Foundations of Physics III
Quantum and Particle Physics
Lecture 5

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1 Potential scattering in Particle Physics

2 Scattering processes in Particle Physics

Reminder: Definition in classical mechanics

- Cross section: $$\boxed{\sigma(\Omega) = |f(\Omega)|^2}$$

  with **transition amplitude** $f(\Omega)$ in a centrally symmetric potential

  $$f(\Omega) = \frac{2l+1}{k} e^{i\delta_l} \sin\delta_l \,.$$

- Alternatively write $f(\Omega)$ as transtion from an incoming state with momentum $\underline{k}$ to an outgoing state with momentum $\underline{k}'$:

  $$f(\Omega) \equiv f(\underline{k}', \underline{k}) \propto \langle \underline{k}' | \mathcal{H}(\underline{r}) | \underline{k} \rangle \,.$$

- Replace integration over Ω by integration over outgoing momenta – identical for potential scattering, as absolute value of momentum does not change there.
  This of course changes when going to more complicated scattering processes like the ones we're interested in.

Perturbative expansion in Quantum Mechanics

- Want to calculate cross section/scattering amplitude $f(\Omega)$ in a complicated potential.
- Remember: transition amplitude $f(\underline{k},\, \underline{k}') \propto \langle \underline{k}'|\mathcal{H}|\underline{k}\rangle$
- Sometimes (in fact, in most realistic/interesting cases) exact solution with full Hamiltonian inaccessible.
- Write Hamiltonian as

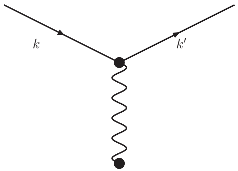
$$\mathcal{H} = \mathcal{H}_0 + \lambda\mathcal{V}$$

  and expand in **small parameter** λ – this works if solution for "unperturbed Hamiltonian" $\mathcal{H}_0$ is known.
- From now on: $\hat{H}_0 =$ freely propagating particles and $\lambda\hat{V} =$ all interactions.

Electron scattering in a potential

- Typically, cross sections etc. are evaluated in momentum space, related by a Fourier transform with position space.
- Also denote particle states by their momentum $|\underline{k}\rangle$.
- Fourier transform of Coulomb potential

$$\int \frac{\mathrm{d}^3 x e^{-\underline{q}\cdot\underline{x}}}{|\underline{x}|} = \frac{4\pi}{|\underline{q}|^2}$$



and therefore, with $\underline{q} = \underline{k}' - \underline{k}$,

$$|\langle\underline{k}'|V(\underline{r})|\underline{k}\rangle|^2 = \left|\frac{(-iZe)(-ie)}{|\underline{q}|^2}\right|^2 = \frac{Z^2 e^4}{|\underline{q}|^4}\,.$$

- Elastic scattering (no energy transfer): Include a factor

$$(2\pi)\delta(E' - E)\,.$$

- Consider relativistic invariance $E^2 - \underline{k}^2 = m^2$ and physical energies $E > 0$ for physical particle states.
- Thus (up to some normalisation)

$$\mathrm{d}\Omega \quad \longrightarrow \quad \frac{\mathrm{d}E\mathrm{d}\underline{k}}{(2\pi)^4}(2\pi)\delta(E^2 - \underline{k}^2 - m^2)\Theta(E) = \frac{\mathrm{d}\underline{k}}{2E(2\pi)^3}$$

where $E = \sqrt{\underline{k}^2 + m^2} \geq 0$ is now explicit.

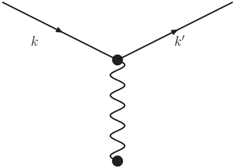
- Needs to be combined with transition amplitude squared:

$$\mathrm{d}\sigma \propto \frac{1}{2}\frac{E}{|\underline{k}|} \cdot \frac{\mathrm{d}\underline{k}'}{2E'(2\pi)^3}\frac{Z^2e^4}{|\underline{k}' - \underline{k}|^4}(2\pi)\delta(E' - E)\,,$$

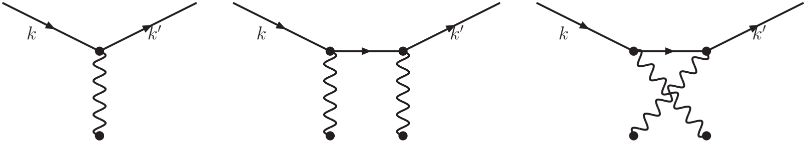
where first factor of $1/2$ averages over the two incoming electron spins and the second factor of $E/\underline{k} = 1/\underline{v}$ represents the incoming flux of an electron moving relativistically.

Feynman diagrams

- Check the "Feynman diagram" related to the potential scattering (right)
- Exhibits a fermion line (with arrows) connecting (interacting) with a wavy line (the photon) in some "vertex".
- Wavy line emitted from another "vertex".



- Such a diagram represents one contribution to a transition amplitude. Can, of course, add more contributions, like, e.g., double scattering etc..



From Feynman diagrams to cross sections

- Each of the building blocks of such a Feynman diagram represents an analytic expression:
  (a) Vertices $\propto (-ie)$

  (or $(-ie_q e)$ for particles/potentials representing charges e_q in units of the electron charge.)

