Foundations of Physics III
Quantum and Particle Physics
Lecture 4

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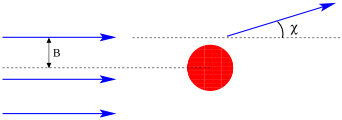
1 Cross sections in Quantum Mechanics

Reminder: Definition in classical mechanics



- Consider a beam of particles, approaching a target at rest with velocity v_∞. Describe the target by a potential centered at the origo.
- Particles are scattered at different solid angles $\Omega(\chi, \phi)$.
- Define **cross section** $\mathrm{d}\sigma(\Omega) = \mathrm{d}N(\Omega)/n$ with
  - $\mathrm{d}N(\Omega) =$ number of particles scattered per unit time into an interval of size $\mathrm{d}\Omega$ physical units: s^{-1}.
  - $n =$ number of particles passing per unit time through a unit area perpendiculr to the beam - physical units: $m^{-2}s^{-1}$
  - Total cross section: $\sigma_{\rm tot} = \int \mathrm{d}\sigma = \int \mathrm{d}\Omega\sigma(\Omega)$.

Formulation in Quantum Mechanics: Potential scattering

- Consider a potential $V(\underline{r})$, located at the origin

(Vanishing faster than Coulomb potential $1/r$ for $r \to \infty$)

- Schrödinger equation for a particle with initial momentum $(E, \underline{k})$:

$$\left[-\frac{1}{2m}\nabla^2 + V(\underline{r})\right]\psi_k(\underline{r}) = E\psi_k(\underline{r})\,.$$

Solution for $r \to \infty$-behaviour: (See also lecture by A.Ciavarella.)

$$\psi_k(\underline{r}) \stackrel{r\to\infty}{=} A\left[e^{i\underline{k}\cdot\underline{r}} + f(\Omega)\frac{e^{ikr}}{r}\right]$$

- Want to interpret this in terms of a cross section. Define flux:

$$\underline{j}(\underline{r}) = -\frac{i}{2m}\left\{\psi_k^*(\underline{r})\left[\underline{\nabla}\psi_k(\underline{r})\right] - \left[\underline{\nabla}\psi_k^*(\underline{r})\right]\psi_k(\underline{r})\right\}\,.$$

- Analyse solution from last slide, $\psi_k(\underline{r}) \stackrel{r\to\infty}{=} A\left[e^{i\underline{k}\cdot\underline{r}} + f(\Omega)\frac{e^{ikr}}{r}\right]$
- First term $A\exp[i\underline{k}\cdot\underline{r}]$
  $\longrightarrow$ monoenergetic plane wave with unit density $\rho = 1$ (normalised)

$$\underline{j}_{\rm in} = \underline{k}/m$$

- Second term $Af(\Omega)\exp[ikr]/r$:
  particles emitted radially off the scattering centre with density $\rho = |f(\Omega)/r|^2$,

$$\underline{j}_{\rm out} = \frac{k\underline{e}_r}{m}\left|\frac{f(\Omega)}{r}\right|^2 .$$

- Cross section: $\boxed{\sigma(\Omega) = |f(\Omega)|^2}$

Partial wave analysis of $f(\Omega)$

- Assuming a centrally symmetric potential:
  $V(\underline{r}) = V(r) \longrightarrow f(\Omega) = f(\cos\theta,\,\phi) \longrightarrow f(\cos\theta)$

  (Remember $\mathrm{d}\Omega = \mathrm{d}\cos\theta\mathrm{d}\phi$.)

- Can expand $f(\theta)$ through Legendre polynomials as

$$f(\theta) = \sum_{l=0}^{\infty} f_l P_l(\cos\theta)$$

- After comparing coefficients etc.: Scattering amplitudes

$$f(\theta) = \frac{1}{k}\sum_{l=0}^{\infty}\left[(2l+1)e^{i\delta_l}\sin\delta_l P_l(\cos\theta)\right]$$

  with potential-dependent *scattering phases* δ_l.

Interpretation for one *partial wave* (index l):

$$\psi_l = \left[\left(e^{-ikr+i(l+1)\pi} + e^{ikr}\right)\frac{2l+1}{2ik} + f_l e^{ikr}\right]\frac{P_l(\cos\theta)}{r}$$

- No scatter ($f_l = 0$):
  Spherical incoming and outgoing waves with $e^{-ikr+i(l+1)\pi}/r$ and e^{ikr}/r modulated by P_l.
- With scatter change of coefficient for outgoing wave:

$$1 \longrightarrow 1 + \frac{2ik}{2l+1} f_l$$

Enforcing **probability conservation (unitarity)**: Demand

$$1 = \left|1 + \frac{2ik}{2l+1} f_l\right|^2 = e^{2i\delta_l} \longleftrightarrow f_l = \frac{2l+1}{k} e^{i\delta_l}\sin\delta_l\,.$$

Total cross section

- Write the (total) cross section as

$$
\begin{aligned}
\sigma_{\rm tot} &= \int \mathrm{d}\phi \mathrm{d}\cos\theta |f(\theta)|^2 \\
&= 2\pi \sum_{ll'} \left[\frac{(2l+1)(2l'+1)}{k^2} e^{i(\delta_l - \delta_{l'})} \sin\delta_l \sin\delta_{l'} \right. \\
&\qquad \left. \int \mathrm{d}\cos\theta P_l(\cos\theta) P_{l'}(\cos\theta) \right] \\
&= 2\pi \sum_{ll'} \left[\frac{(2l+1)(2l'+1)}{k^2} e^{i(\delta_l - \delta_{l'})} \sin\delta_l \sin\delta_{l'} \frac{\delta_{ll'}}{2l+1} \right] \\
&= \sum_l \frac{4\pi(2l+1)}{k^2} \sin\delta_l \,.
\end{aligned}
$$

Two specific cases

- Scattering off a hard sphere with radius a:

$$\sigma_{\rm tot} = 2\pi a^2 = 2\sigma_{\rm classical}\,.$$

- Rutherford scattering off a potential α/r:

$$\sigma(\Omega) = \left(\frac{\alpha}{4E}\right)^2 \frac{1}{\sin^4\frac{\theta}{2}} = \sigma_{\rm classical}(\Omega)\,.$$

Learning outcomes

- Cross section by **summing over scattering amplitudes** $f(\theta)$ and **integrating over outgoing phase space**, given by Ω.
- Check for/enforce probability conservation: scattering amplitudes become merely phase shifts in spherical waves.
- If potential has infinite reach (Coulomb potential), cross section diverges. This is true in both classical and quantum physics.