Gauge theory

Lecture given by Frank Krauss, winter term 2003/04,
at Dresden university for technology

Disclaimer:

This script is, to a large extent, influenced by a number of excellent books on group theory, quantum field theory, and gauge theory. Specifically, the following books have been used:

- H. F. Jones, "Groups, representations and physics";
- H. Georgi, "Lie algebras and particle physics";
- T. Kugo, "Gauge theory";
- B. Hatfield, "Quantum field theory of point particles and strings".
- P. Ramond, "Field theory, a modern primer";

Contents

Notation

Throughout this lecture the following notation will be used:

1. Units:
   Natural units are used,
$$\hbar = c = 1\,. \tag{1}$$

2. Four-vectors:
   Four-vectors in space time are written as
$$x^\mu = (t,\,\vec{x}) = (x^0,\,x^1,\,x^2,\,x^3),\ \mu = 0,\,1,\,2,\,3\,, \tag{2}$$
   where the first coordinate, labelled as x^0 is the time coordinate. In general, in the framework of this lecture, greek indices μ, ν, ... will run from 0 to 3 whereas latin indices i, j, ... run from 1 to 3. In other words,
$$x^\mu = (t,\,\vec{x}) = (x^0,\,x^i)\,. \tag{3}$$
   The scalar product of two four-vectors is
$$x\cdot y = x^\mu y_\mu = x_\mu y^\mu = x^0 y^0 - \vec{x}\cdot\vec{y} = x^0 y^0 - x^i y^i\,, \tag{4}$$
   where the Einstein convention of summing over repeated Lorentz-indices is used unless stated otherwise.

3. Metric:
   The scalar product above can also be written as
$$x\cdot y = x^\mu y^\nu g_{\mu\nu} = x_\mu y_\nu g^{\mu\nu}\,, \tag{5}$$

where the Minkowski-metric is

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{6}$$

and satisfies

$$g_{\mu\nu} g^{\nu\rho} = g_{\mu\nu} g^{\rho\nu} = g_\mu^\rho := \delta_\mu^\rho \tag{7}$$

with the Kronecker-delta

$$\delta_\mu^\nu = \begin{cases} 1 & \text{for} \quad \mu = \rho \\ 0 & \text{for} \quad \mu \neq \rho \end{cases} \tag{8}$$

4. Co- and contravariant four-vectors:
   The Minkowski-metric above us used to raise and lower indices, i.e.

   $$x_\mu = g_{\mu\nu} x^\nu = g_{\mu\rho} g^{\rho\nu} x_\nu = \delta_\mu^\nu x_\nu \,. \tag{9}$$

   Four-vectors with lower or upper indices, x_μ and x^μ are called covariant and contravariant, respectively. Their connection can be visualized employing their components. Obviuosly, contravariant four-vectors are

   $$x^\mu = (t,\, \vec{x}) \tag{10}$$

   and the corresponding contravariant four-vectors are given by

   $$x_\mu = (t,\, -\vec{x})\,, \tag{11}$$

   i.e. they have negative space-components.

5. Transformation of tensors:
   In the same fashion, the indices of tensors can be lowered or raised,

   $$T_{\mu\nu} = g_{\mu\rho} g_{\nu\sigma} T^{\rho\sigma} \,. \tag{12}$$

   Similarly, individual indices of tensors can be treated.

6. Totally anti-symmetric tensor:
   The totally anti-symmetric tensor $\epsilon^{\mu\nu\rho\sigma}$ is fixed by

   $$\epsilon^{0123} = 1 \,. \tag{13}$$

   Odd permutations of the indices result in the corresponding component to change its sign.

Chapter 1

Brief review of group theory

Algebra, and especially group theory, provide the mathematical framework in which many aspects of symmetry can be formulated in an elegant and compact fashion. Modern day particle physics is - to a large extent - based on external or internal symmetries such as Lorentz or gauge invariance. It is fair to say that these foundations form the most important cornerstone in the formulation of, e.g., the Standard Model and its super-symmetric extensions. To gain insight into this structure and into the beauty of theoretical particle physics some terms of group theory are necessary. Therefore, in this first chapter of the lecture, basics of group theory and Lie algebras will be recalled. Starting out with finite groups, the concept of the representation of groups will be briefly discussed, before the focus shifts onto Lie groups and algebras that are used to formalise continuous symmetries like the ones mentioned above. Finally, the concepts introduced so far will be exemplified by analysing the structure of $SU(2)$.

1.1 Finite Groups and their Representations

1.1.1 Finite Groups

Definition of a group

A group consists of a set $\mathcal{G}$, on which an operation "$\cdot$" ("multiplication") is defined. This operation exhibits the following properties

1. Closure : If g_i and g_j are elements of $\mathcal{G}$, then also $g_k = g_i \cdot g_j$ is a member of $\mathcal{G}$.

2. Identity element (existence of a "1") : There is exactly one identity element g_1 in $\mathcal{G}$, that maps all elements g_i exactly on themselves, $g_i = g_i \cdot g_1 = g_1 \cdot g_i$.

3. Inverse element : For every element g_i of $\mathcal{G}$ there is exactly one inverse element g_i^{-1} with the property $g_i^{-1} \cdot g_i = g_i \cdot g_i^{-1} = g_1$.

4. Associativity : $g_i \cdot (g_j \cdot g_k) = (g_i \cdot g_j) \cdot g_k$ is fulfilled.

The property of commutativity, i.e. the validity of $g_i \cdot g_j \stackrel{?}{=} g_j \cdot g_i$, decides, whether a group as defined above is Abelian or non–Abelian.
Some examples of groups include :

- $\{1, -1, i, -i\}$ together with the ordinary multiplication (note here, that the subset $\{-1, 1\}$ with the multiplication forms a subgroup),

- The integer numbers with the addition as operation,

- The set $\{1, 2, \ldots, p-1\}$ forms a group under multiplication modulo p, if p is a prime number.

