Introduction to particle physics Lecture 6: Cross sections in Quantum Field Theory

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Outline

Cross Section in Quantum Field Theory

Reminder: cross section in Quantum Mechanics

- Reminder: cross section in quantum mechanics given by

$$\sigma_{\rm tot} = \int \mathrm{d}\Omega\sigma(\Omega) = \int \mathrm{d}\Omega |f(\Omega)|^2$$

with transition amplitude $f(\Omega) = \frac{2l+1}{k} e^{i\delta_l} \sin \delta_l$.

(must be normalised to incident flux, and then per unit time and volume.)

- An alternative representation of $f(\Omega)$:
  - rewrite with incoming and outgoing momentum, $\underline{k}$ and $\underline{k}'$:
    $$f(\Omega) \equiv f(\underline{k}', \underline{k})$$
  - remember: scattering amplitude due to potential, therefore
    $$f(\underline{k}', \underline{k}) \propto \langle \underline{k}' | V(r) | \underline{k} \rangle,$$
    where the states have been labelled with their momenta.
    Note: In elastic scattering (considered so far), $|\underline{k}'| = |\underline{k}|$
- Total cross section obtained by integrating over all final state configurations; can be parametrised by $\underline{k}'$ or Ω.

Amplitudes & states in Quantum Field Theory

(Details not exminable)

- Consider transition amplitude between asymptotic (at large times $T = \pm\infty$) "in"- and "out"-states

$$A_{\mathrm{in}\to\mathrm{out}} = \langle \mathrm{out}, T = +\infty | \mathrm{in}, T = -\infty \rangle .$$

- Meaningful states must obey completeness & orthogonality relations:

(then they're base vectors for the respective Hilbert/Fock space!)

$$\begin{aligned} \sum_\alpha |\alpha, T = \pm\infty\rangle\langle\alpha, T = \pm\infty| &= 1 \\ \langle\alpha, T = \pm\infty|\beta, T = \pm\infty\rangle &= \delta_{\alpha\beta} . \end{aligned}$$

- However, $|\alpha, T = +\infty\rangle$ not identical to $|\alpha, T = -\infty\rangle$:

$$\langle\alpha, T = \pm\infty|\alpha, T = \mp\infty\rangle = ?$$

they live in two completely different vector spaces.

Constructing states in Quantum Field Theory

(Not exminable)

- Consider one-particle states $|\underline{p}\rangle$, labelled by momentum $\underline{p}$.
- Completeness:

$$1 = \int \frac{\mathrm{d}E\,\mathrm{d}^3p}{(2\pi)^4}\,(2\pi)\delta(E^2 - p^2 - m^2)\Theta(E)\,|\underline{p}\rangle\langle\underline{p}|$$
$$= \int \frac{\mathrm{d}^3p}{2E(2\pi)^3}\,|\underline{p}\rangle\langle\underline{p}|$$

  - Integrate over all possibilities (add summation over spins if needed);
  - Guarantee correct relativistic energy-momentum relation;
  - Ensure physical states only.
- Normalisation:

$$\langle\underline{p}|\underline{p}'\rangle = 2E\,(2\pi)^3\delta^3(\underline{p} - \underline{p}')$$

after using completeness relation.

Constructing states in Quantum Field Theory (cont'd)

(Details not exminable)

- Aim: Construct multi-particle states from one-particle states.
- Idea: use creation/annihilation operators:

$$|\underline{p}\rangle = \hat{a}^\dagger(\underline{p})|0\rangle \text{ and } \hat{a}(\underline{p})|\underline{p}\rangle = |0\rangle.$$

- Can quantise particle states with them: "2nd quantisation". Use commutators $[\hat{a}^\dagger, \hat{a}] = \hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger = 1$ for bosons, use anti-commutators $[\{\hat{a}^\dagger, \hat{a}\} = \hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger = 1$ for fermions;
- Use them repeatedly for N-particle states:

$$|\underline{p}_1\underline{p}_2\dots\underline{p}_N\rangle = \hat{a}^\dagger(\underline{p}_1)\hat{a}^\dagger(\underline{p}_2)\dots\hat{a}^\dagger(\underline{p}_N)|0\rangle.$$

- In principle: would have to add symmetry factors etc..
- This works for asymptotic states at $T = \pm\infty$ only.

The S-matrix and transition amplitudes

(Details not exminable)

- Introduce S-matrix: $\hat{S}|\mathrm{in},\ T=-\infty\rangle = |\mathrm{out},\ T=+\infty\rangle$
- S-matrix is unitary: $\hat{S}^\dagger\hat{S}=1$

(from $1=\sum|\mathrm{out},\ T=+\infty\rangle\langle\mathrm{out},\ T=+\infty| = \sum\hat{S}|\mathrm{in},\ T=-\infty\rangle\langle\mathrm{in},\ T=-\infty|\hat{S}^\dagger = \hat{S}1\hat{S}^\dagger$)

- Rewrite $\hat{S}=1+i\hat{T}$:

(optical theorem from unitarity: $\hat{T}-\hat{T}^\dagger = -i\hat{T}\hat{T}^\dagger$, relates cross sections with discontinuity in amplitude)

- Write transition amplitude $\mathcal{A}_{\alpha\to\beta}$ with states in same $T=\pm\infty$-basis

$$\begin{aligned}\mathcal{A}_{\alpha\to\beta} &= \langle\beta,+\infty|\alpha,-\infty\rangle = \langle\beta,-\infty|\hat{S}^\dagger|\alpha,-\infty\rangle\\ &= \delta_{\alpha\beta} - i\langle\beta,-\infty|\hat{T}^\dagger|\alpha,-\infty\rangle\\ &= \delta_{\alpha\beta} - i(2\pi)^4\delta(E_\alpha-E_\beta)\delta^3(\underline{p}_\alpha-\underline{p}_\beta)\langle\beta,-\infty|\hat{T}^\dagger|\alpha,-\infty\rangle\,,\end{aligned}$$

with explicit Lorentz-invariance (energy-momentum conservation).

