

Foundations of Physics III

Quantum and Particle Physics

Lecture 3

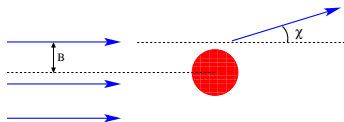
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1 Cross sections

2 Connection to observables

Cross sections in classical physics



- Consider a beam of particles, approaching a target at rest with velocity v_∞ . Describe the target by a potential centered at the origo.
- Due to different impact parameters B , different particles of the beam are scattered at different angles χ .
- Define **cross section** $d\sigma(\chi) = dN(\chi)/n$ with
 - $dN(\chi)$ = number of particles scattered per unit time into the interval $[\chi, \chi + d\chi]$ - physical units: s^{-1} .
 - n = number of particles passing per unit time through a unit area perpendicular to the beam - physical units: $m^{-2}s^{-1}$

Rewriting the differential cross section $d\sigma$

- Assume unique relation $\chi = \chi(B)$. (Fulfilled if χ decreases with B increasing - a typical setup!)
- Assume homogenous beams: $n = \text{constant}$.
- Assume symmetry around beam axis:

$$dN(\chi) = 2\pi n B dB \implies d\sigma = 2\pi B dB.$$

- Can rewrite to expose dependence on scattering angle:

$$d\sigma = 2\pi B(\chi) \left| \frac{dB}{d\chi} \right| d\chi.$$

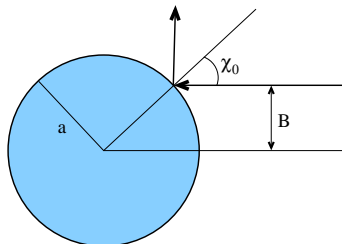
- Use solid angle $d\Omega = \sin \chi d\chi d\phi$ and azimuthal symmetry $d\phi \rightarrow 2\pi$:

$$d\sigma = \frac{B(\chi)}{\sin \chi} \left| \frac{dB}{d\chi} \right| d\Omega.$$

Example: Scattering of a hard sphere with radius a

- Read off from sketch:

$$\begin{aligned} B &= a \sin \chi_0 \\ &= a \sin \frac{\pi - \chi}{2} = a \cos \frac{\chi}{2} \end{aligned}$$



- Therefore:

$$\begin{aligned} d\sigma &= 2\pi B \left| \frac{dB}{d\chi} \right| d\chi = 2\pi a^2 \cos \frac{\chi}{2} \left| \frac{1}{2} \sin \frac{\chi}{2} \right| d\chi \\ &= \frac{\pi a^2}{2} \sin \chi d\chi = \frac{a^2}{4} d\Omega \implies \sigma = a^2 \pi. \end{aligned}$$

- Particle must “hit” the target.

Example: Scattering in a central potential $V = \alpha/r^n$

- Consider now scattering in a central potential $V(r) = \alpha/r^n$.
- Obviously: Need to determine trajectories in dependence on B .
- Use energy and angular momentum conservation:

$$E = \frac{m}{2} \left(\dot{r}^2 + r^2 \dot{\theta}^2 \right) + V(r) = \frac{m \dot{r}^2}{2} + \frac{J^2}{2mr^2} + V(r),$$

where 2-dim spherical coordinates r and ϕ and the angular momentum $J = mr^2 \dot{\theta}$ have been used.

- Therefore:

$$\dot{r} = \frac{dr}{dt} = \sqrt{\frac{2}{m} [E - V(r)] - \frac{J^2}{m^2 r^2}}.$$

- Rewrite (with $dt = dr/\dot{r}$ from above):

$$d\theta = \frac{J dt}{mr^2} = \frac{\frac{J}{r^2} dr}{\sqrt{2m[E - V(r)] - \frac{J^2}{r^2}}}.$$

- Use that energy in infinite distance is purely kinetic: $E = \frac{mv_\infty^2}{2}$ and angular momentum is given by $J = mv_\infty B$.
- Assume specific form of potential: $V(r) = \alpha/r$

(typical for gravity/electromagnetism)

$$\chi_0 = \int d\theta = \int \frac{\frac{B}{r^2} dr}{\sqrt{1 - \frac{B^2}{r^2} - \frac{2V(r)}{mv_\infty^2}}} = \cos^{-1} \frac{\frac{\alpha}{mv_\infty^2 B}}{\sqrt{1 + \left(\frac{\alpha}{mv_\infty^2 B}\right)^2}}$$

(Finite terms absorbed in definition of angle/orientation of coordinate system)

