Introduction to particle physics
Lecture 2: Special relativity

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Outline

1. Galilei vs. Einstein
2. Lorentz transformations
3. Mass, momentum and energy
4. Two-body decays
5. Two-body reactions
Frames of inertia: Galilei vs. Einstein

Galilei relativity

- Relativity discusses how changes in the coordinate set influence physical events. Coordinates specify positions in time \( t \) and space \( x \) - meaningful only if system (reference frame) is specified.
- Example (a la Galilei): A man dropping a stone from a ship’s mast. The man’s perspective: trajectory accelerates along a straight line. A bystander at the shore: trajectory is a parabola.
- Since both perspectives describe the same event, the math behind the respective description must be connected: Transformations.
Galilei transformations

- Basic idea: Space and time are decoupled.
- Consequence: A time interval of one hour remains invariant, irrespective of the choice of reference frame. This allows only transformations of the type \( t \rightarrow t' = t + \Delta t \).
- Similarly, at a time \( t_0 \) the origins of the two reference systems may be displaced: \( x(t_0) \rightarrow x'(t_0) = x(t_0) + \Delta x \), and only a constant velocity \( u \) between them is allowed.
- Ignoring \( \Delta t \) and \( \Delta x \), therefore \( x(t) \rightarrow x'(t) = x(t) + ut \).
- Consequence: Velocities are strictly additive. Assume system \( A \) (man on mast) is at rest and \( B \) (man on shore) moves with velocity \( u \) w.r.t. \( A \), then velocities are related by \( \mathbf{v}_B = \mathbf{v}_A + u \).
Lorentz transformations, once more

- **Basic idea:** Space and time are entangled.
- **Consequence:** Relative velocities between reference frames affect both space and time coordinates (remember: $c = 1$).

$$x \rightarrow x' = \frac{x - ut}{\sqrt{1 - u^2}} \quad \text{and} \quad t \rightarrow t' = \frac{t - ux}{\sqrt{1 - u^2}}.$$  

- Can write in matrix form – acting on vector $(t, x)$:

$$\hat{M}_{u_z} = \begin{pmatrix} \cosh \eta & 0 & 0 & -\sinh \eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh \eta & 0 & 0 & \cosh \eta \end{pmatrix},$$

with $\tanh \eta = u_z$ for a “boost” in $z$-direction.
- **Note the similarity to a rotation in space!**
Adding velocities

- Consequence of this: Velocities below $c$ can never add up to a result larger than $c$:

$$v_{\text{tot}} = \frac{v_1 + v_2}{1 + v_1 \cdot v_2}.$$ 

- Remark: This limits the maximal transmission velocity of information to $c$, therefore a perfectly rigid body cannot exist.

- Remark (2): Space-time is divided into causally connected ("time-like distances") and disconnected ("space-like distances") regions.
Mass, momentum and energy

- Demand conservation of mass, momentum and energy to be invariant under Lorentz transformations:

  \[
  m(v) = \frac{m_0}{\sqrt{1 - v^2}} = m_0 + m_0 \frac{v^2}{2} + \ldots.
  \]

  This is the original reason for the identification \( E = mc^2 \) - the second term in the expansion is just the kinetic energy.

- Using \( p = mv \) therefore \( E^2 = m_0^2 + p^2 \).

- This implies that for particles with no rest mass \( E/|p| = 1 \).
Example: Relativistic two-body decay

- Consider the decay of a massive particle into two lighter ones, such that rest masses satisfy $M > m_1 + m_2$.

- To calculate energies and momenta of decay products use:
  - Rest frame of decaying particle: $E = M$, $\vec{P} = 0$;
  - Energy conservation: $E = E_1 + E_2$;
  - Momentum conservation: $\vec{P} = \vec{p}_1 + \vec{p}_2 \implies \vec{p}_1 = \vec{p}_2$;

- Case 1: $m_1 = m_2 = 0 \implies E_1 = p_1 = p_2 = E_2 = M/2$

- Case 2: Arbitrary masses, $m_1 \neq 0$, $m_2 \neq 0$:
  
  $E_{1,2} = \frac{M^2 \pm (m_1^2 - m_2^2)}{2M}$ and $p_1 = p_2 = \frac{\sqrt{(M^2 - m_1^2 - m_2^2)^2 - 4m_1^2m_2^2}}{2M}$. 
Example: Relativistic two-body reactions

- Central reaction type in particle physics: **2-body scattering**: \( a + b \rightarrow c + d \)
- Convenient frame of inertia for description: centre-of-momentum frame, characterised by \( p_a + p_b = p_c + p_d = 0 \)
- Calculate Lorentz-invariant mass (energy) from:
  \[
  s = M_{\text{inv}}^2 = (E_a + E_b)^2 - (p_a + p_b)^2 \\
  = (E_c + E_d)^2 - (p_c + p_d)^2.
  \]
- This is the energy squared in the c.m.-frame: \( s = E_{\text{c.m.}}^2 \).
Example: Relativistic two-body reactions (cont’d)

- Can also calculate the (Lorentz-invariant) momentum transfer from $a$ to $c$, called $t$ and from $a$ to $d$, called $u$:

\[
t = (E_a - E_c)^2 - (p_a - p_c)^2 = (E_b - E_d)^2 - (p_b - p_d)^2
\]
\[
u = (E_a - E_d)^2 - (p_a - p_d)^2 = (E_b - E_c)^2 - (p_b - p_c)^2.
\]

- Properties:
  - $s > 0$, and $t, u \leq 0$
  - $s + t + u = m_a^2 + m_b^2 + m_c^2 + m_d^2$.

Therefore, for massless particles $s + t + u = 0$.

- In the c.m.-frame, and for massless particles:

\[
t = -\frac{E_{c.m.}^2}{2} \left(1 - \cos \theta_{ac}\right) \quad \text{and} \quad u = -\frac{E_{c.m.}^2}{2} \left(1 + \cos \theta_{ac}\right).
\]

$\theta_{ac}$ is called the “scattering angle”.

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Consider a special case of $2 \rightarrow 2$-scattering:

**Production of intermediate particle:**

$$a + b \rightarrow M \rightarrow c + d$$

Energy and momentum of $M$ in c.m.-frame:

$$E = E_a + E_b, \quad P = 0$$

We will see that the probability for this process “resonates”, if

$$s = E_{c.m.}^2 = M^2 \text{ (resonance production).}$$

The production cross section will yield a peak.

Note: Cross section is a way to quantify the probability for a process to happen, more on this in Lecture 3.
Example for resonance production: $e^+e^- \rightarrow \text{hadrons}$
Summary

- Reviewed more of special relativity.
- Calculated kinematics of relativistic two-body decays.
- To read: Primer on special relativity.
- To read: Coughlan, Dodd & Gripaios, “The ideas of particle physics”, Sec 1-2.