- the set of all permutations of three objects a, b, and c, sitting on positions 1, 2, and 3. The group of permutations contains the six operations

( )	do nothing, $(a, b, c) \to (a, b, c)$
(12)	exchange objects on positions 1 and 2, $(a, b, c) \to (b, a, c)$
(13)	exchange objects on positions 1 and 3, $(a, b, c) \to (c, b, a)$
(23)	exchange objects on positions 2 and 3, $(a, b, c) \to (a, c, b)$
(123)	cyclic permutation, and $(a, b, c) \to (c, a, b)$
(321)	anti-cyclic permutation $(a, b, c) \to (b, c, a)$.

- The group D_4 of symmetry transformations on a square with corners labeled clockwise from 1 to 4. This group consists of the following elements :

E	Identity element.
$\mathcal{R}_{90,180,270}$	Clockwise rotations around the center of the square, with angle as indicated.
$\mathcal{M}_{x,y}$	Reflections around the horizontal (x) and vertical (y) axis.
$\mathcal{M}_{13,24}$	Reflections around the axis (13) or (24), respectively.

The "multiplication" of two such operations is defined as their sequential application. Here and in the following it is understood that in the product $A \cdot B$ first B and then A is executed, i.e. the order of operation is from the right to the left.

Generators of finite groups

In general it is possible to generate all the elements of a finite group by starting from a certain limited set of elements subject to some relations. Consider now the smallest set of elements generating all other elements by repeated application of the operation belonging to the group. The elements of this minimal set are called the generators of the group. Note that this set is not necessarily unique. Coming back to the example of the permutation group over three elements, it is quite easy to check that for instance any two out of the three (ij) generate the whole group.

Direct Products

The direct product of two groups $\mathcal{H}_1$ and $\mathcal{H}_2$ of order h_1 and h_2, respectively, is defined as

$$\mathcal{G} = \mathcal{H}_1 \otimes \mathcal{H}_2 = \{h_1^i \cdot h_2^j\} \ . \tag{1.1}$$

This definition holds true only, if both groups have only the identity element in common and if the elements of both groups commute mutually. As an example, consider

$$\{E\,,\ \mathcal{M}_x\} \otimes \{E\,,\ \mathcal{M}_y\} = \{E\,,\ \mathcal{M}_x\,,\ \mathcal{M}_y\,,\ \mathcal{R}_{180}\}\,. \tag{1.2}$$

1.1.2 Representations of finite groups

Definitions

Any mapping $\mathcal{D}$ of the group $\mathcal{G}$ with the property

$$\mathcal{D}(g_i) \cdot \mathcal{D}(g_j) = \mathcal{D}(g_i \cdot g_j) \tag{1.3}$$

for all elements g_i and g_j of $\mathcal{G}$ is called a representation of this group. More particularly, the name representation is reserved for a collection of square matrices T with the property above. The dimension of the representation is then identified with the rank of the matrices.
So, returning once more to one of the favourite examples, a representation of the permutation group of three elements in terms of square 3×3 matrices is given by,

$$\begin{aligned}
\mathcal{D}(\ ) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \mathcal{D}(12) &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
\mathcal{D}(13) &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} & \mathcal{D}(23) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
\mathcal{D}(321) &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} & \mathcal{D}(123) &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} .
\end{aligned}$$

In this particular representation, the operation "$\cdot$" can be identified with the multiplication of matrices, for instance $(12) \cdot (23) = (123)$ or

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} . \tag{1.4}$$

Since the matrices T do not need to be distinct, the set of the T does not necessarily form a group under matrix multiplication.

Equivalent representations

Two representations T_1 and T_2 of a group are called equivalent, if there is a nonsingular matrix S with

$$T_1 = S^{-1} T_2 S \,. \tag{1.5}$$

In other words if the two representations can be mapped onto each other by a similarity transformation within the same vector space, in which both representations (matrices) are defined and act, then the two representations are equivalent. A very useful theorem on representations is that each representation T of a finite group is equivalent to a representation by unitary matrices. This can be used to cast any representation T into a block diagonal form,

$$\tilde{T} = \begin{pmatrix} \tilde{t}^{(1)} & & & \\ & \tilde{t}^{(2)} & & \\ & & \ddots & \\ & & & \tilde{t}^{(n)} \end{pmatrix} . \tag{1.6}$$

The block diagonal elements $\tilde{t}^{(i)}$ are called irreducible representations of the group.

1.2 Lie Groups and Lie Algebras

1.2.1 Lie Groups, the basics

Definition

Stated in an intuitive manner (suitable for physicists), a Lie group is a continuous group, i.e. the elements are labeled by some continuous real parameter, where the multiplication law depends smoothly on these parameters. In other words, the multiplication law can be rewritten as a function of these parameters, the corresponding inverse elements exist and can be labeled as smooth functions of the parameters. This translates into the possibility to generate group elements from each other that differ only by an infinitesimal amount of one of the real parameters:

$$x(0,0,\dots,\epsilon_j,\dots,0) = x(0,0,\dots,0) + i\epsilon_j I_j(0,0,\dots,0) + \mathcal{O}(\epsilon^2) \tag{1.7}$$

For obvious reasons, the r operators I_j are called the generators of the Lie group, since their multiple application spans the group. Therefore, as indicated above, any group element, which can be derived from the identity element by continuous changes in the parameters can be re-expressed as

$$\exp(i\alpha_l I_l) \, , \tag{1.8}$$

where the summation over l is understood. The I_l are the linearly independent generators defined by

$$I_l = \lim_{\epsilon_l \to 0} \left[\frac{x(0, \ldots, \epsilon_l, \ldots, 0) - x(0, 0, \ldots, 0)}{i\epsilon_l} \right] \tag{1.9}$$

and the α_l are real parameters.