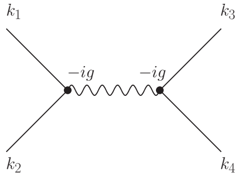
  In general: Vertices $\propto -ig$ for coupling g.

  (b) Propagators $\propto i/(q^2 - m^2)$ for particles with mass m;

  (Propagators = internal lines between two vertices.)

  (c) Four-momentum conservation $(2\pi)^4\delta^4(\sum p_{\rm in} - \sum p_{\rm out})$ at each vertex and in total.
- In addition must perform the following steps with amplitude:
  (a) Squaring (absolute value),
  (b) Summing/integrating over all final state configurations (momenta, spins, other unobserved internal degrees of freedom) and averaging over unobserved incoming degrees of freedom.
  (c) Multiply with incoming flux.
- These rules will be generalised later.

Example: $2 \to 2$ scattering in s-channel



- Consider a Feynman diagram for process $k_1 + k_2 \to k_3 + k_4$ (right picture)
- Assume wavy line represents a massive particle with mass M.
- Amplitude can be written as

$$\mathcal{M} \propto (-ig)^2 \frac{i}{(E_1+E_2)^2 - (\underline{k}_1 + \underline{k}_2)^2 - M^2} = \frac{-ig^2}{\hat{s} - M^2}$$

  giving rise to the name s-channel, with $\hat{s} = (E_1 + E_2)^2$ in the centre-of-mass frame, or $\hat{s} = E^2_{\mathrm{c.m.}}$.
- If this is the only contributor, then

$$|\mathcal{M}_{s-\mathrm{channel}}|^2 \propto \frac{g^4}{(\hat{s} - M^2)^2}$$

- Putting it all together, the cross section is given by

$$
\begin{aligned}
\mathrm{d}\sigma_{12\to 34} &= \frac{1}{2\hat{s}}\left|\mathcal{M}_{12\to 34}\right|^2 (2\pi)^4\delta^4(k_1+k_2-k_3-k_4) \\
&\quad \frac{\mathrm{d}^4 k_3}{(2\pi)^4}(2\pi)\delta(k_3^2-m_3^2)\Theta(E_3)\frac{\mathrm{d}^4 k_4}{(2\pi)^4}(2\pi)\delta(k_4^2-m_4^2)\Theta(E_4) \\
&= \frac{1}{2E_{\mathrm{c.m.}}^2}\int\limits_0^{E_{\mathrm{c.m.}}}\mathrm{d}E_3\int\limits_0^{\infty}\mathrm{d}|\underline{k}_3||\underline{k}_3|^2\delta(E_3^2-|\underline{k}_3|^2-m_3^2)\int\limits_0^{4\pi}\frac{\mathrm{d}\Omega}{4\pi^2}\delta[(k_1+k_2-k_3)^2-m_4^2]\left|\mathcal{M}_{12\to 34}\right|^2 \\
&= \frac{\mathrm{d}\Omega}{8\pi^2 E_{\mathrm{c.m.}}^2}\int\limits_0^{E_1+E_2}\mathrm{d}E_3\frac{|\underline{k}_3|}{2}\delta[(E_{\mathrm{c.m.}}+E_3)^2-E_3^2+m_3^2-m_4^2]|\mathcal{M}_{12\to 34}|^2 \\
&= \frac{\mathrm{d}\Omega}{16\pi^2 E_{\mathrm{c.m.}}^2}\int\limits_0^{E_1+E_2}\mathrm{d}E_3|\underline{k}_3|\delta[(E_{\mathrm{c.m.}}+2E_{\mathrm{c.m.}}E_3^2+m_3^2-m_4^2]|\mathcal{M}_{12\to 34}|^2 \\
&= \left|\mathcal{M}_{12\to 34}\right|^2\frac{|\underline{k}_3|\mathrm{d}\Omega}{32\pi^2 E_{\mathrm{c.m.}}^3}\,.
\end{aligned}
$$

- For our specific example and massless particles in the final state.

$$
\sigma_{12\to 34} = \frac{g^4}{(\hat{s}-M^2)^2}\,\frac{1}{16\pi\hat{s}}\,.
$$

Learning outcomes

- Perturbation series in interactions
- A first glimpse to Feynman diagrams

(Will return to them ...)

- Feynman diagrams as building blocks of transition amplitudes
- Cross section from transition amplitudes and the corresponding expression for $2 \to 2$ scattering:

$$\begin{aligned} \mathrm{d}\sigma_{AB\to 12} &= |\mathcal{M}_{AB\to 12}|^2 \frac{\mathrm{d}\Omega}{64\pi^2 E_{\mathrm{c.m.}}^2} \frac{(2|\underline{p}_1|)}{E_{\mathrm{c.m.}}} \\ &\xrightarrow{m_1=m_2=0} |\mathcal{M}_{AB\to 12}|^2 \frac{\mathrm{d}\Omega}{64\pi^2 E_{\mathrm{c.m.}}^2}\,. \end{aligned}$$