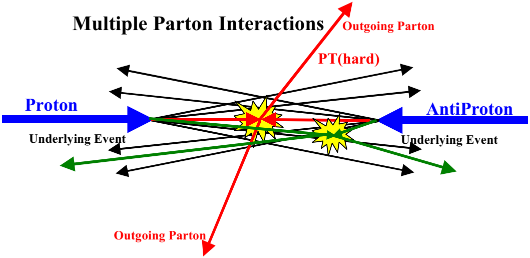


- more measurements, also at LHC
- ATLAS results from $W+2$ jets









but: how to define the underlying event?

1. everything apart from the hard interaction, but including IS showers, FS showers, remnant hadronisation.
2. remnant-remnant interactions, soft and/or hard.
3. lesson: hard to define

- origin of MPS: parton–parton scattering cross section exceeds hadron–hadron total cross section

$$\sigma_{\rm hard}(p_{\perp,\rm min}) = \int\limits_{p^2_{\perp,\rm min}}^{s/4} {\rm d}p_\perp^2 \frac{{\rm d}\sigma(p_\perp^2)}{{\rm d}p_\perp^2} > \sigma_{pp,\rm total}$$

  for low $p_{\perp,\rm min}$

- remember

$$\frac{{\rm d}\sigma(p_\perp^2)}{{\rm d}p_\perp^2} = \int\limits_0^1 {\rm d}x_1 {\rm d}x_2 f(x_1,\, q^2) f(x_2,\, q^2) \frac{{\rm d}\hat\sigma_{2\to 2}}{{\rm d}p_\perp^2}$$

- $\langle \sigma_{\rm hard}(p_{\perp,\rm min})/\sigma_{pp,\rm total}\rangle \geq 1$
- depends strongly on cut-off $p_{\perp,\rm min}$ (energy-dependent)!

Modelling the underlying event

- take the old PYTHIA model as example:
  - start with hard interaction, at scale $Q^2_{\rm hard}$.
  - select a new scale $p_\perp^2$ from

$$\exp\left[-\frac{1}{\sigma_{\rm norm}}\int\limits_{p_\perp^2}^{Q^2_{\rm hard}} {\rm d}{p'_\perp}^2\,\frac{{\rm d}\sigma(p_\perp^2)}{{\rm d}{p'_\perp}^2}\right]$$

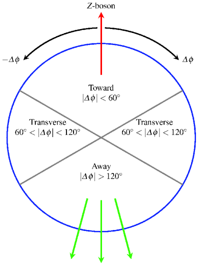
    with constraint $p_\perp^2 > p^2_{\perp,\rm min}$
  - rescale proton momentum ("proton-parton = proton with reduced energy").
  - repeat until no more allowed $2\to 2$ scatter

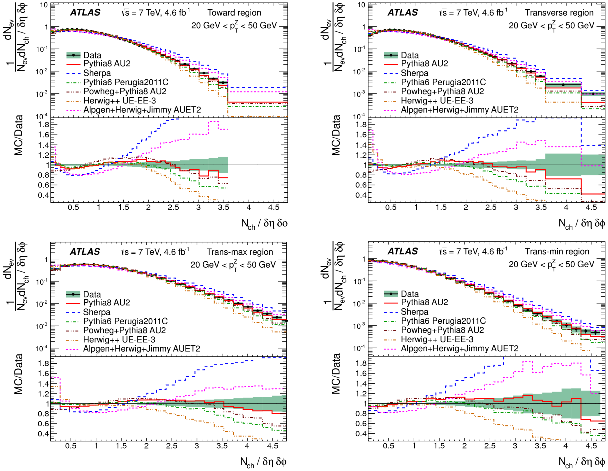
Modelling the underlying event

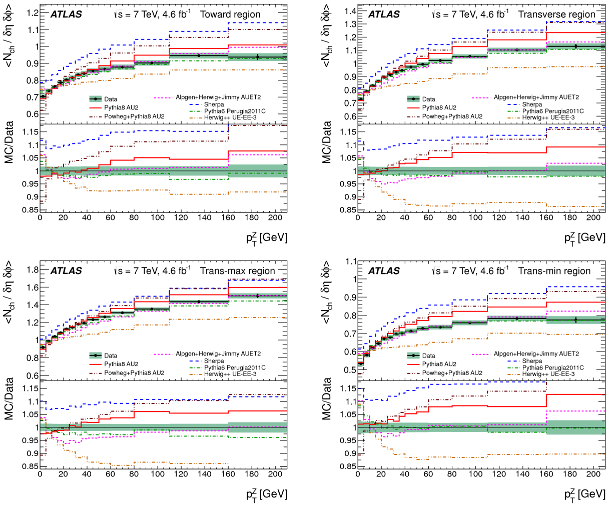
- possible refinements:
  - may add impact-parameter dependence $\longrightarrow$ more fluctuations
  - add parton showers to UE
  - "regularisation" to dampen sharp dependence on $p_{\perp,\mathrm{min}}$: replace $1/\hat{t}$ in MEs by $1/(t+t_0)$, also in α_s.
  - treat intrinsic $k_\perp$ of partons ($\to$ parameter)
  - model proton remnants ($\to$ parameter)

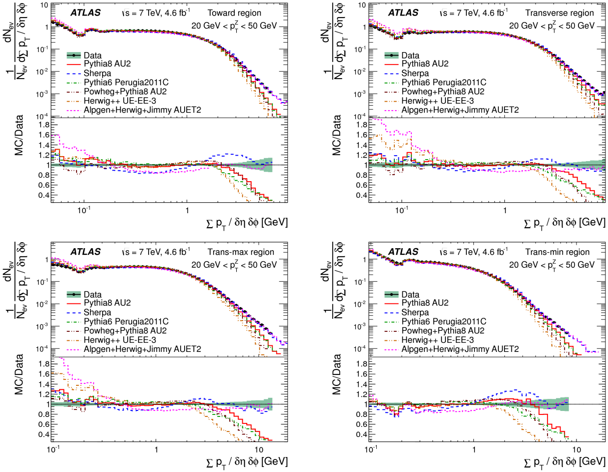
Some results for MPS in Z production

- observables sensitive to MPS
- classical analysis: transverse regions in QCD/jet events
- idea: find the hardest system, orient event into regions:
    - toward region along system
    - away region back-to-back
    - transverse regions
- typically each in 120°




ATLAS
√s = 7 TeV, 4.6 fb⁻¹
Toward region
Transverse region
Trans-max region
Trans-min region
20 GeV < p_T^Z < 50 GeV
Data
Pythia8 AU2
Sherpa
Pythia6 Perugia2011C
Powheg+Pythia8 AU2
Herwig++ UE-EE-3
Alpgen+Herwig+Jimmy AUET2
MC/Data
N_ch / δη δφ


ATLAS
√s = 7 TeV, 4.6 fb-1
Toward region
Transverse region
Trans-max region
Trans-min region
<Nch / δη δφ>
MC/Data
Data
Pythia8 AU2
Powheg+Pythia8 AU2
Alpgen+Herwig+Jimmy AUET2
Sherpa
Pythia6 Perugia2011C
Herwig++ UE-EE-3
pTZ [GeV]


ATLAS
√s = 7 TeV, 4.6 fb⁻¹
Toward region
Transverse region
Trans-max region
Trans-min region
20 GeV < p_T^Z < 50 GeV
Data
Pythia8 AU2
Sherpa
Pythia6 Perugia2011C
Powheg+Pythia8 AU2
Herwig++ UE-EE-3
Alpgen+Herwig+Jimmy AUET2
MC/Data
Σ p_T / δη δφ [GeV]

- see some data comparison in Minimum Bias
- practicalities of underlying event models: parameters
  - profile in impact parameter space — 2-3 parameters
  - IR cut-off at reference energy, its energy evolution, dampening paramter and normalisation cross section — 4 parameters
  - treating colour connections to rest of event — 2-5 parameters
- tuned to LHC data, overall agreement satisfying
- energy extrapolation not exactly perfect, plus other process categories such as diffraction etc..

SIMULATING SOFT QCD

MINIMUM BIAS

Contents

Eikonal models

- optical theorem relates total cross section $\sigma_{\rm tot}$ with elastic scattering amplitude $\mathcal{A}(s,t)$ through

$$\sigma_{\rm tot}(s) \;=\; \frac{1}{s}\,{\rm Im}[\mathcal{A}(s,t=0)]$$

- rewriting $\mathcal{A}(s,t)$ as $a(s,b)$ in impact parameter space

$$\mathcal{A}(s,t=-\vec{q}_\perp^{\,2}) = 2s\int {\rm d}^2 b_\perp e^{i\vec{q}_\perp\cdot\vec{b}_\perp} a(s,\vec{b}_\perp)$$

yields (real part of $a(s,\vec{b}_\perp)$ vanishing)

$$\begin{aligned}
\sigma_{\rm tot}(s) &= 2\int {\rm d}^2 b_\perp {\rm Im}[a(s,\vec{b}_\perp)]\\
\sigma_{\rm el}(s) &= 2\int {\rm d}^2 b_\perp |a(s,\vec{b}_\perp)|^2\\
\sigma_{\rm inel}(s) &= \sigma_{\rm tot}(s)-\sigma_{\rm el}(s)
\end{aligned}$$

- good old strong interaction physics (Regge poles): cross section given by sum over Regge exchanges
- leading Regge pole (Pomeron) yields

$$\sigma_{\rm tot} = \sigma_0 \left(\frac{s}{s_0}\right)^{\alpha_P - 1}$$

- Donnachie-Landshoff:

$$\sigma_0 = 21.7\,\mathrm{mb}\;,\quad s_0 = 1\,\mathrm{GeV}^2\;,\quad \alpha_P = 1.0808$$

- CDF: $\sigma_0 = 24.36$ mb, rest like DL.
- two-pomeron fit:

$$\sigma_{\rm tot} = 0.0139\mathrm{mb}\left(\frac{s}{s_0}\right)^{0.452} + 24.22\mathrm{mb}\left(\frac{s}{s_0}\right)^{0.0667}$$

- predictions for $\sigma_{\rm tot}(14{\rm TeV})$:

$$\sigma_{\rm tot}(14{\rm TeV}) = \left\{ \begin{array}{ll} 101.5\,{\rm mb} & ({\rm DL}) \\ 114.0{\rm mb} & ({\rm CDF}) \\ 164.4{\rm mb} & (\text{two-pom}) \end{array} \right.$$

- common problem: all violate unitarity (because $\alpha_P > 1$)
- possible solution: use parametrisation for (see next slide) eikonal rather than for cross section

$$\Omega(s) \simeq \left(\frac{s}{s_0}\right)^{\alpha_P - 1}$$

- N.B.: pomeron intercept t-dependent

$$\alpha_P(t) = \alpha_P + \alpha'_P t \qquad (\alpha' = 0.25\ {\rm GeV}^{-2})$$

- common: parametrisation of amplitude through eikonal Ω

$$a(s,\,\vec{b}_\perp) = \frac{e^{-\Omega(s,\vec{b}_\perp)}-1}{2i}$$

Then:

$$\begin{aligned}
\sigma_{\rm tot}(s) &= 2\int {\rm d}^2b_\perp[1-e^{-\Omega(s,\vec{b}_\perp)}]\\
\sigma_{\rm el}(s) &= \int {\rm d}^2b_\perp[1-e^{-\Omega(s,\vec{b}_\perp)}]^2\\
\sigma_{\rm inel}(s) &= \int {\rm d}^2b_\perp[1-e^{-2\Omega(s,\vec{b}_\perp)}]
\end{aligned}$$