Representations of a Lie group

This leads immediately to the issue of how to represent a Lie group. The answer to this is pretty straightforward and already indicated above : Lie groups are represented by finite dimensional matrices. Extending the previous results on finite groups, representations of Lie groups have the following properties :

1. For any representation of a Lie group an equivalent representation with unitary matrices can be found.
2. Any unitary representation can be cast into block diagonal form as displayed in Eq.(1.6).
3. Any irreducible representation is finite dimensional.

Generator algebra

The more interesting thing about the generators I_l is that they define to a large extend the structure of the group. The basic point is that they satisfy simple commutation relations

$$[I_a, I_b] = I_a I_b - I_b I_a = i f_{abc} I_c \,, \tag{1.10}$$

and

$$[I_a, [I_b, I_c]] + \text{ cycl.} = 0 \,. \tag{1.11}$$

The commutator relation above, Eq.(1.10), determines completely the structure of the Lie group. Thus, the constants f_{abc} are called structure constants of the group.

1.2.2 Lie Algebras

Definition

The r generators of a Lie group form the basis vectors of an r–dimensional vector space. The set of real linear combinations of the generators is called a Lie algebra. Such an algebra L has the following properties :

1. If $x, y \in L$, then $[x, y] \in L$.
2. $[x, y] = -[y, x]$.
3. The Jacobi identity holds, $[x, [y, z]]$+cycl.$= 0$.

Since the generators of a Lie group together with the commutation relation form a Lie algebra, it is easy to see that the structure constants fulfil

$$f_{abc} = -f_{bac}\,. \tag{1.12}$$

Even more, Eq.(1.11) translates into

$$f_{bcd}f_{ade} + f_{abd}f_{cde} + f_{cad}f_{bde} = 0\,. \tag{1.13}$$

and defining a set of new matrices J via

$$(J_a)_{bc} = -if_{abc} \tag{1.14}$$

into

$$[J_a, J_b] = if_{abc}J_c\,. \tag{1.15}$$

Stated in less formal words, the structure constants f_{abc} themselves generate a representation of the algebra, which is called the adjoint representation. In contrast, the representation generated by the I is called fundamental representation. The dimension of this adjoint representation equals the number of generators, since this is the space, on which it acts. This is exactly the number of real parameters needed to describe every group element.

Construction of representations

As mentioned before, the structure constants can not be defined uniquely. Instead, they depend on the choice of the basis for the vector space of the generators. A suitable choice can be obtained via the adjoint representation, defined in Eq.(1.14). Consider the trace

$$Tr(J_aJ_b) = \sum_{d,e} f_{ade}f_{edb}\,. \tag{1.16}$$

J_aJ_b is a real, symmetric matrix. Therefore it can be diagonalised with real eigenvalues by appropriate choices of the generators I, i.e. by replacing the original I with suitable linear combinations, and, correspondingly, by linear combinations of the generators of the adjoint representation J. Thus,

$$Tr(J_aJ_b) = k_a\delta_{ab}\,, \tag{1.17}$$

and the freedom of rescaling the generators can be used to choose all k_a in a way that $|k_a| = 1$. In the framework of this lecture, mainly compact, semi–simple Lie–groups will be encountered. They are characterised by positive k_a that can be rescaled to some common κ. In this particular basis, the f_{abc} are completely antisymmetric, because

$$f_{abc} = -\frac{i}{\kappa}\,\mathrm{Tr}([J_a, J_b]\,J_c) \tag{1.18}$$

and by using the cyclic property of the trace. In this particular basis, the J_a are hermitian matrices.

Casimir operators

Finally, note that often operators can be constructed that commute with all generators of a Lie group. Such operators are dubbed Casimir operators, and the number of independent Casimir operators defines the rank of the group. For $SU(2)$, it can be shown that the operator

$$J^2 = \vec{J}^2 = J_1^2 + J_2^2 + J_3^2 \tag{1.19}$$

commutes with all the J_a. For $SO(3)$ the only Casimir operator looks similarly. Since the Casimir operators can then be diagonalised simultaneously with the generators, their eigenvalues may be used to label the irreducible representations of the group.

1.2.3 Fun with $SO(2)$ and $SU(2)$

$SO(2)$

The group $SO(2)$ describes the set of all rotations of a circle around the axis perpendicular to its plane through its center. Each element can be characterised by one angle $\theta \in [0, 2\pi[$. Integer multiples of 2π added to the interval can obviously be ignored, since they do not add any new physical information. The law of composition for group elements $T(\theta)$ is

$$T(\theta) \cdot T(\phi) = T(\phi) \cdot T(\theta) = \begin{cases} T(\theta + \phi) & \text{if } \theta + \phi < 2\pi \\ T(\theta + \phi - 2\pi) & \text{if } \theta + \phi \geq 2\pi \end{cases} \tag{1.20}$$