- From now on: $\alpha\neq\beta$ — true scattering processes.

From amplitudes to probabilities

(Details not exminable)

- Obviously $\mathcal{P}_{\alpha\to\beta} = |\mathcal{A}_{\alpha\to\beta}|^2$.
- Problem: need to square δ-functions:

$$\begin{aligned} &|(2\pi)^4\delta(E_\alpha - E_\beta)\delta^3(\underline{p}_\alpha - \underline{p}_\beta)|^2 \\ &= (2\pi)^4\delta(E_\alpha - E_\beta)\delta^3(\underline{p}_\alpha - \underline{p}_\beta)\cdot(2\pi)^4\delta(0)\delta(\underline{0}) \\ &= (2\pi)^4\delta(E_\alpha - E_\beta)\delta^3(\underline{p}_\alpha - \underline{p}_\beta)\cdot V_{\text{space}-\text{time}}\,. \end{aligned}$$

- Therefore (unnormalised) transition rate per unit space volume:

$$\frac{\mathcal{P}_{\alpha\to\beta}}{V} = (2\pi)^4\delta(E_\alpha - E_\beta)\delta^3(\underline{p}_\alpha - \underline{p}_\beta)\left|\langle\beta, -\infty|\,\hat{T}^\dagger\,|\alpha, -\infty\rangle\right|^2.$$

From probabilities to cross sections

(Details not exminable)

- Consider now processes $a + b \to 1 + 2 + 3 + \cdots + N$
- Must:
  - normalise states, (divide by $4E_aE_b$),
  - divide by incident flux, $|\underline{v}_{ab}| = \dfrac{\sqrt{(E_aE_b - \underline{p}_a\underline{p}_b)^2 - m_a^2m_b^2}}{E_aE_b}$,
  - sum/integrate over all d.o.f. in final state, $\prod_{i=1}^{N} \dfrac{\mathrm{d}^3 p_i}{2E_i(2\pi)^3}$

Putting it all together

(Details not exminable)

- Master formula:

$$\mathrm{d}\sigma_{ab\to 123\ldots N} = \frac{\left|\langle 123\ldots N,\, -\infty|\hat{T}^\dagger|ab,\, -\infty\rangle\right|^2}{4\sqrt{(E_a E_b - \underline{p}_a \underline{p}_b)^2 - m_a^2 m_b^2}} \prod_{i=1}^{N} \frac{\mathrm{d}^3 p_i}{2E_i(2\pi)^3}(2\pi)^4 \delta\left(E_a + E_b - \sum_{i=1}^{N} E_i\right) \delta^3\left(\underline{p}_a + \underline{p}_b - \sum_{i=1}^{N} \underline{p}_i\right)$$

- Individual parts:
    - QM transition amplitude,
    - incoming flux & normalisation of states,
    - summation/integration over all final state configurations,
    - energy-momentum conservation.

Final formulae: Scattering cross sections

(Remember them!)

- Consider $2 \longrightarrow 2$-scattering in c.m.-frame
- Ignore incoming masses, $m_a = m_b = 0$
- Kinematics almost completely fixed (up to angles):
  - $\sqrt{s} = E_{\mathrm{c.m.}} = E_a + E_b$, $\underline{p}_a = -\underline{p}_b$
  - $\underline{p}_1 = -\underline{p}_2 = p \cdot \underline{\hat{p}}(\theta, \phi)$ with $p = \frac{\sqrt{(s - m_1^2 - m_2^2)^2 - 4m_1^2 m_2^2}}{2E_{\mathrm{c.m.}}}$
  - $E_{1,2} = \dfrac{s \pm (m_1^2 - m_2^2)}{2E_{\mathrm{c.m.}}}$
- Use $E\mathrm{d}E = p\mathrm{d}p$ (from $E^2 = p^2 + m^2 \implies \mathrm{d}E^2 = \mathrm{d}p^2$)
- Then:

$$\mathrm{d}\sigma_{ab\to 12} = \frac{\left|\mathcal{M}_{ab\to 12}\right|^2}{32\pi^2 E_{\mathrm{c.m.}}^3} \left|\underline{p}\right| \mathrm{d}\Omega$$

Final formulae: Decay widths and lifetimes

(Remember them!)

- In a similar way can express a particles partial width related to a decay $M \to 12$:

$$\mathrm{d}\Gamma_{M\to 12} = \frac{|\mathcal{M}_{M\to 12}|^2}{32\pi^2 M^2}\,|\underline{p}|\,\mathrm{d}\Omega$$

  with decay matrix element $\mathcal{M}_{M\to 12}$ and identical kinematics: to construct momenta just go to c.m.-frame of M and replace $E_{\mathrm{c.m.}}$ with M in equations on previous slide.

(yields nearly identical final result)

- Total width Γ of a particle is given by all partial widths Γ_i

  $\Gamma = \sum_i \Gamma_i$ and lifetime $\tau = 1/\Gamma$

Summary

- Discussed scattering processes in relativistic quantum mechanics: This is the basis of most of our understanding of particle physics.
- Introduced S-matrix and related to scattering amplitude.
- Derived master formula for cross sections and specialised for 2-body scattering and decay processes.