- impossible to describe "diffractive excitation" (like e.g. $p \to N(1440)$) with one eikonal only:
  such processes are a consequence of the
  internal structure of the colliding hadrons
- for description employ high-energy limit:
  in this limit the Fock states of the hadrons are "frozen",
  (lifetime of fluctuations $\tau = E/m^2$ large)
  and each component can interact separately, destroying coherence of the colliding hadrons
- way out: multi–channel eikonals
  assumption: proton is linear combination of diffractive eigenstates

- introduce Good-Walker states (diffractive eigenstates):

$$|p\rangle = \sum_i a_i|\phi_i\rangle, \text{ where } \langle\phi_i|\phi_k\rangle = \delta_{ik} \text{ and } \sum_i |a_i|^2 = 1$$

- these states diagonalise the $\mathcal{T}$-matrix:

$$\langle\phi_i|\mathrm{Im}\mathcal{T}|\phi_k\rangle = T_k^D\delta_{ik}$$

- therefore only "elastic scattering" of these states
- N.B.: use two states

$$|p,\,N^*\rangle = \frac{1}{\sqrt{2}}\left[|\phi_1\rangle \pm |\phi_2\rangle\right],$$

related to two different form factors,

$$F_{1,2}(q_\perp) = \beta_0^2(1\pm\kappa)\frac{\exp\left[-\frac{(1\pm\kappa)\xi q_\perp^2}{\Lambda^2}\right]}{\left[1+\frac{(1\pm\kappa)q_\perp^2}{\Lambda^2}\right]^2}$$

allows to write cross sections as

$$\begin{aligned}
\frac{\mathrm{d}\sigma_{\rm tot}}{\mathrm{d}^2 b} &= 2\mathrm{Im}\langle j|\mathcal{T}|j\rangle = 2\langle T\rangle \\
\frac{\mathrm{d}\sigma_{\rm el}}{\mathrm{d}^2 b} &= |\langle j|\mathcal{T}|j\rangle|^2 = \langle T\rangle^2 \\
\frac{\mathrm{d}\sigma_{\rm el+SD}}{\mathrm{d}^2 b} &= |\langle \phi_k|\mathcal{T}|j\rangle|^2 = \sum_k |a_{jk}|^2 T_k^2 = \langle T^2\rangle \\
\frac{\mathrm{d}\sigma_{\rm SD}}{\mathrm{d}^2 b} &= \langle T^2\rangle - \langle T\rangle^2
\end{aligned}$$

single diffraction given by statistical dispersion of absorption probabilities of diffractive eigenstates

Comments:

- if all eigenstates ϕ_k experienced same absorption:
  $\longrightarrow$ dispersion vanishes
  $\longrightarrow$ diffractive production xsec vanishes
- this happens in the black disc limit ($T_k = 1$), at small $b_\perp$
- consequence: already at Tevatron energies
  diffractive production processes due to large $b_\perp$
- this region is responsible for small t components
- also: there eikonal (equivalent to optical density, opacity) is small
  $\longrightarrow$ large rapidity gap survival probability

(due to small density, only few scatters)

Eikonals in partonic language

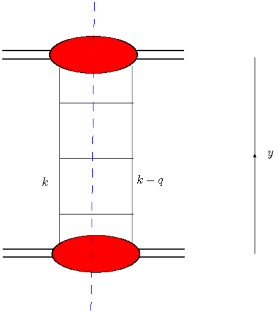
or: Regge physics rules

- now: need to construct meaningful eikonals
- preferable: make (loose) contact to QCD
- want to use Regge physics, in particular pomeron exchange, for amplitudes
- naively write $A(s, \vec{b}_\perp) \propto (s/s_0)^{\alpha_P(t)}$
- want to connect Regge picture to QCD
- unitarisation (naive): identify amplitude with eikonal, related to sum over all possible pomeron-exchanges between hadronic states

- trivial interpretation of pomeron in QCD
- pomeron as the sum of ladder-type diagrams, i.e. a sum over all (inelastic) $2 \to N$ processes

  - treat diagram (right) as amplitude squared for the production of N particles, homogeneously distributed in rapidity y with $y \in [-Y/2, Y/2]$ and $Y = \ln s/m_p^2$:

$$\sigma_{2\to N} = A_{2\to N}(Y) A^*_{2\to N}(Y)$$



- amplitude then reads

$$A(Y) = \sum_n \frac{1}{n!} \prod_{i=1}^{N} \int\limits_{-Y/2}^{Y/2} \mathrm{d}y_i \alpha = \sum_n \frac{(\alpha Y)^n}{n!} = e^{\alpha Y} = s^\alpha$$

- with the $k_\perp$-integral over each cell of the ladder:

$$\alpha(q_\perp^2) = \frac{g^2}{16\pi^2} \int \frac{\mathrm{d}^2 k_\perp}{[k_\perp^2 - m^2][(k_\perp - q_\perp)^2 - m^2]}$$

- alternative way to obtain result:

$$\frac{\mathrm{d}A(y)}{\mathrm{d}y} = \alpha A(y)$$

- similar to DGLAP equation, but for evolution in y - α plays role of splitting kernel

(describing with uniform emission probability)

- typically written as recursion from the amplitude for the $n-1$ to the n particles final state:

$$f_n(x,k_\perp) = \frac{N_c\alpha_s}{\pi}\int\limits_x^1 \frac{\mathrm{d}x'}{x'}\int\frac{\mathrm{d}^2k'_\perp}{\pi}K(k_\perp,k'_\perp)f_{n-1}(x',k'_\perp)$$

- rewrite recursively

$$\frac{\mathrm{d}f}{\mathrm{d}\ln 1/x} = \frac{N_c\alpha_s}{\pi}\,K\otimes f$$

therefore $f\propto x^{-\Delta_0}$, where $\alpha_P = 1+\Delta$ and

$$\Delta_0 = \frac{N_c\alpha_s}{\pi}\langle K\rangle = \frac{N_c\alpha_s}{\pi}4\ln 2\approx 0.5$$

- including higher order effects,

$$\Delta\approx\Delta_0\exp\left(-6\frac{N_c\alpha_s}{\pi}\right)\approx 0.3$$

- interesting reinterpretation of eikonal
- start with the evolution equation for the bare pomeron contribution to the elastic amplitude:

$$\frac{\mathrm{d}f(y)}{\mathrm{d}y} = \Delta f(y)$$

- this can also be regarded as the evolution equation for the parton density of hadron i (entering at $y = -Y/2$) in the presence of hadron k:

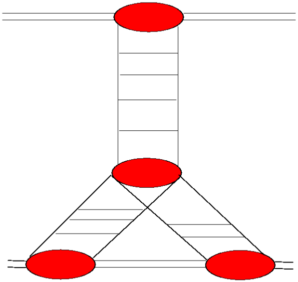
$$\frac{\mathrm{d}\Omega_{i(k)}(y)}{\mathrm{d}y} = \Delta\Omega_{i(k)}(y) \iff \Omega_{i(k)}(y) = g_i^2 e^{\Delta(y+Y/2)}$$

- g_i is fixed by the initial condition, i.e. the interaction probability or the parton density at $y = -Y/2$

(given by the Fourier transform related to the eigenstate ϕ_i)

Rescattering and other effects

- high-density, strong coupling regime:
  fusion of initial state partons
  rescattering effects important
- in other words: large triple pomeron vertex g_{3P}
- physical effect: high-mass dissociation through diagrams like the one ot the right
- note: other cuts apart from naive vertical possible



- resumming all rescattering contributions
- multi-pomeron contributions (rescattering) arise from absorption of intermediate s-channel partons (the rungs in the ladder during evolution
- allowing for all possible rescatterings/absorptions, must sum over all possible ladder configurations:

$$\frac{\mathrm{d}f(y)}{\mathrm{d}y} = \exp[-\lambda f(y)]\Delta f(y)$$

where $\lambda \propto g_{3P} \approx 0.25g$, the pomeron-nucleon coupling

The Khoze-Martin-Ryskin model (naive version)

- construct eikonal from individual parton densities:

$$\Omega(\vec{b}_\perp) = \frac{1}{2\beta_0^2}\int \mathrm{d}^2 b_\perp^{(1)} \mathrm{d}^2 b_\perp^{(2)} \delta^2(\vec{b}_\perp - \vec{b}_\perp^{(1)} - \vec{b}_\perp^{(2)}) \cdot \Omega_{i(k)}(\vec{b}_\perp^{(1)}, \vec{b}_\perp^{(2)}, y)\Omega_{(i)k}(\vec{b}_\perp^{(1)}, \vec{b}_\perp^{(2)}, y)$$

(the value of the integral will not depend on y)

- where the parton densities fulfil the evolution equations

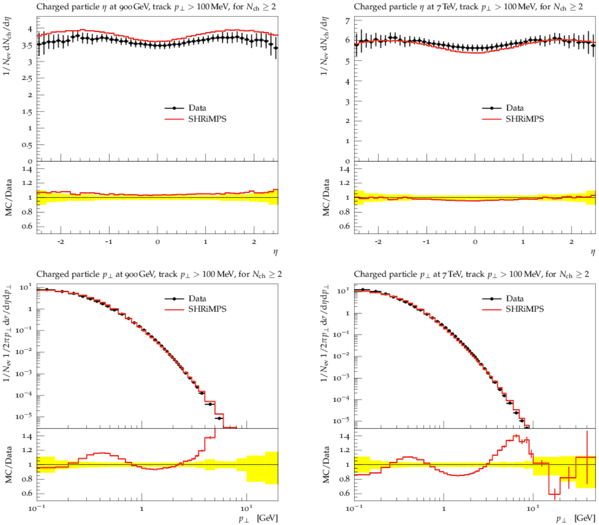
$$\frac{\mathrm{d}\ln\Omega_{i(k)}(y)}{\mathrm{d}y} = +\exp\left\{-\frac{\lambda}{2}\left[\Omega_{i(k)}(y) + \Omega_{(i)k}(y)\right]\right\}\Delta$$
$$\frac{\mathrm{d}\ln\Omega_{(i)k}(y)}{\mathrm{d}y} = -\exp\left\{-\frac{\lambda}{2}\left[\Omega_{i(k)}(y) + \Omega_{(i)k}(y)\right]\right\}\Delta$$

- boundary conditions

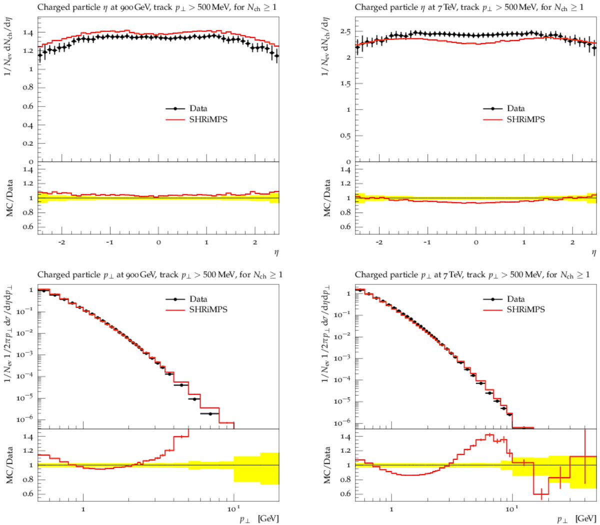
$$\Omega_{i(k)}(\vec{b}_\perp^{(1)}, \vec{b}_\perp^{(2)}, -Y/2) \; = F_i(b_\perp(1)^2)$$
$$\Omega_{(i)k}(\vec{b}_\perp^{(1)}, \vec{b}_\perp^{(2)}, +Y/2) \; = \; F_k(b_\perp(2)^2)$$