The identity element is $T(0)$, inverse elements are

$$T^{-1}(\phi) = T(2\pi - \phi)\,. \tag{1.21}$$

A representation of this group can be constructed by studying the effect of the group elements on some point (x, y) in the plane. Then,

$$T(\phi) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{1.22}$$

which is clearly an orthogonal 2×2–matrix with determinant 1. Coming back to Eq.(1.20), it is obvious that this group is Abelian, and following previous remarks, some alternative representation through $U(1)$ factors, namely $\exp(ic\phi)$ is recovered.
Due to its property as one–parameter group, $SO(2)$ has one generator, namely either

$$I = \lim_{\phi \to 0} \frac{1}{i\phi} \left[\exp(im\phi) - 1\right] = m \tag{1.23}$$

or

$$I = \lim_{\phi \to 0} \frac{1}{i\phi} \left[\begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{1.24}$$

depending on the representation chosen for the group elements. Having identified the result for the generator in the second choice with the Pauli–matrix σ_y, the generator reads

$$T(\phi) = \exp(i\phi\sigma_y)\,. \tag{1.25}$$

Yet another generator can be found that has some more physical meaning by considering the effect of the $T(\phi)$ on some function $f = f(x, y)$ defined on the plane, the group acts on. Then,

$$T(\phi)f(x,y) = f(x\cos\phi + y\sin\phi, -x\sin\phi + y\cos\phi) \tag{1.26}$$

and the generator can be found via

$$\begin{aligned} I \cdot f(x,y) &= \lim_{\phi\to 0} \frac{1}{i\phi} \left[f(x\cos\phi + y\sin\phi, -x\sin\phi + y\cos\phi) - f(x,y) \right] \\ &= -i \left(\frac{y\partial}{\partial x} - \frac{x\partial}{\partial y} \right) f(x,y)\,. \end{aligned} \tag{1.27}$$

Identifying the component of the angular momentum normal to the (x, y)–plane L_z with

$$L_z = i\hbar \left(\frac{y\partial}{\partial x} - \frac{x\partial}{\partial y} \right) = -i\hbar \frac{\partial}{\partial\phi} \tag{1.28}$$

the generator finally reads

$$T(\phi) = \exp\left(-\frac{i\phi L_z}{\hbar} \right)\,. \tag{1.29}$$

$SU(2)$

The simplest non–Abelian Lie–algebra consists of three generators J_a with structure constants given by the completely antisymmetric tensor, $f_{abc} = \epsilon_{abc}$. Then the commutation relations read

$$[J_a, J_b] = i\epsilon_{abc}J_c\,, \tag{1.30}$$

the angular momentum algebra. To construct the representations of the algebra (1.30), let us forget everything about the special role of the Casimir operator $\vec{L}^2$ in angular momentum or analogously of J^2 for $SU(2)$. Assume an N–dimensional irreducible representation of $SU(2)$, which can be diagonalised by one of the hermitian operators, say J_3. Now let us look for the α eigenstates $|j, \alpha\rangle$ of J_3 with maximal eigenvalue j.

$$J_3|j,\alpha\rangle = j|j,\alpha\rangle\,, \tag{1.31}$$

and reshuffle them in a way that they're orthogonal,

$$\langle j,\beta|j,\alpha\rangle = \delta_{\alpha\beta}\,. \tag{1.32}$$

As usual the non–hermitian ladder operators

$$J_\pm = \frac{1}{\sqrt{2}}(J_1 \pm iJ_2) \tag{1.33}$$

are constructed, where

$$[J_3, J_\pm] = \pm J_\pm\,,\quad [J_+, J_-] = J_3\,. \tag{1.34}$$

By acting on eigenstates $|m\rangle$ of J_3, these ladder operators raise or lower the eigenvalue m of them by one unit,

$$J_3J_\pm|m\rangle = J_\pm J_3|m\rangle \pm J_\pm|m\rangle = (m\pm 1)J_\pm|m\rangle\,. \tag{1.35}$$

Consequently, when acting on the eigenstate with the highest eigenvalue, J_+ yields

$$J_+|j,\alpha\rangle = 0\,. \tag{1.36}$$

Conversely, with some appropriate normalisation $N_j(\alpha)$ the states

$$J_-|j,\alpha\rangle = N_j(\alpha)|j-1,\alpha\rangle \tag{1.37}$$

are orthogonal,

$$\begin{aligned} N_j^*(\beta)N_j(\alpha)\langle j-1,\beta|j-1,\alpha\rangle &= \langle j,\beta|J_+J_-|j,\alpha\rangle \\ &= \langle j,\beta|[J_+,J_-]|j,\alpha\rangle \\ &= \langle j,\beta|J_3|j,\alpha\rangle = j\delta_{\alpha\beta}\,. \end{aligned} \tag{1.38}$$

When replacing the product with the commutator in the equation above, the fact has been used that J_+ annihilates the state $|j,\alpha\rangle$. Thus,

$$|N_j(\alpha)|^2 = j \tag{1.39}$$

and the phases of the $N_j(\alpha)$ can be chosen in order to make them real and positive numbers,

$$N_j(\alpha) = \sqrt{j}\,. \tag{1.40}$$

Therefore,

$$\begin{aligned} J_+|j-1,\alpha\rangle &= \frac{1}{N_j} J_+ J_- |j,\alpha\rangle \\ &= \frac{1}{N_j}[J_+, J_-]|j,\alpha\rangle = N_j |j,\alpha\rangle \end{aligned} \tag{1.41}$$

This procedure can be repeated, and step by step orthonormal states $|j-k,\alpha\rangle$ are found which satisfy

$$\begin{aligned} J_-|j-k,\alpha\rangle &= N_{j-k}|j-k-1,\alpha\rangle \\ J_+|j-k-1,\alpha\rangle &= N_{j-k}|j-k,\alpha\rangle \end{aligned} \tag{1.42}$$

with a recursion relation among the N_{j-k},

$$\begin{aligned} N_{j-k}^2 &= \langle j-k,\alpha|J_+J_-|j-k,\alpha\rangle \\ &= \langle j-k,\alpha|[J_+,J_-]|j-k,\alpha\rangle + \langle j-k,\alpha|J_-J_+|j-k,\alpha\rangle \\ &= j-k+N_{j-k+1}^2 \,. \end{aligned} \tag{1.43}$$