- Solve for B and use that $\chi_0 = \frac{\pi - \chi}{2}$:

$$B^2 = \frac{\alpha^2}{m^2 v_\infty^4} \tan^2 \chi_0 = \frac{\alpha^2}{m^2 v_\infty^4} \cot^2 \frac{\chi}{2}.$$

- To express cross section $d\sigma = 2\pi B \left| \frac{dB}{d\chi} \right| d\chi$

use that

$$dB/d\chi = -\frac{\alpha}{2mv_\infty^2} \frac{1}{\sin^2 \frac{\chi}{2}}.$$

- Rediscover Rutherford formula

$$d\sigma = \frac{\pi\alpha^2}{m^2 v_\infty^4} \frac{\cos \frac{\chi}{2}}{\sin^3 \frac{\chi}{2}} d\chi = \frac{\alpha^2}{4m^2 v_\infty^4} \frac{d\Omega}{\sin^4 \frac{\chi}{2}}$$

- Observation: for small angles $\chi \rightarrow 0$ differential cross section infinite
total cross section diverges for Rutherford scattering
- This is a consequence of the infinite range of the potential!

(Will need to “cut” for minimal scattering angle . . .)

Event rates, luminosity and cross sections

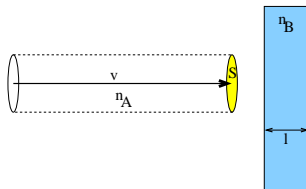
- Introduce (*instantaneous*) *luminosity* \mathcal{L} ,
the number of incident particles per unit area per unit time times
the opacity of the target; units $[\mathcal{L}] = m^{-2}s^{-1}$
- Then the **instantaneous** number of events dN_{ev}/dt reads

$$\frac{dN_{\text{ev}}}{dt} = \sigma \mathcal{L}$$

- Therefore the total number of events during a time T given by the
integral over time of \mathcal{L} , the **integrated luminosity** – cross section σ
depends on the particles and the reaction type only.
- Typically a year of beam in an accelerator experiment: $1 \text{ yr} = 10^7 \text{ s}$.

Determining luminosity

- Consider a simple case: scattering off a fixed target.
Beam consisting of particles A , travelling with velocity v , uniformly distributed in area S with constant density n_A along beam axis.



- Therefore: number of particles in the beam per unit length is given by $dN_A = n_A S dx$, and the incident flux (number of particles per unit time) on target is

$$\Phi_A = \frac{dN_A}{dt} = n_A S \frac{dx}{dt} = n_A S v$$

- Assume thin (only one scatter per particle) target with size l :
Area density of particles in target able to participate in reaction

$$\rho_B = n_B l$$

- Define **luminosity**

$$\mathcal{L} = \Phi_A \rho_B = n_A n_B S v l.$$

Luminosity in collider experiments

- Consider now two quadratic beams each with area h^2 , colliding under an angle θ , with densities n_A and n_B along their beam axes, and velocities $v_A \geq v_B$.
- Look at beam B as “target”, and beam A remaining the “beam”.
- Need only to evaluate N_B – can use flux Φ_B :

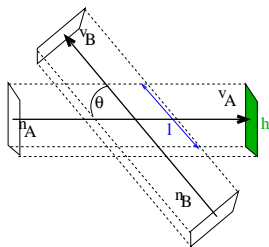
$$n_B = \frac{\Phi_B}{h^2 v_B}$$

- Therefore density per area

$$\rho_B = n_B \frac{h}{\sin \theta} = \frac{\Phi_B}{h v_B \sin \theta}$$

and

$$\mathcal{L} = \Phi_A \rho_B = \frac{\Phi_A \Phi_B}{\sin \theta h v_B}.$$



Practicalities

- Hard to determine luminosity - geometry dependent and fluctuations with quality/homogeneity of the beams
- Therefore: Measure them with “standard candles”, i.e. well understood processes with good theoretical accuracy in calculation of cross section σ .

- Recap event rate for events of type $AB \rightarrow X$:

$$\dot{N}_{AB \rightarrow X} = \frac{dN_{AB \rightarrow X}}{dt} = \mathcal{L}_{AB} \sigma_{AB \rightarrow X}$$

- Units: $[\dot{N}_{AB \rightarrow X}] = \text{Hz} = 1/\text{s}, \implies [\mathcal{L}] = 1/([\sigma_{AB \rightarrow X}] \cdot \text{s})$.

$$[\sigma] = 1 \text{ barn} = (10 \text{ fm})^2 = 10^{-24} \text{ cm}^2$$

$$\approx (2 \text{ GeV})^{-2} = (2000 \text{ MeV})^{-2}$$

Learning outcomes

- Cross sections and their interpretation
- Connection to observables: Luminosity, event rates etc..
- How to determine luminosities