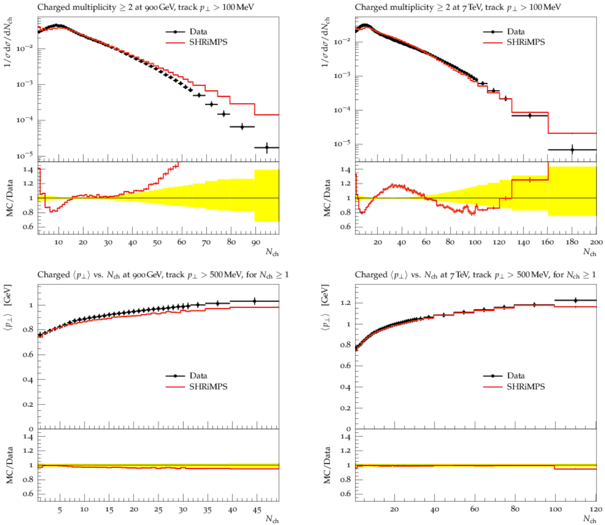
- Form factor parameters:
  $\beta_0^2 \approx 30$ mb, $\kappa \approx 0.5$, $\Lambda^2 \approx 1.5$ GeV^2, $\xi \approx 0.2$
- Evolution parameters: $\Delta \approx 0.3$, $\lambda \approx 0.25$

Some results for Minimum Bias

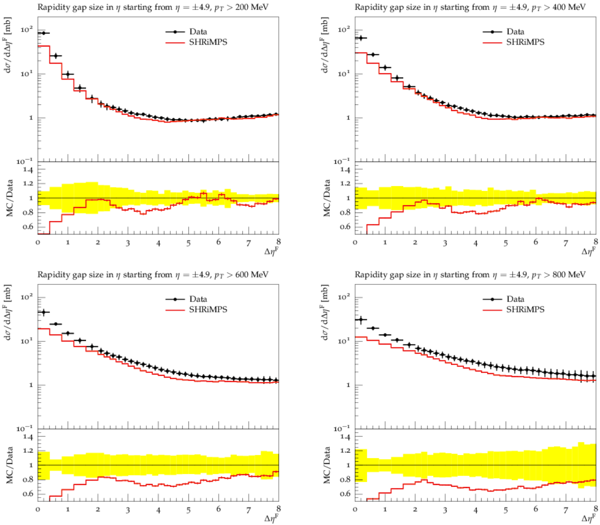


Some results




Charged multiplicity ≥ 2 at 900 GeV, track $p_\perp$ > 100 MeV
Charged multiplicity ≥ 2 at 7 TeV, track $p_\perp$ > 100 MeV
Charged ⟨$p_\perp$⟩ vs. N_{ch} at 900 GeV, track $p_\perp$ > 500 MeV, for N_{ch} ≥ 1
Charged ⟨$p_\perp$⟩ vs. N_{ch} at 7 TeV, track $p_\perp$ > 500 MeV, for N_{ch} ≥ 1
Data
SHRiMPS
$1/\sigma\, d\sigma/dN_{ch}$
⟨$p_\perp$⟩ [GeV]
MC/Data
N_{ch}

Some results



ROUND IV: PRECISION MONTE CARLO

PRECISION MONTE CARLO

PARTON SHOWERS, ONCE MORE

Contents

Improving event generators

The inner working of event generators

. . . simulation: divide et impera

- hard process:
  fixed order perturbation theory

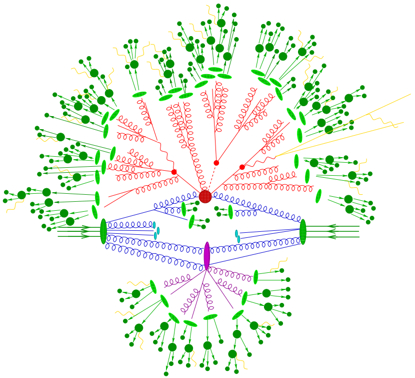
  traditionally: Born-approximation

- bremsstrahlung:
  resummed perturbation theory

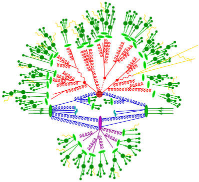
- hadronisation:
  phenomenological models

- hadron decays:
  effective theories, data

- "underlying event":
  phenomenological models



... and possible improvements
possible strategies:



- improving the phenomenological models:
  - "tuning" (fitting parameters to data)
  - replacing by better models, based on more physics

    (my hot candidate: "minimum bias" and "underlying event" simulation)

- improving the perturbative description:
  - inclusion of higher order exact matrix elements and correct connection to resummation in the parton shower:

    "NLO-Matching" & "Multijet-Merging"

  - systematic improvement of the parton shower: next-to leading (or higher) logs & colours

- remember structure of NLO calculation for N-body production

$$\begin{aligned} \mathrm{d}\sigma &= \mathrm{d}\Phi_{\mathcal{B}}\mathcal{B}_N(\Phi_{\mathcal{B}}) + \mathrm{d}\Phi_{\mathcal{B}}\mathcal{V}_N(\Phi_{\mathcal{B}}) + \mathrm{d}\Phi_{\mathcal{R}}\mathcal{R}_N(\Phi_{\mathcal{R}}) \\ &= \mathrm{d}\Phi_{\mathcal{B}}\left(\mathcal{B}_N + \mathcal{V}_N + \mathcal{I}_N^{(\mathcal{S})}\right) + \mathrm{d}\Phi_{\mathcal{R}}\left(\mathcal{R}_N - \mathcal{S}_N\right) \end{aligned}$$

- phase space factorisation assumed here ($\Phi_{\mathcal{R}} = \Phi_{\mathcal{B}} \otimes \Phi_1$)

$$\int \mathrm{d}\Phi_1 \mathcal{S}_N(\Phi_{\mathcal{B}} \otimes \Phi_1) = \mathcal{I}_N^{(\mathcal{S})}(\Phi_{\mathcal{B}})$$

- process independent subtraction kernels

$$\begin{aligned} \mathcal{S}_N(\Phi_{\mathcal{B}} \otimes \Phi_1) &= \mathcal{B}_N(\Phi_{\mathcal{B}}) \otimes \mathcal{S}_1(\Phi_{\mathcal{B}} \otimes \Phi_1) \\ \mathcal{I}_N^{(\mathcal{S})}(\Phi_{\mathcal{B}} \otimes \Phi_1) &= \mathcal{B}_N(\Phi_{\mathcal{B}}) \otimes \mathcal{I}_1^{(\mathcal{S})}(\Phi_{\mathcal{B}}) \end{aligned}$$

  with universal $\mathcal{S}_1(\Phi_{\mathcal{B}} \otimes \Phi_1)$ and $\mathcal{I}_1^{(\mathcal{S})}(\Phi_{\mathcal{B}})$

- in Catani–Seymour invertible phase space mapping

$$\Phi_{\mathcal{R}} \longleftrightarrow \Phi_{\mathcal{B}} \otimes \Phi_1$$

Parton showers, again

- Sudakov form factor (**no-decay** probability)

$$\Delta^{(\mathcal{K})}_{ij,k}(t,t_0) = \exp\left[-\int\limits_{t_0}^{t}\frac{\mathrm{d}t}{t}\,\frac{\alpha_s}{2\pi}\int\mathrm{d}z\,\frac{\mathrm{d}\phi}{2\pi}\;\underbrace{\mathcal{K}_{ij,k}(t,z,\phi)}_{\substack{\text{splitting kernel for}\\ (ij)\to ij\ (\text{spectator } k)}}\right]$$

- evolution parameter t defined by kinematics

  generalised angle (HERWIG ++) or transverse momentum (PYTHIA, SHERPA)

- will replace $\frac{\mathrm{d}t}{t}\mathrm{d}z\frac{\mathrm{d}\phi}{2\pi} \longrightarrow \mathrm{d}\Phi_1$
- scale choice for strong coupling: $\alpha_s(k_\perp^2)$

  resums classes of higher logarithms

- regularisation through cut-off t_0

- "compound" splitting kernels $\mathcal{K}_n$ and Sudakov form factors $\Delta_n^{(\mathcal{K})}$ for emission off n-particle final state:

$$\mathcal{K}_n(\Phi_1) = \frac{\alpha_s}{2\pi} \sum_{\text{all }\{ij,k\}} \mathcal{K}_{ij,k}(\Phi_{ij,k}) \,, \quad \Delta_n^{(\mathcal{K})}(t, t_0) = \exp\left[-\int\limits_{t_0}^{t} \mathrm{d}\Phi_1 \, \mathcal{K}_n(\Phi_1)\right]$$

- consider first emission only off Born configuration

$$\mathrm{d}\sigma_B = \mathrm{d}\Phi_N \, \mathcal{B}_N(\Phi_N) \cdot \underbrace{\left\{\Delta_N^{(\mathcal{K})}(\mu_N^2, t_0) + \int\limits_{t_0}^{\mu_N^2} \mathrm{d}\Phi_1 \left[\mathcal{K}_N(\Phi_1)\Delta_N^{(\mathcal{K})}(\mu_N^2, t(\Phi_1))\right]\right\}}_{\text{integrates to unity} \longrightarrow \text{"unitarity" of parton shower}}$$

- further emissions by recursion with $Q^2 = t$ of previous emission

- analyse connection to Q_T resummation formalism
- consider standard Collins-Soper-Sterman formalism (CSS):

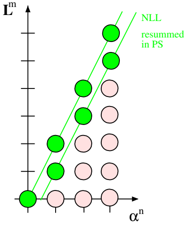
$$\frac{\mathrm{d}\sigma_{AB\to X}}{\mathrm{d}y\mathrm{d}Q_\perp^2} = \mathrm{d}\Phi_X\,\mathcal{B}_{ij}(\Phi_X)\cdot\underbrace{\int\frac{\mathrm{d}^2b_\perp}{(2\pi)^2}\exp(i\vec{b}_\perp\cdot\vec{Q}_\perp)}_{\text{guarantee 4-mom conservation}}\underbrace{\tilde{W}_{ij}(b;\,\Phi_X)}_{\text{higher orders}}$$

with

$$\tilde{W}_{ij}(b;\,\Phi_X) = \overbrace{C_i(b;\,\Phi_X,\,\alpha_s)C_j(b;\,\Phi_X,\,\alpha_s)}^{\text{collinear bits}}\overbrace{H_{ij}(\alpha_s)}^{\text{loops}}$$
$$\underbrace{\exp\left[-\int\limits_{1/b_\perp^2}^{Q_X^2}\frac{\mathrm{d}k_\perp^2}{k_\perp^2}\left(A(\alpha_s(k_\perp^2))\log\frac{Q_X^2}{k_\perp^2}+B(\alpha_s(k_\perp^2))\right)\right]}_{\text{Sudakov form factor, } A,\,B \text{ expanded in powers of } \alpha_s}$$

- analyse structure of emissions above
- logarithmic accuracy in $\log\frac{\mu_N}{k_\perp}$ (a la CSS)
  possibly up to next-to leading log,
  - if evolution parameter $\sim$ transverse momentum,
  - if argument in α_s is $\propto\ k_\perp$ of splitting,
  - if $K_{ij,k}\to$ terms $A_{1,2}$ and B_1 upon integration