Adding up these normalisation factors from N_j^2 down to N_{j-k}^2,

$$N_{j-k}^2 = \frac{(k+1)(2j-k)}{2} \tag{1.44}$$

or, introducing $m = j-k$

$$N_m = \sqrt{\frac{(j-m+1)(j+m)}{2}}\,. \tag{1.45}$$

Climbing down the ladder, sooner or later some state $|j-l\rangle$ will be reached that gets annihilated by J_-, $J_-|j-l\rangle = 0$. But this requires

$$N_{j-l} = 0 \implies l = 2j\,. \tag{1.46}$$

Since l is an integer number, the j are half–integers, and the space breaks up into α subspaces of dimension $2j+1$ plus possibly a subspace which might not have been found yet. This is exactly, what we would have found taking into account our knowledge about the Casimir operator J^2 and its eigenvalues. On the subspaces labeled by α, however, the J's act in a block–diagonal form,

since the orthogonality in the indices α and β, Eq.(1.38), remains conserved throughout our ladder procedure. But it was assumed that we already worked in some irreducible representation, therefore there is only one α. This can be used to break down any representation into its irreducible components. Starting with any representation the block–diagonal form is reconstructed, where the highest j–representation is explicitly singled out. This procedure can be repeated with the remaining block until the representation has been decomposed completely.
The eigenvalues of J_3 are called weights, and j is the highest weight. The irreducible representations are characterised by the highest weight j and they are called spin–j representations. In fact, they are associated with a particle at rest and spin angular momentum j, given by $J^2 = \hbar^2 j(j+1)$. The existence of hermitian angular momentum operators in a problem allows to decompose the Hilbert space of the full problem into states $|j, m, \alpha\rangle$ in the spin–j representation, where m is the eigenvalue of J_3, j is related to the eigenvalues of J^2 and α stands for all other labels, which have to be introduced, in order to characterise the states. States can be constructed which satisfy

$$\langle m, j, \alpha | m', j' \alpha' \rangle = \delta_{mm'} \delta_{jj'} \delta_{\alpha\alpha'} \, . \tag{1.47}$$

The δ–function in the α is in analogy with what has been discussed above, the δ–function in m and m' results from the fact that they are eigenvalues of an hermitian operator (which has different eigenvectors and eigenvalues). Even without employing the fact that the operator J^2 is hermitian, too, it can be shown that the δ–function in j and j' holds. Suppose that $j' > j$ and consider

$$\langle j, j, \alpha | j, j', \beta \rangle = \sqrt{\frac{2}{(j'+j)(j'-j+1)}} \langle j, j, \alpha | J_- | j+1, j', \beta \rangle = 0 \tag{1.48}$$

Now, some irreducible representations of $SU(2)$ will be constructed explicitly:

- <u>$j = 1/2$:</u>
  The simplest non–trivial representation of this group has dimension 2, i.e. $j = 1/2$. The vectors, in which J_3 is already diagonal are then $|1/2\rangle = (1,0)^T$ and $|-1/2\rangle = (0,1)^T$. Hence, the ladder operators have the form

$$J_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \, , \quad J_- = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \, . \tag{1.49}$$

Thus,

$$J_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \ \ J_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \qquad (1.50)$$

and by use of the commutator relations,

$$J_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ . \qquad (1.51)$$

Not unsurprisingly, the common Pauli–matrices σ_i can be recovered. So, the simplest, non–trivial representation of $SU(2)$ is generated by $\sigma_i/2$.

- $\underline{j = 1 :}$
  The spin–1 representation is spanned by the following basis vectors : $|1\rangle = (1,0,0)^T$, $|0\rangle = (0,1,0)^T$ and $|-1\rangle = (0,0,1)^T$. The ladder operators are

$$J_+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \ , \ \ J_- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \ . \qquad (1.52)$$

and

$$\begin{aligned} J_1 &= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \ , \ \ J_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \ , \\ J_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \ . \qquad (1.53) \end{aligned}$$

Chapter 2

Gauge symmetries and Lagrangians

In this chapter basic symmetry principles will be discussed, and their implications to the construction of theories and their Lagrangians obeying these symmetries.
In particular this lecture starts out with the basic property of fields in quantum field theory, namely their invariance under Lorentz transformations leading to the notion of the Lorentz group. The general form of its generators will be constructed, and, with their help, representations of this group will be classified before some of the representations such as, e.g. spinors, will be discussed in some more detail. Then the effect of symmetries in general will be highlighted and the implications of the Noether theorem will be summarised. Finally, some basic building blocks of Lagrange densities for fields prevalent in particle physics will be constructed. This chapter closes with a discussion of how the gauge principle comes into play when constructing Lagrange densities for gauge theories. This is exemplified in some detail for the simplest case, electromagnetism, which is connected to the group $U(1)$ as gauge group. The more involved case of $SU(N)$ gauge theories is only briefly highlighted.

2.1 Space-time symmetries: Lorentz group

2.1.1 Lorentz transformations

Recalling special relativity

The structure of space-time in standard quantum field theory and, thus, in gauge theories is usually given by the one of special relativity determined by the following principles:

1. Space and time are homogenous,
2. space is isotropic,
3. in all inertial systems of reference the speed of light has the same value c (put to 1 in the framework of this lecture).

This has a number of consequences. Remember that the space-time distance of two events 1 and 2 is the same in two frames S and S' that move relative to each other with constant velocity v,

$$(x_1 - x_2)^2 = (x'_1 - x'_2)^2 \,. \tag{2.1}$$

Assuming the velocity v to point along an axis labelled by $\|$, the relation between points in the two systems reads

$$t' = \gamma \cdot (t - x_{\|} \cdot v) \,, \quad x'_{\|} = \gamma(x_{\|} - v \cdot t) \,, \quad x'_{\perp} = x_{\perp} \,. \tag{2.2}$$

(Remember, in any case, that this lecture uses natural units, i.e. $c \equiv 1$). In the equations above, Eq. (2.2), the boost γ is given by

$$\gamma = 1/\sqrt{1 - v^2} \,. \tag{2.3}$$

The Lorentz transformation of Eq.(2.2) can be cast into a form similar to the following one for rotations, namely

$$\begin{aligned} t' &= t \\ x'_1 &= x_1 \cos\theta - x_2 \sin\theta \\ x'_2 &= x_1 \sin\theta + x_2 \cos\theta \\ x'_3 &= x_3 \,, \end{aligned} \tag{2.4}$$

where the rotation angle is θ and the rotation is around the z- (or 3-) axis. For rotations the spatial distance is constant,

$$\vec{x'}^2 = \vec{x}^2 \,. \tag{2.5}$$

In three dimensions, the set of all rotations forms a non-commutative or non-Abelian group under consecutive application called $SO(3)$, the special orthogonal group in three dimensions. In this framework, orthogonal denotes such groups that can be represented by real matrices and leave Euclidean distances invariant, i.e. have determinant equal to one; the group is special because its determinant equals plus one. In contrast, the inclusion of reflections around a plane leads to matrices with determinant equal to minus one.

Returning to Lorentz transformations and identifying the boost to be along the z-axis it is easy to check that Eq.(2.2) can be written as

$$\begin{aligned} x'_0 &= x_0 \cosh\xi - x_3 \sinh\xi \\ x'_1 &= x_1 \\ x'_2 &= x_2 \\ x'_3 &= -x_0 \sinh\xi + x_3 \cosh\xi \,, \end{aligned} \tag{2.6}$$

with $\gamma = \cosh\xi$, or, equivalently, $\tanh\xi = v$.

Definition

Lorentz transformations are defined as linear transformations

$$x^\mu \longrightarrow x'^\mu = \Lambda^\mu_{\ \nu} x^\nu \tag{2.7}$$

of four-vectors that leave their norm invariant:

$$x^\mu \to x'^\mu = \Lambda^\mu_{\ \nu} x^\nu , \ \ x'^2 = x'^\mu x'_\mu =: x^2 \,. \tag{2.8}$$

Invariance of the norm implies that

$$x'^\mu x'^\nu g_{\mu\nu} = g_{\mu\nu} \Lambda^\mu_{\ \rho} \Lambda^\nu_{\ \sigma} x^\rho x^\sigma = x^\rho x^\sigma g_{\rho\sigma} \Longrightarrow \Lambda^\mu_{\ \rho} \Lambda^\nu_{\ \sigma} g_{\mu\nu} = g_{\rho\sigma} \,. \tag{2.9}$$

Multiplication of the second part of this equation with $g^{\rho\tau}$ leads to

$$\Lambda^\mu_{\ \rho} \Lambda^\nu_{\ \sigma} g_{\mu\nu} g^{\rho\tau} = \Lambda_\mu^{\ \tau} \Lambda^\mu_{\ \sigma} = g_{\rho\sigma} g^{\rho\tau} = \delta^\tau_\sigma \,. \tag{2.10}$$

This together with

$$\Lambda^{\mu}_{\ \rho} g_{\mu\nu} g^{\rho\tau} = \Lambda_{\nu}^{\ \tau} \tag{2.11}$$

immediately implies that

$$\Lambda_{\mu}^{\ \tau} = (\Lambda^{\mu}_{\ \tau})^{-1} = \left(\Lambda^{-1}\right)^{\tau}_{\ \ \mu} , \tag{2.12}$$

fixing the inverse of Λ. Using the inverse the Lorentz transformation of a covariant four-vector x_μ can thus be written as

$$x_\mu \;\longrightarrow\; x'_{\ \mu} = \Lambda_{\mu}^{\ \nu} x_\nu = x_\nu \left(\Lambda^{-1}\right)^{\nu}_{\ \ \mu} . \tag{2.13}$$

For the Lorentz transformation of an arbitrary tensor all its indices have to be transformed:

$$T'^{\ \ \nu_1\nu_2\ldots\nu_m}_{\mu_1\mu_2\ldots\mu_n} = \Lambda_{\mu_1}^{\ \rho_1} \Lambda_{\mu_2}^{\ \rho_2} \ldots \Lambda_{\mu_n}^{\ \rho_n} \; \Lambda^{\nu_1}_{\ \sigma_1} \Lambda^{\nu_2}_{\ \sigma_2} \ldots \Lambda^{\nu_m}_{\ \sigma_m} \; T_{\rho_1\rho_2\ldots\rho_n}^{\ \ \sigma_1\sigma_2\ldots\sigma_m} . \tag{2.14}$$