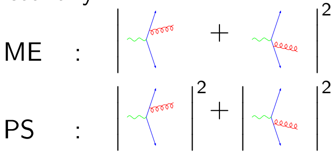
  (okay, if soft gluon correction is included, and if $K_{ij,k}\ \to$ AP splitting kernels)

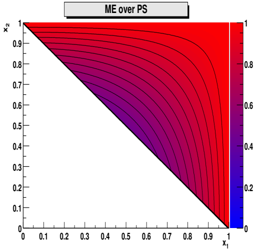


- in CSS $k_\perp$ typically is the transverse momentum of produced system, in parton shower of course related to the cumulative effect of explicit multiple emissions
- resummation scale $\mu_N\approx\mu_F$ given by (Born) kinematics –
  simple for cases like $q\bar{q}'\to V$, $gg\to H$, ...
  tricky for more complicated cases

Matrix element corrections

- parton shower ignores interferences typically present in matrix elements
- pictorially



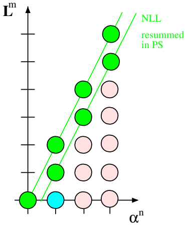


- form many processes $\mathcal{R}_N < \mathcal{B}_N \times \mathcal{K}_N$
- typical processes: $q\bar{q}' \to V$, $e^-e^+ \to q\bar{q}$, $t \to bW$
- practical implementation: shower with usual algorithm, but reject first/hardest emissions with probability $\mathcal{P} = \mathcal{R}_N/(\mathcal{B}_N \times \mathcal{K}_N)$

- analyse **first** emission, given by

$$\mathrm{d}\sigma_B = \mathrm{d}\Phi_N\,\mathcal{B}_N(\Phi_N)$$
$$\cdot\underbrace{\left\{\Delta_N^{(\mathcal{R}/\mathcal{B})}(\mu_N^2,t_0)+\int\limits_{t_0}^{\mu_N^2}\mathrm{d}\Phi_1\left[\frac{\mathcal{R}_N(\Phi_N\times\Phi_1)}{\mathcal{B}_N(\Phi_N)}\Delta_N^{(\mathcal{R}/\mathcal{B})}(\mu_N^2,t(\Phi_1))\right]\right\}}_{\text{once more: integrates to unity}\,\longrightarrow\,\text{“unitarity” of parton shower}}$$

- radiation given by $\mathcal{R}_N$ (correct at $\mathcal{O}(\alpha_s)$)
  (but modified by logs of higher order in α_s from $\Delta_N^{(\mathcal{R}/\mathcal{B})}$)
- emission phase space constrained by μ_N
- also known as “soft ME correction”
  hard ME correction fills missing phase space
- used for “power shower”:
  $\mu_N\to E_{pp}$ and apply ME correction



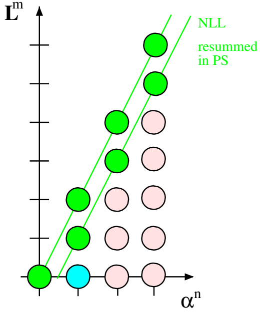
PRECISION MONTE CARLO

NLO MATCHING

Contents

NLO matching: Basic idea

- parton shower resums logarithms
  fair description of collinear/soft emissions
  jet evolution (where the logs are large)
- matrix elements exact at given order
  fair description of hard/large-angle emissions
  jet production (where the logs are small)
- adjust ("match") terms:
  - cross section at NLO accuracy &
    correct hardest emission in PS to exactly
    reproduce ME at order α_s
    ($\mathcal{R}$-part of the NLO calculation)
    (this is relatively trivial)
  - maintain (N)LL-accuracy of parton shower
    (this is not so simple to see)



PowHeg

- reminder: $\mathcal{K}_{ij,k}$ reproduces process-independent behaviour of $\mathcal{R}_N/\mathcal{B}_N$ in soft/collinear regions of phase space

$$\mathrm{d}\Phi_1\,\frac{\mathcal{R}_N(\Phi_{N+1})}{\mathcal{B}_N(\Phi_N)} \xrightarrow{\text{IR}} \mathrm{d}\Phi_1\,\frac{\alpha_s}{2\pi}\,\mathcal{K}_{ij,k}(\Phi_1)$$

- define modified Sudakov form factor (as in ME correction)

$$\Delta_N^{(\mathcal{R}/\mathcal{B})}(\mu_N^2, t_0) = \exp\left[-\int\limits_{t_0}^{\mu_N^2}\mathrm{d}\Phi_1\,\frac{\mathcal{R}_N(\Phi_{N+1})}{\mathcal{B}_N(\Phi_N)}\right],$$

- assumes factorisation of phase space: $\Phi_{N+1} = \Phi_N \otimes \Phi_1$
- typically will adjust scale of α_s to parton shower scale

- define local K-factors
- start from Born configuration Φ_N with NLO weight:

("local K-factor")

$$
\begin{aligned}
\mathrm{d}\sigma_N^{(\mathrm{NLO})} &= \mathrm{d}\Phi_N\,\bar{\mathcal{B}}(\Phi_N) \\
&= \mathrm{d}\Phi_N \Bigg\{ \mathcal{B}_N(\Phi_N) + \underbrace{\mathcal{V}_N(\Phi_N) + \mathcal{B}_N(\Phi_N)\otimes\mathcal{S}}_{\tilde{\mathcal{V}}_N(\Phi_N)} \\
&\qquad + \int \mathrm{d}\Phi_1 \left[\mathcal{R}_N(\Phi_N\otimes\Phi_1) - \mathcal{B}_N(\Phi_N)\otimes\mathrm{d}\mathcal{S}(\Phi_1)\right] \Bigg\}
\end{aligned}
$$

- by construction: exactly reproduce cross section at NLO accuracy
- note: second term vanishes if $\mathcal{R}_N \equiv \mathcal{B}_N \otimes \mathrm{d}\mathcal{S}$

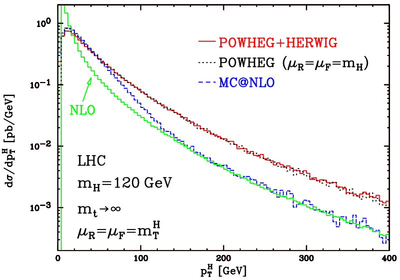
(relevant for MC@NLO)

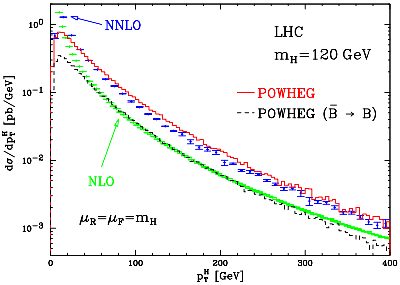
- analyse accuracy of radiation pattern
- generate emissions with $\Delta_N^{(\mathcal{R}/\mathcal{B})}(\mu_N^2,\, t_0)$:

$$\mathrm{d}\sigma_N^{\rm (NLO)} = \mathrm{d}\Phi_N\,\bar{\mathcal{B}}(\Phi_N)$$
$$\times \underbrace{\left\{\Delta_N^{(\mathcal{R}/\mathcal{B})}(\mu_N^2,\, t_0) + \int\limits_{t_0}^{\mu_N^2}\mathrm{d}\Phi_1\,\frac{\mathcal{R}_N(\Phi_N\otimes\Phi_1)}{\mathcal{B}_N(\Phi_N)}\,\Delta_N^{(\mathcal{R}/\mathcal{B})}(\mu_N^2,\, k_\perp^2(\Phi_1))\right\}}_{\text{integrating to yield 1 - ``unitarity of parton shower''}}$$

- radiation pattern like in ME correction
- pitfall, again: choice of upper scale μ_N^2 (this is vanilla POWHEG!)
- apart from logs: which configurations enhanced by local K-factor

(K-factor for inclusive production of X adequate for $X+$ jet at large $p_\perp$?)



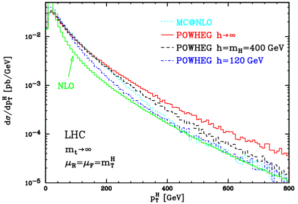


- large enhancement at high $p_{T,h}$
- can be traced back to large NLO correction
- fortunately, NNLO correction is also large $\to\sim$ agreement

- improving PowHeg
- split real-emission ME as

$$\mathcal{R} = \mathcal{R}\left(\underbrace{\frac{h^2}{p_\perp^2 + h^2}}_{\mathcal{R}^{(S)}} + \underbrace{\frac{p_\perp^2}{p_\perp^2 + h^2}}_{\mathcal{R}^{(F)}}\right)$$

- can "tune" h to mimick NNLO - or maybe resummation result



- differential event rate up to first emission

$$\mathrm{d}\sigma = \mathrm{d}\Phi_B \bar{\mathcal{B}}^{(\mathrm{R}^{(\mathrm{S})})} \left[\Delta^{(\mathcal{R}^{(S)}/\mathcal{B})}(s,\, t_0) + \int\limits_{t_0}^{s} \mathrm{d}\Phi_1 \frac{\mathcal{R}^{(S)}}{\mathcal{B}} \Delta^{(\mathcal{R}^{(S)}/\mathcal{B})}(s,\, k_\perp^2)\right]$$

$$+\mathrm{d}\Phi_R\, \mathcal{R}^{(F)}(\Phi_R)$$

MC@NLO

- MC@NLO paradigm: divide $\mathcal{R}_N$ in soft (“S”) and hard (“H”) part:

$$\mathcal{R}_N = \mathcal{R}_N^{(S)} + \mathcal{R}_N^{(H)} = \mathcal{B}_N \otimes \mathrm{d}\mathcal{S}_1 + \mathcal{H}_N$$

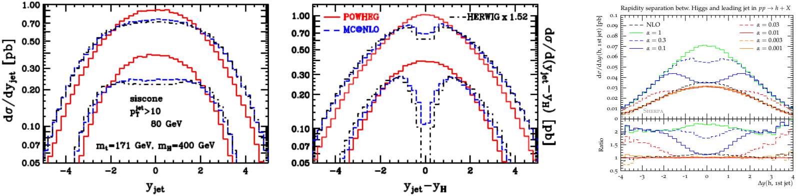
- identify subtraction terms and shower kernels $\mathrm{d}\mathcal{S}_1 \equiv \sum\limits_{\{ij,k\}} \mathcal{K}_{ij,k}$

(modify $\mathcal{K}$ in 1^{st} emission to account for colour)

$$\mathrm{d}\sigma_N = \mathrm{d}\Phi_N \underbrace{\tilde{\mathcal{B}}_N(\Phi_N)}_{\mathcal{B}+\tilde{\mathcal{V}}} \left[\Delta_N^{(\mathcal{K})}(\mu_N^2,\, t_0) + \int\limits_{t_0}^{\mu_N^2} \mathrm{d}\Phi_1\, \mathcal{K}_{ij,k}(\Phi_1)\, \Delta_N^{(\mathcal{K})}(\mu_N^2,\, k_\perp^2)\right] + \mathrm{d}\Phi_{N+1}\, \mathcal{H}_N$$

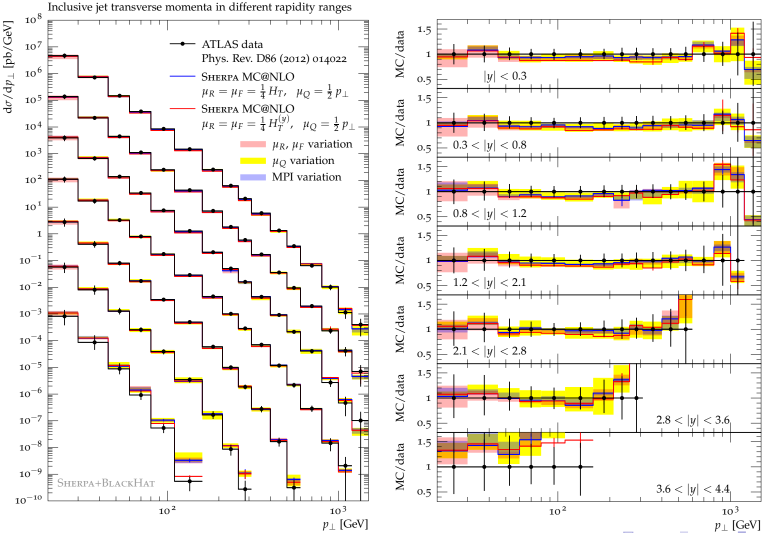
- effect: only resummed parts modified with local K-factor

- phase space effects: shower vs. fixed order

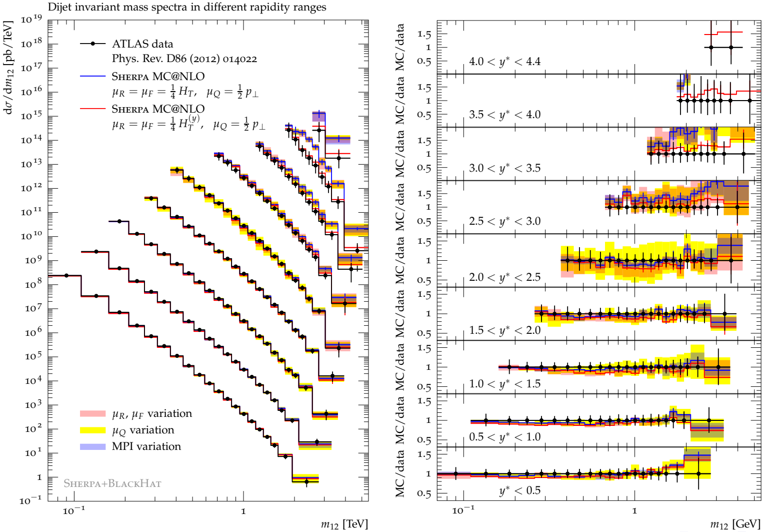


- problem: impact of subtraction terms on local K-factor (filling of phase space by parton shower)
- studied in case of $gg \to H$ above
- proper filling of available phase space by parton shower paramount

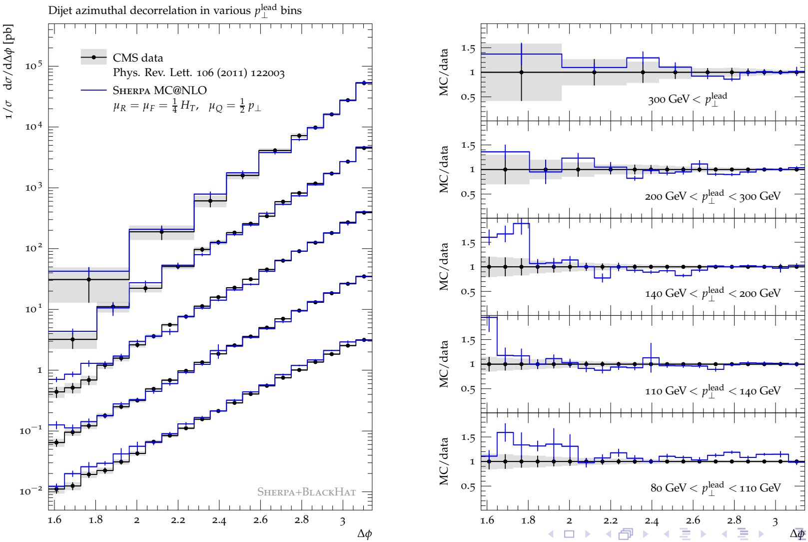
MC@NLO for light jets: jet-$p_\perp$



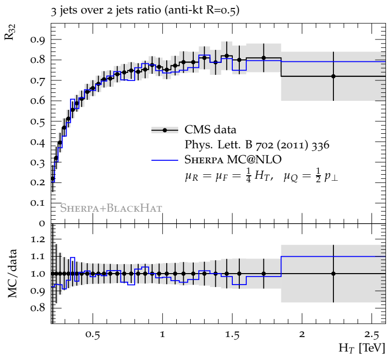
MC@NLO for light jets: dijet mass

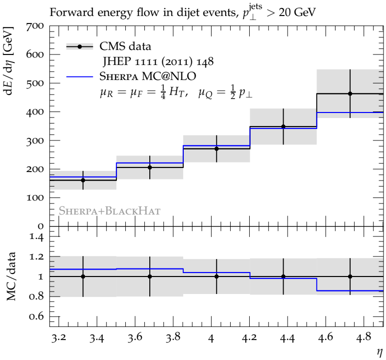


MC@NLO for light jets: azimuthal decorrelations

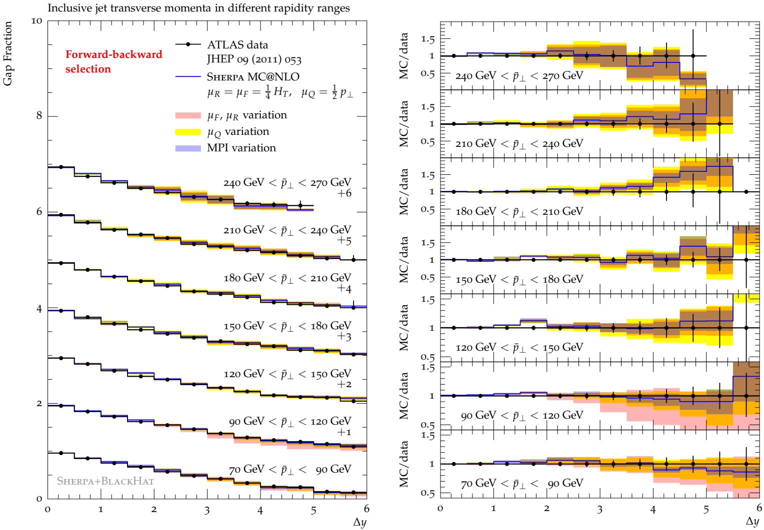


MC@NLO for light jets: R_{32} & forward energy flow



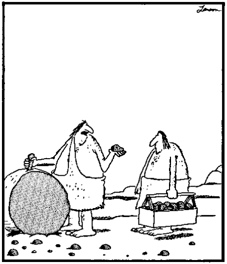


MC@NLO for light jets: jet vetoes



Summary

- reviewed higher order calculations and parton shower, with emphasis on accuracy
- these things are not blackboxes and can be understood
- systematic improvement of event generators by including higher orders has been at the core of QCD theory and developments in the past decade:
  - ME corrections
  - NLO matching ("MC@NLO", "PowHeg")



"So what's this? I asked for a *hammer*! A hammer! *This* is a crescent wrench! ... Well, maybe it's a hammer. ... Damn these stone tools."

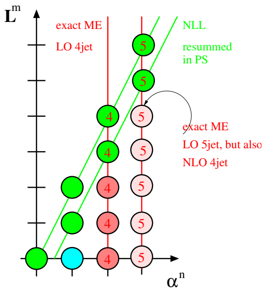
PRECISION MONTE CARLO

MULTIJET MERGING

Contents

Multijet merging: basic idea

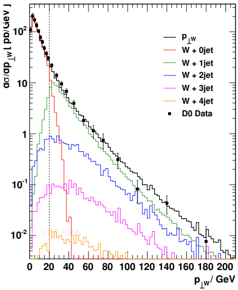
- parton shower resums logarithms
  fair description of collinear/soft emissions
  jet evolution (where the logs are large)
- matrix elements exact at given order
  fair description of hard/large-angle emissions
  jet production (where the logs are small)
- combine ("merge") both:
  result: "towers" of MEs with increasing number of jets evolved with PS
  - multijet cross sections at Born accuracy
  - maintain (N)LL accuracy of parton shower



- separate regions of jet production and jet evolution with jet measure Q_J

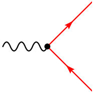
  ("truncated showering" if not identical with evolution parameter)

- matrix elements populate hard regime
- parton showers populate soft domain

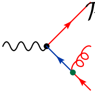


Why it works: jet rates with the parton shower

- consider jet production in $e^+e^- \to$ hadrons
  Durham jet definition: relative transverse momentum $k_\perp > Q_J$
- fixed order: one factor α_S and up to $\log^2 \frac{E_{\rm c.m.}}{Q_J}$ per jet
- use Sudakov form factor for resummation &
  replace approximate fixed order by exact expression:



$$\mathcal{R}_2(Q_J) = \left[\Delta_q(E^2_{\rm c.m.},\, Q_J^2)\right]^2$$



$$\mathcal{R}_3(Q_J) = 2\Delta_q(E^2_{\rm c.m.},\, Q_J^2) \int\limits_{Q_J^2}^{E^2_{\rm c.m.}} \frac{{\rm d}k_\perp^2}{k_\perp^2} \left[\frac{\alpha_s(k_\perp^2)}{2\pi}{\rm d}z\mathcal{K}_q(k_\perp^2, z)\right.$$
$$\left.\times\Delta_q(E^2_{\rm c.m.},\, k_\perp^2)\Delta_q(k_\perp^2,\, Q_J^2)\Delta_g(k_\perp^2,\, Q_J^2)\right]$$

Multijet merging at LO

- expression for first emission

$$d\sigma = d\Phi_N \mathcal{B}_N \left[\Delta_N^{(\mathcal{K})}(\mu_N^2, t_0) + \int_{t_0}^{\mu_N^2} d\Phi_1 \mathcal{K}_N \Delta_N^{(\mathcal{K})}(\mu_N^2, t_{N+1})\Theta(Q_J - Q_{N+1})\right]$$
$$+ d\Phi_{N+1} \mathcal{B}_{N+1} \Delta_N^{(\mathcal{K})}(\mu_{N+1}^2, t_{N+1})\Theta(Q_{N+1} - Q_J)$$

- note: $N+1$-contribution includes also $N+2$, $N+3$, ...