Classification of Lorentz transformations

There are only two constant, Lorentz invariant tensors, namely the metric tensor $g^{\mu\nu}$ and its co- and contravariant versions, especially the unity matrix, and the totally anti-symmetric tensor $\epsilon^{\mu\nu\rho\sigma}$. Lorentz invariance can be checked easily,

$$\begin{aligned} g'^{\mu\nu} &= \Lambda^{\mu}_{\ \rho} \Lambda^{\nu}_{\ \sigma} g^{\rho\sigma} = \Lambda^{\mu\sigma} \Lambda^{\nu}_{\ \sigma} = g^{\mu\rho} \Lambda_{\rho}^{\ \sigma} \Lambda^{\nu}_{\ \sigma} = g^{\mu\rho} \delta^{\nu}_{\rho} = g^{\mu\nu} , \\ \epsilon'^{\mu\nu\rho\sigma} &= \Lambda^{\mu}_{\ \alpha} \Lambda^{\nu}_{\ \beta} \Lambda^{\rho}_{\ \gamma} \Lambda^{\sigma}_{\ \delta} \epsilon^{\alpha\beta\gamma\delta} = \epsilon^{\mu\nu\rho\sigma} \cdot \det \Lambda . \end{aligned} \tag{2.15}$$

The identities above can be easily constructed using Eq.(2.12) and the definition of the determinant. Due to Eq.(2.10) this determinant is fixed such that

$$|\det \Lambda| = 1 \;\longrightarrow\; \det \Lambda = \pm 1 . \tag{2.16}$$

In other words, if the determinant equals plus one, the totally anti-symmetric tensor does not change its sign under the corresponding Lorentz transformations, otherwise it does. This in addition with the sign of the $\mu = \nu = 0$ component of the Lorentz transformation allows to classify any such transformation Λ. The $\mu = \nu = 0$ component of any Lorentz transformation Λ is given by

$$\left(\Lambda_0^{\ 0}\right)^2 = 1 + \sum_{i=1,2,3} \left(\Lambda_0^{\ i}\right)^2 \geq 1 . \tag{2.17}$$

According to the determinant and the sign of the 00 component Lorentz transformations can be classified as:

1. Proper, orthochronous:
   Transformations connected with the identity or unity matrix **1**, given below. These transformations have $\det \Lambda = +1$ and $\Lambda_0^0 \geq 1$.

2. Improper, orthochronous:
   Transformations connected with the parity transformation matrix **P**, given below. These transformations have $\det \Lambda = -1$ and $\Lambda_0^0 \geq 1$.

3. Improper, non-orthochronous:
   Transformations connected with the time reversal matrix **T**, given below. These transformations have $\det \Lambda = -1$ and $\Lambda_0^0 \leq 1$.

4. Proper, non-orthochronous:
   Transformations connected with the combination of both, **PT**, given below. These transformations have $\det \Lambda = +1$ and $\Lambda_0^0 \leq 1$.

The matrices named above are

$$\mathbf{1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$\mathbf{T} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{PT} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{2.18}$$

For pseudo-tensors the Lorentz transformation, Eq.(2.14), reads

$$T'{}_{\mu_1\mu_2\ldots\mu_n}{}^{\nu_1\nu_2\ldots\nu_m} = (\det\Lambda)\cdot\Lambda_{\mu_1}^{\rho_1}\Lambda_{\mu_2}^{\rho_2}\ldots\Lambda_{\mu_n}^{\rho_n}\;\Lambda_{\sigma_1}^{\nu_1}\Lambda_{\sigma_2}^{\nu_2}\ldots\Lambda_{\sigma_m}^{\nu_m}\;T_{\rho_1\rho_2\ldots\rho_n}{}^{\sigma_1\sigma_2\ldots\sigma_m}\,, \tag{2.19}$$

i.e., the determinant of the transformation has to be added. Its effect is that pseudo-tensors transform like ordinary tensors under Lorentz transformation with determinant equals to one, but they transform differently otherwise and gain an extra sign. Vectors having this behaviour under Lorentz transformations are called axial vectors, an example is the angular momentum.
From the considerations so far it is obvious that all Lorentz indices of a term have to be contracted to ensure its Lorentz -invariance. In other words, such

terms have to be Lorentz scalars or pseudo-scalars, the latter emerge when the indices are contracted through a pseudo-tensor. This can be exemplified by terms of the form

$$\epsilon_{\mu\nu\rho\sigma}\, x^\mu x^\nu x^\rho x^\sigma \,, \tag{2.20}$$

which changes sign when Lorentz transformations are applied that are connected with either **P** or **T**.

Infinitesimal Lorentz transformations

Consider a Lorentz transformation that differs from the identity only by infinitesimal amounts,

$$\Lambda^\mu_\nu = \delta^\mu_\nu + \varepsilon^\mu_\nu \,, \tag{2.21}$$

where

$$|\varepsilon^\mu_\nu| \ll 1 \,. \tag{2.22}$$

Plugging this into Eq.(2.10) and expanding up to first order in ε yields

$$g_{\mu\nu} + g_{\mu\sigma}\varepsilon^\sigma_{\ \nu} + g_{\tau\nu}\varepsilon^\tau_{\ \mu} = g_{\mu\nu} \,, \tag{2.23}$$

resulting in

$$\varepsilon_{\mu\sigma} + \varepsilon_{\sigma\mu} = 0 \,. \tag{2.24}$$

This proves that the tensor ε is an anti-symmetric tensor which can be expressed through six independent parameters (for the six off-diagonal elements forming either the upper or the lower triangle).
Applying such an infinitesimal Lorentz transformation looks like

$$x'^\mu = \Lambda^\mu_{\ \nu} x^\nu = x^\mu + \varepsilon^\mu_{\ \nu} x^\nu := \left(1 - \frac{i}{2}\varepsilon^{\rho\sigma} M_{\rho\sigma}\right)^\mu_{\ \nu} x^\nu \,, \tag{2.25}$$

where the six totally antisymmetric 4×4 matrices $M_{\rho\sigma} = -M_{\rho\sigma}$ have been introduced. They read

$$(M_{\rho\sigma})^\mu_{\ \nu} = i\left(\delta^\mu_\rho g_{\sigma\nu} - \delta^\mu_\sigma g_{\rho\nu}\right) \,. \tag{2.26}$$