(no Sudakov suppression below t_{n+1}, see further slides for iterated expression)

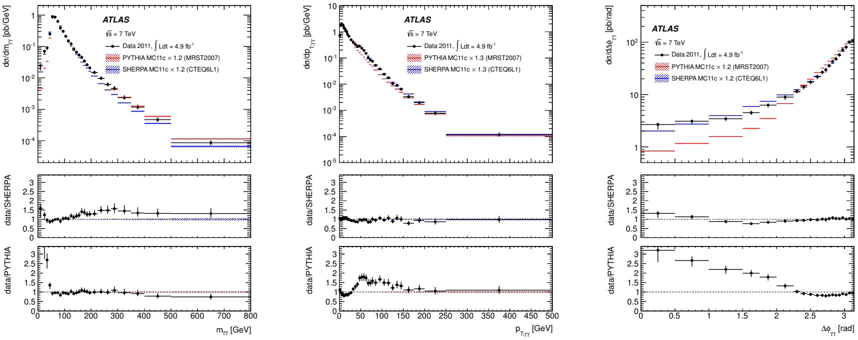
- potential occurence of different shower start scales: $\mu_{N,N+1,\ldots}$
- "unitarity violation" in square bracket: $\mathcal{B}_N \mathcal{K}_N \longrightarrow \mathcal{B}_{N+1}$

(cured with UMEPS formalism, L. Lonnblad & S. Prestel, JHEP 1302 (2013) 094 & S. Platzer, arXiv:1211.5467 [hep-ph] & arXiv:1307.0774 [hep-ph])

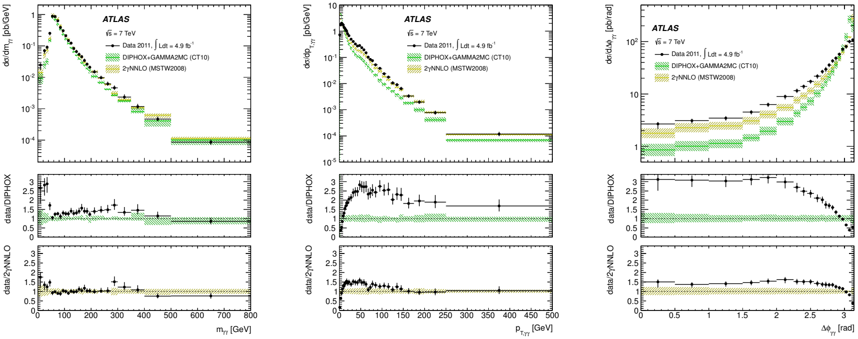
$$
\begin{aligned}
\mathrm{d}\sigma \;=\; & \sum_{n=N}^{n_{\max}-1} \Bigg\{ \mathrm{d}\Phi_n\,\mathcal{B}_n \overbrace{\left[\prod_{j=N}^{n-1} \Theta(Q_{j+1}-Q_J)\right]}^{(n-N)\text{ extra jets}} \overbrace{\left[\prod_{j=N}^{n-1} \Delta_j^{(\mathcal{K})}(t_j,\,t_{j+1})\right]}^{\text{no emissions off internal lines}} \\
& \times \Bigg[\underbrace{\Delta_n^{(\mathcal{K})}(t_n,t_0)}_{\text{no emission}} + \underbrace{\int\limits_{t_0}^{t_n} \mathrm{d}\Phi_1\,\mathcal{K}_n \Delta_n^{(\mathcal{K})}(t_n,t_{n+1})\Theta(Q_J-Q_{n+1})}_{\text{next emission no jet \& below last ME emission}} \Bigg] \\
& + \mathrm{d}\Phi_{n_{\max}}\,\mathcal{B}_{n_{\max}} \left[\prod_{j=N}^{n_{\max}-1} \Theta(Q_{j+1}-Q_J)\right] \left[\prod_{j=N}^{n_{\max}-1} \Delta_j^{(\mathcal{K})}(t_j,\,t_{j+1})\right] \\
& \times \left[\Delta_{n_{\max}}^{(\mathcal{K})}(t_{n_{\max}},t_0) + \int\limits_{t_0}^{t_{n_{\max}}} \mathrm{d}\Phi_1\,\mathcal{K}_{n_{\max}} \Delta_{n_{\max}}^{(\mathcal{K})}(t_{n_{\max}},t_{n_{\max}+1}) \right]
\end{aligned}
$$

Di-photons @ ATLAS: $m_{\gamma\gamma}$, $p_{\perp,\gamma\gamma}$, and $\Delta\phi_{\gamma\gamma}$ in showers

(arXiv:1211.1913 [hep-ex])



Aside: Comparison with higher order calculations



A step towards multijet-merging at NLO: MeNloPs

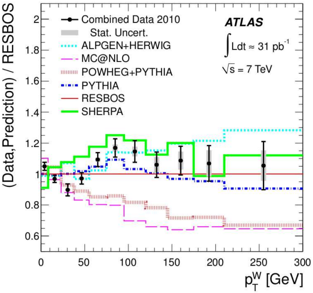
- combine matching for lowest multiplicity with multijet merging
- interpolating local K-factor for reweighting hard emissions

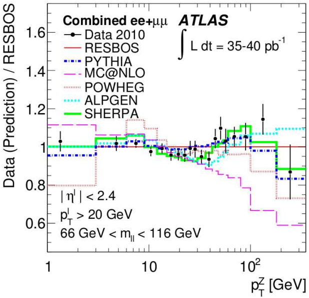
$$k_N(\Phi_{N+1}) = \frac{\tilde{\mathcal{B}}_N}{\mathcal{B}_N}\left(1 - \frac{\mathcal{H}_N}{\mathcal{B}_{N+1}}\right) + \frac{\mathcal{H}_N}{\mathcal{B}_{N+1}} \longrightarrow \begin{cases} \tilde{\mathcal{B}}_N/\mathcal{B}_N & \text{for soft emission} \\ 1 & \text{for hard emission} \end{cases}$$

$$\begin{aligned} \mathrm{d}\sigma \;=\; & \mathrm{d}\Phi_N\,\tilde{\mathcal{B}}_N\left[\Delta_N^{(\mathcal{K})}(\mu_N^2,t_0) + \int\limits_{t_0}^{\mu_N^2}\mathrm{d}\Phi_1\,\mathcal{K}_N\Delta_N^{(\mathcal{K})}(\mu_N^2,t_{N+1})\Theta(Q_J-Q_{N+1})\right] \\ & +\mathrm{d}\Phi_{N+1}\,\mathcal{H}_N\Delta_N^{(\mathcal{K})}(\mu_N^2,t_{N+1})\Theta(Q_J-Q_{N+1}) \\ & +\mathrm{d}\Phi_{N+1}\,k_N\,\mathcal{B}_{N+1}\Delta_N^{(\mathcal{K})}(\mu_N^2,t_{N+1})\Theta(Q_{N+1}-Q_J) \end{aligned}$$

Transverse momentum of W & Z boson

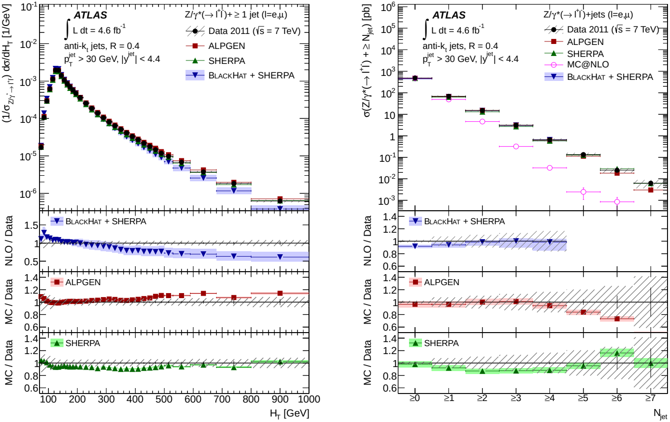
ATLAS, arXiv:1108.6308, arXiv:1107.2381





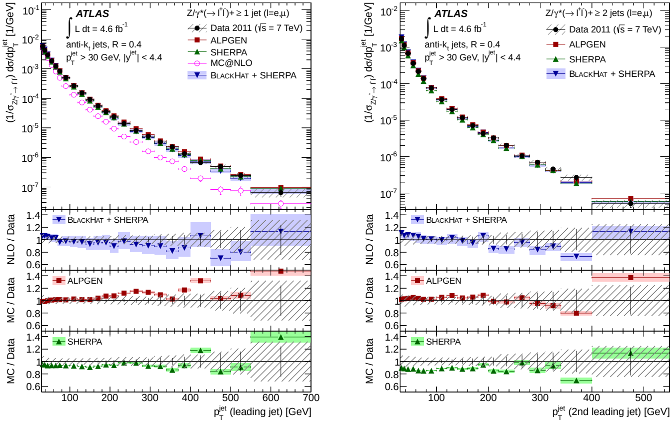
Z+jets: inclusive quantities

ATLAS, arXiv:1111.2690



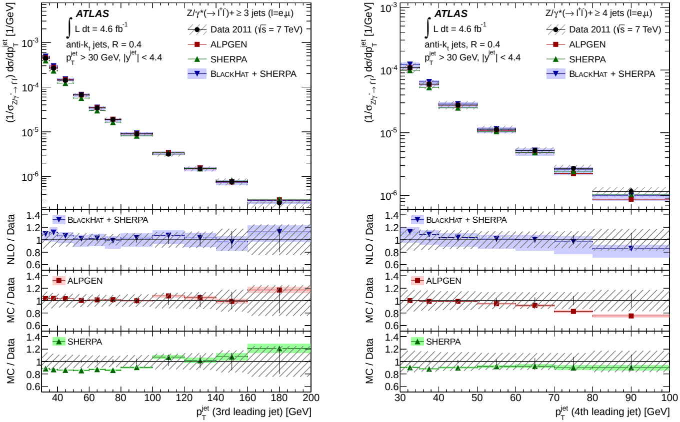
Z+jets: jet transverse momenta

ATLAS, arXiv:1111.2690



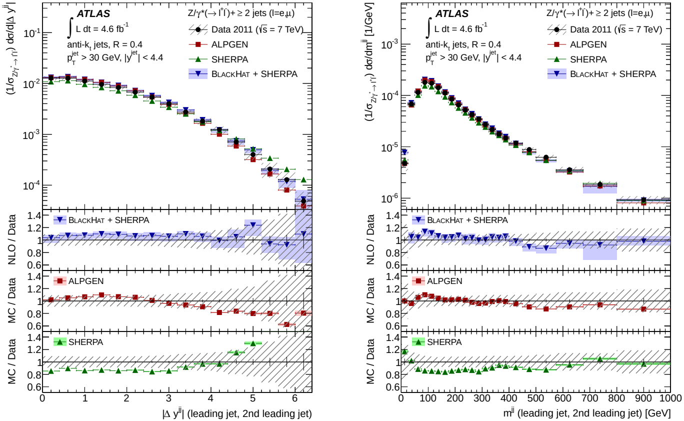
Z+jets: jet transverse momenta

ATLAS, arXiv:1111.2690



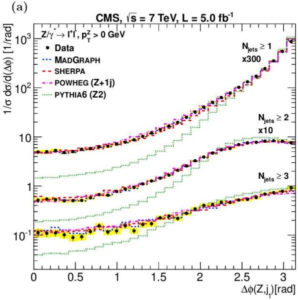
Z+jets: correlation of leading jets

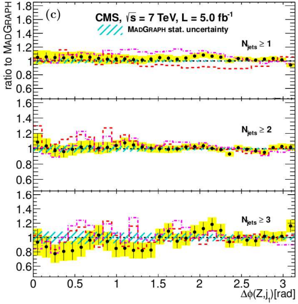
ATLAS, arXiv:1111.2690



Z+jets: $\Delta\phi_{Zj}$ in unboosted sample

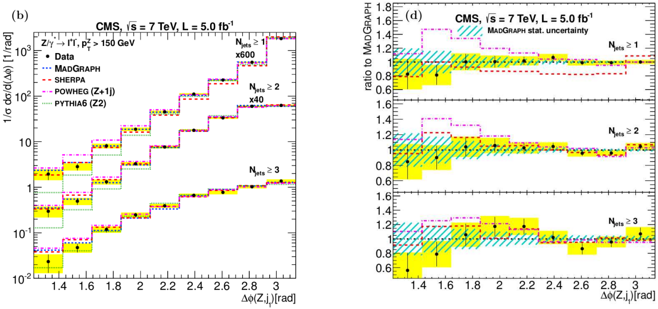
CMS, arXiv:1301.1646





Z+jets: $\Delta\phi_{Zj}$ in boosted sample

CMS, arXiv:1301.1646



Multijet-merging at NLO: MEPS@NLO

- basic idea like at LO: towers of MEs with increasing jet multi (but this time at NLO)
- combine them into one sample, remove overlap/double-counting

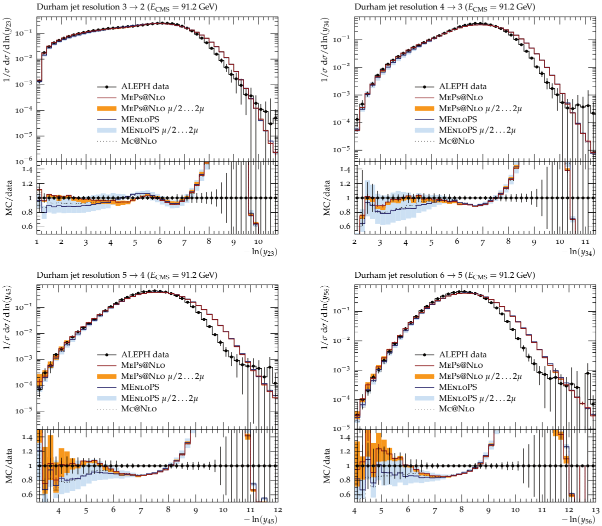
  maintain NLO and (N)LL accuracy of ME and PS

- this effectively translates into a merging of MC@NLO simulations and can be further supplemented with LO simulations for even higher final state multiplicities

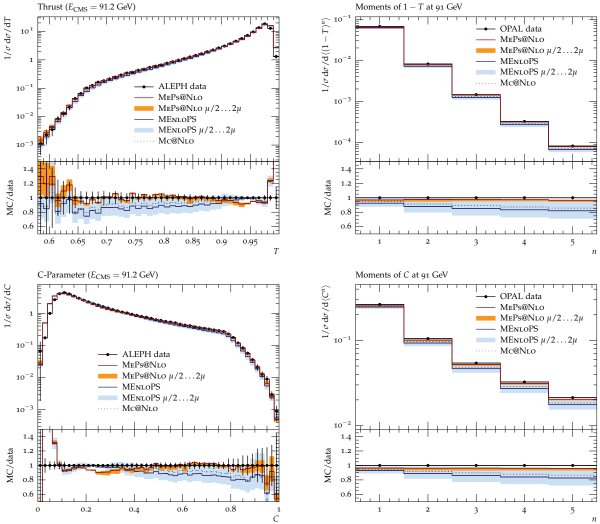
First emission(s), once more

$$
\begin{aligned}
d\sigma \;=\; & d\Phi_N\,\tilde{\mathcal{B}}_N\left[\Delta_N^{(\mathcal{K})}(\mu_N^2,t_0)+\int\limits_{t_0}^{\mu_N^2} d\Phi_1\,\mathcal{K}_N\Delta_N^{(\mathcal{K})}(\mu_N^2,t_{N+1})\Theta(Q_J-Q_{N+1})\right]\\
& +d\Phi_{N+1}\,\mathcal{H}_N\Delta_N^{(\mathcal{K})}(\mu_N^2,t_{N+1})\Theta(Q_J-Q_{N+1})\\
& +d\Phi_{N+1}\,\tilde{\mathcal{B}}_{N+1}\left(1+\frac{\mathcal{B}_{N+1}}{\tilde{\mathcal{B}}_{N+1}}\int\limits_{t_{N+1}}^{\mu_N^2} d\Phi_1\,\mathcal{K}_N\right)\Theta(Q_{N+1}-Q_J)\\
& \quad\cdot\left[\Delta_{N+1}^{(\mathcal{K})}(t_{N+1},t_0)+\int\limits_{t_0}^{t_{N+1}} d\Phi_1\,\mathcal{K}_{N+1}\Delta_{N+1}^{(\mathcal{K})}(t_{N+1},t_{N+2})\right]\\
& +d\Phi_{N+2}\,\mathcal{H}_{N+1}\Delta_N^{(\mathcal{K})}(\mu_N^2,t_{N+1})\Delta_{N+1}^{(\mathcal{K})}(t_{N+1},t_{N+2})\Theta(Q_{N+1}-Q_J)+\dots
\end{aligned}
$$

MEPS@NLO: example results for $e^-e^+ \to$ hadrons

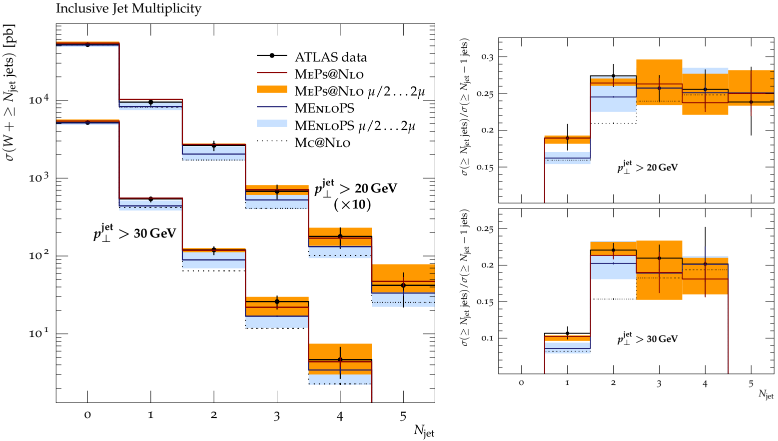


MEPS@NLO: example results for $e^-e^+ \to$ hadrons

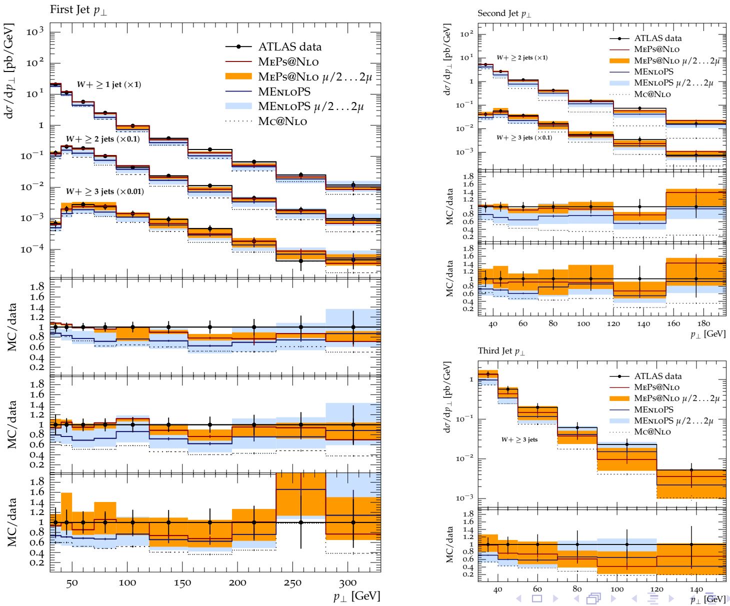


Example: MEPS@NLO for W+jets

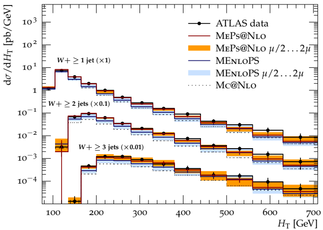
(up to two jets @ NLO, from BlackHat, see arXiv: 1207.5031 [hep-ex])

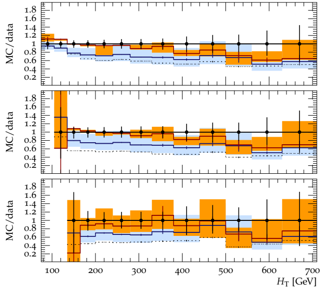


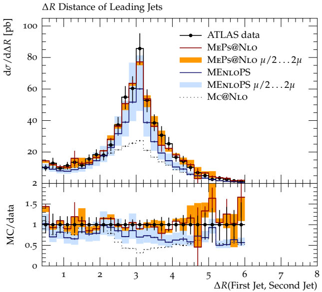
Multijet merging at NLO

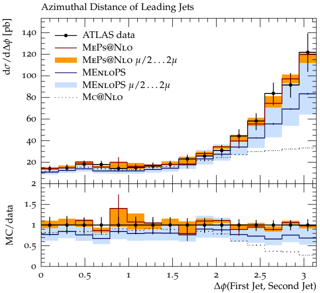


Multijet merging at NLO









Results for Higgs boson production through gluon fusion

- parton-shower level, Higgs boson does not decay
- setup & cuts:

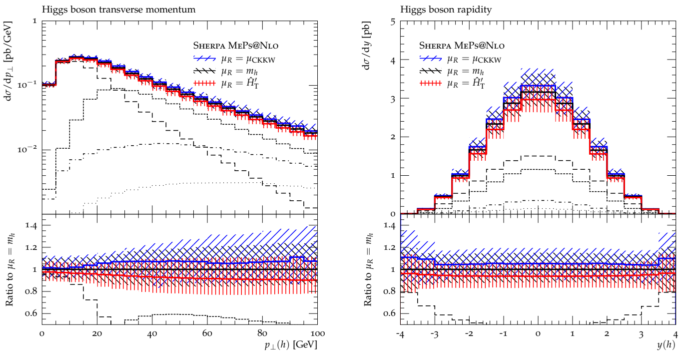
jets:	anti-kt, $p_\perp \geq 20$ GeV, $R = 0.4$, $\lvert\eta\rvert \leq 4.5$
dijet cuts:	at least 2 jets with $p_\perp \geq 25$ GeV
WBF cuts:	$m_{jj} \geq 400$ GeV, $\Delta y_{jj} \geq 2.8$

- jet multiplicity plots:

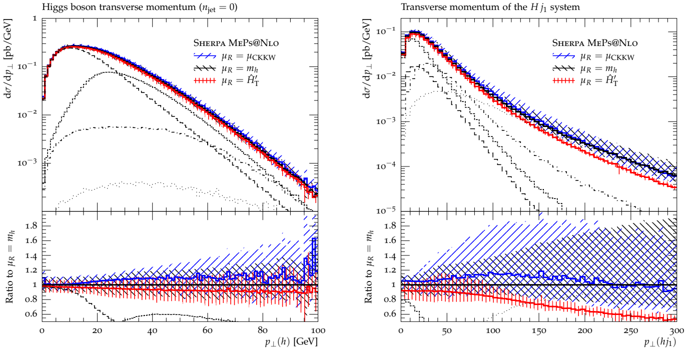
0-jet excl.:	no jet with $p_\perp \geq \{20, 25, 30\}$ GeV
2-jet incl.:	at least two jets with $p_\perp \geq \{20, 25, 30\}$ GeV

- SHERPA with $H + \{0,1,2\}^{(NLO)} + \{3\}^{(LO)}$ jets, $Q_{\rm cut} = 20 GeV$

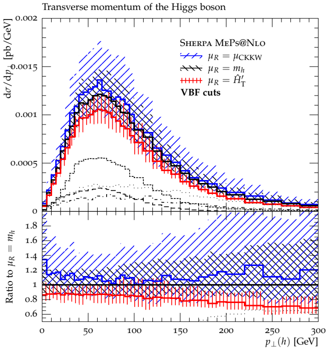
Inclusive observables for $gg \to H$

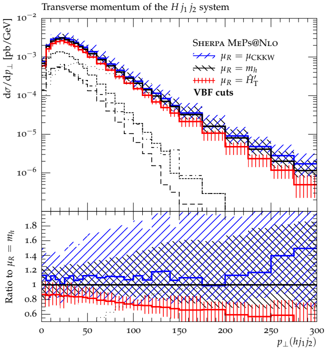


Exclusive observables for $gg \to H$

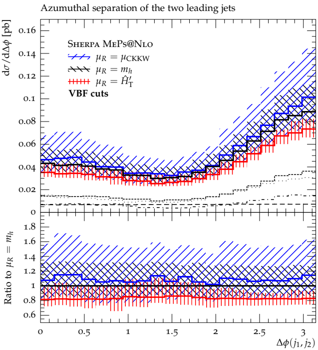


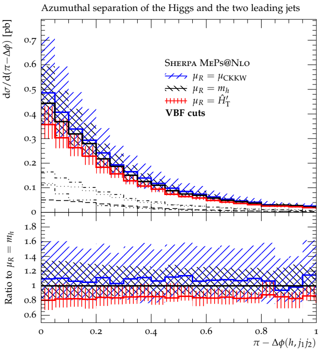
$gg \to H$ after WBF cuts





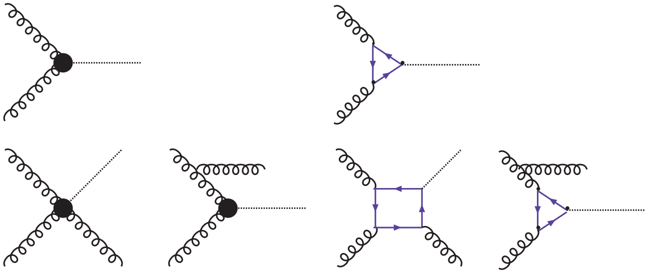
$gg \to H$ after WBF cuts





Quark mass effects

- include effects of quark masses

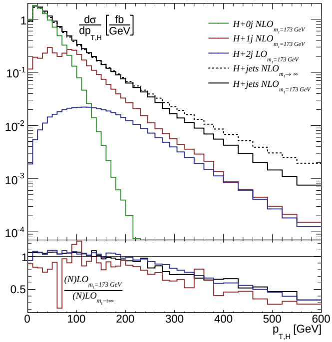


- reweight NLO HEFT with LO ratio:

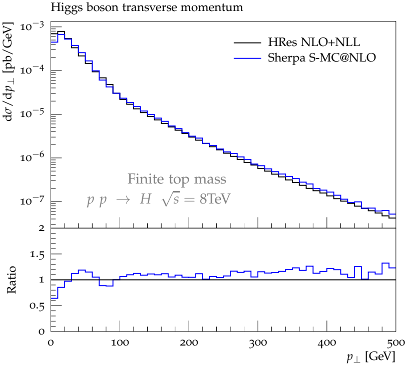
$$\mathrm{d}\sigma^{(\mathrm{NLO})}_{\mathrm{mass}} \approx \mathrm{d}\sigma^{(\mathrm{NLO})}_{\mathrm{HEFT}} \times \frac{\mathrm{d}\sigma^{(\mathrm{LO})}_{\mathrm{mass}}}{\mathrm{d}\sigma^{(\mathrm{LO})}_{\mathrm{HEFT}}}$$

Quark mass effects – results

- top mass effect in MEPS@NLO (on Higgs–$p_\perp$)



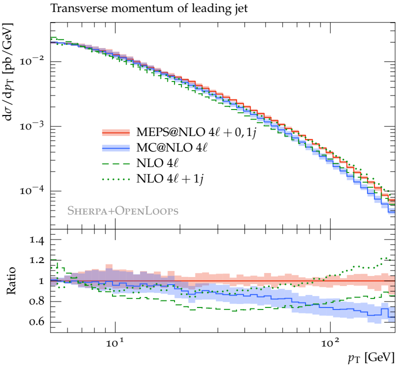
- comparison S-MC@NLO– HRES (top–loop only)

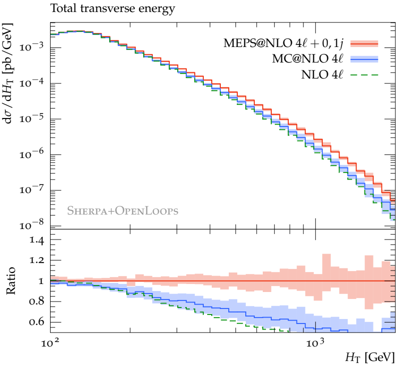


b–mass effects

- b–mass effects more tricky
- relevant only for (negative) interference of top– and bottom–loops (bottom^2 double Yukawa - supressed)
- but: cannot start shower at m_H
  radiation "sees" bottom at all scales above m_b
  $\Longrightarrow$ must use full theory there
- p_T spectrum naively "squeezed" – funny shapes
- LO multijet merging improves situation

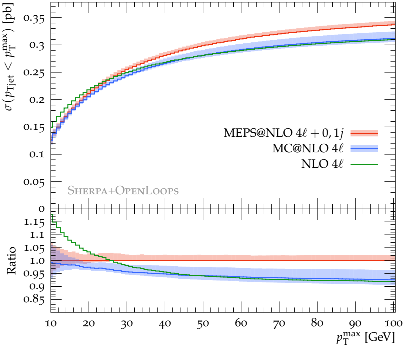
Higgs backgrounds: inclusive observables in W^+W^-+jets



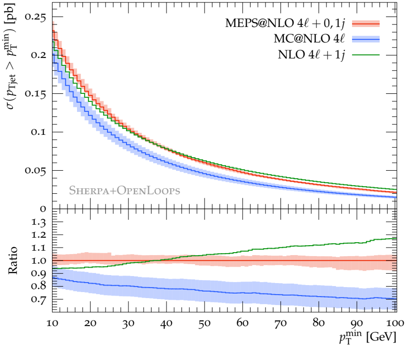


Higgs backgrounds: jet vetoes in W^+W^-+jets

Integrated cross section in the exclusive 0-jet bin

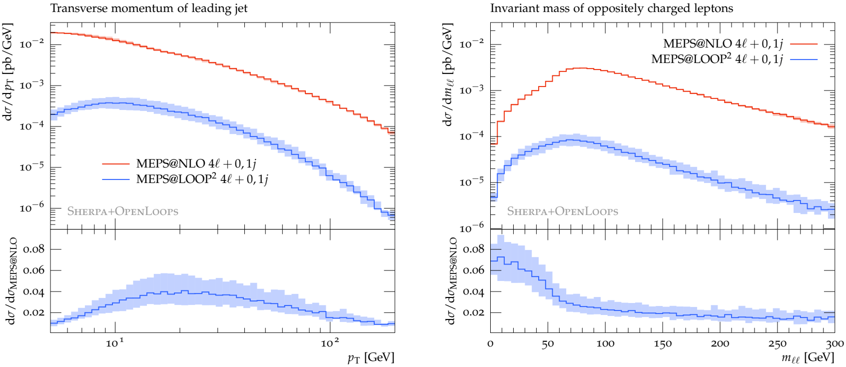


Integrated cross section in the inclusive 1-jet bin

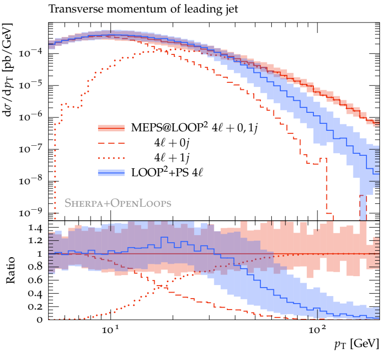


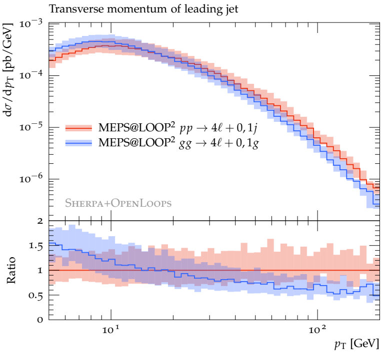
Higgs backgrounds: gluon-induced processes W^+W^-+jets

- include (LO-) merged loop^2 contributions of $gg \to VV$ (+1 jet)

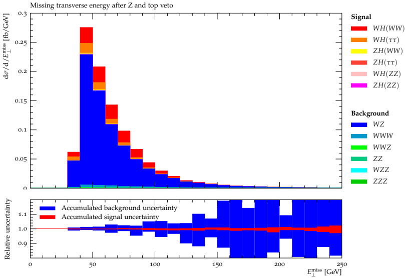


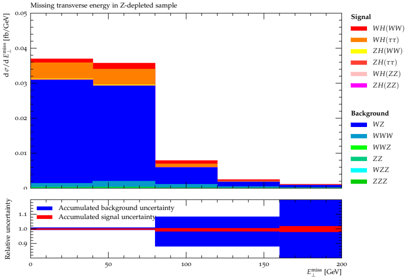
Higgs backgrounds: jet vetoes in W^+W^-+jets



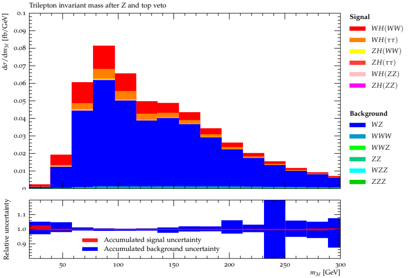


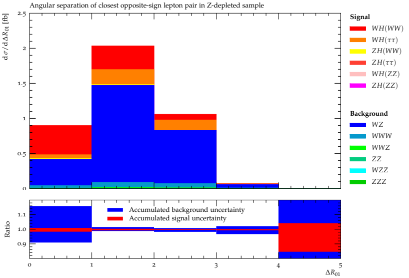
Relevant observables for $VH \to 3\ell$: $\not{E}_T$



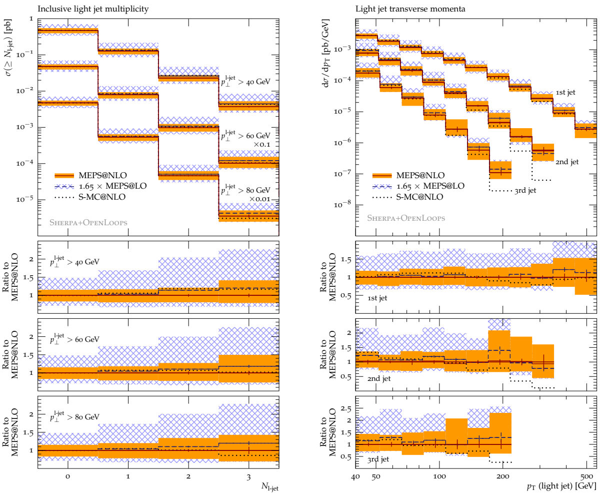


Relevant observables for $VH \to 3\ell$: m_{123} & ΔR_{01}



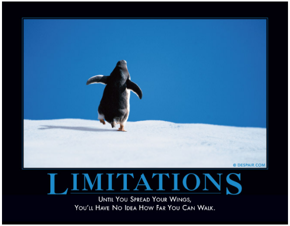


Higgs backgrounds: light jets in $t\bar{t}$ + jets



Summary

- Systematic improvement of event generators by including higher orders has been at the core of QCD theory and developments in the past decade:
  - multijet merging ("CKKW", "MLM")
  - NLO matching ("MC@NLO", "PowHeg")
  - MeNLoPs NLO matching & merging
  - MePs@Nlo ("Sherpa", "UNLoPs", "MINLo", "FxFx")



- multijet merging an important tool for many relevant signals and backgrounds - pioneering phase at LO & NLO over
- complete automation of NLO calculations done
  $\longrightarrow$ must benefit from it!

(it's the precision and trustworthy & systematic uncertainty estimates!)

Famous last screams

- in Run-II we'll be in for a ride:

  more statistics
  more energy
  more channels
  more precision
  more fun

- . . . and all with QCD . . .

oh, and btw.: the first NNLO+PS are out!