The commutator of two such matrices obeys the following relation

$$[M_{\mu\nu}\,, M_{\rho\sigma}] = -i\left(g_{\mu\rho}M_{\nu\sigma} + g_{\nu\sigma}M_{\mu\rho} - g_{\mu\sigma}M_{\nu\rho} - g_{\nu\rho}M_{\mu\sigma}\right) \,. \tag{2.27}$$

Casting Eq.(2.25) into the finite version of a Lorentz transformation,

$$\hat{\Lambda} = \exp\left(-\frac{i}{2}\varepsilon^{\mu\nu}\hat{\mathcal{M}}_{\mu\nu}\right) \approx 1 - \frac{i}{2}\varepsilon^{\mu\nu}\hat{\mathcal{M}}_{\mu\nu} \,. \tag{2.28}$$

the six matrices $\hat{\mathcal{M}}_{\rho\sigma}$ are recognised as the generators of the Lorentz group $SO(3, 1)$. The specific form in Eq.(2.26) is also the one for the generators $\hat{\mathcal{M}}_{\rho\sigma}$ when acting on four-vectors. When acting on other entities or fields, the $\hat{\mathcal{M}}_{\mu\nu}$ might have another form. This specific form depends on the representation and on what they act on. In any case, of course, these generators obey the same commutation relations above, Eq.(2.27). In what follows now, a more general form of the generators $\hat{\mathcal{M}}_{\mu\nu}$ will be reconstructed.
It is easy to check that the operators

$$\hat{L}_{\mu\nu} := i(x_\mu \partial_\nu - x_\nu \partial_\mu) \tag{2.29}$$

also satisfy the same Lie algebra, Eq.(2.27). Therefore, the $\hat{\mathcal{M}}_{\mu\nu}$ can be expressed as

$$\hat{\mathcal{M}}_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) + \hat{S}_{\mu\nu}\,, \tag{2.30}$$

where the hermitian (spin-) operator $\hat{S}_{\mu\nu}$ obeys the same Lie-algebra, Eq.(2.27), and commutes with the $\hat{L}_{\mu\nu}$. The last identity will be discussed later on, in connection with the transformation properties of various fields. For now it should be sufficient to mention that Eq.(2.30) states that the generators of the Lorentz group can be decomposed into two parts. One part, $\hat{L}_{\mu\nu}$, acts on the spatial dependence of the fields, the other part, $\hat{S}_{\mu\nu}$ acts on its internal symmetries, such as spin.

Construction of the generators

To see the form of the $M_{\mu\nu}$, Eq.(2.26), let us step back to the rotation group in three dimensions, $SO(3)$. The generators of this (and - in principle - any other) group can be obtained through a cook-book recipe by differentiating matrices related to finite transformations and taking the limit of the finite parameter going to zero, cf. Eq.(1.9). The generators are then a minimal set of basis-matrices. In the case of SO(3) we have

$$\hat{\mathcal{R}}_j := i\,\frac{d\mathcal{R}_j(\phi_j)}{d\phi_j}\bigg|_{\phi_j \to 0}\,, \tag{2.31}$$

where the $\hat{\mathcal{R}}$ denote the generators and $j = 1, 2, 3$ labels the axis around which the finite rotation of angle ϕ_j is performed. The generators are

$$
\begin{aligned}
\hat{\mathcal{R}}_1 &= i\left[\frac{d}{d\phi_1}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\phi_1 & -\sin\phi_1 \\ 0 & 0 & \sin\phi_1 & \cos\phi_1 \end{pmatrix}\right]_{\phi_1\to 0} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \\
\hat{\mathcal{R}}_2 &= i\left[\frac{d}{d\phi_2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi_2 & 0 & \sin\phi_2 \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\phi_2 & 0 & \cos\phi_2 \end{pmatrix}\right]_{\phi_2\to 0} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \\
\hat{\mathcal{R}}_3 &= i\left[\frac{d}{d\phi_3}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi_3 & -\sin\phi_3 & 0 \\ 0 & \sin\phi_3 & \cos\phi_3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right]_{\phi_3\to 0} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} .
\end{aligned}
\tag{2.32}
$$

But apart from the three rotations the Lorentz group also contains three more transformations related to boosts along the three spatial axes. Their generators $\hat{\mathcal{B}}_i$ are given by

$$
\begin{aligned}
\hat{\mathcal{B}}_1 &= i\left[\frac{d}{d\xi_1}\begin{pmatrix} \cosh\xi_1 & -\sinh\xi_1 & 0 & 0 \\ -\sinh\xi_1 & \cosh\xi_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right]_{\xi_1\to 0} = \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
\hat{\mathcal{B}}_2 &= i\left[\frac{d}{d\xi_2}\begin{pmatrix} \cosh\xi_2 & 0 & -\sinh\xi_2 & 0 \\ 0 & 1 & 0 & 0 \\ -\sinh\xi_2 & 0 & \cosh\xi_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right]_{\xi_2\to 0} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
\hat{\mathcal{B}}_3 &= i\left[\frac{d}{d\xi_3}\begin{pmatrix} \cosh\xi_3 & 0 & 0 & -\sinh\xi_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\xi_3 & 0 & 0 & \cosh\xi_3 \end{pmatrix}\right]_{\xi_3\to 0} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} .
\end{aligned}
\tag{2.33}
$$

The commutation relations can now be checked explicitly; for instance we have

$$\begin{aligned}\left[\hat{\mathcal{B}}_3\,,\hat{\mathcal{R}}_1\right] &= i\hat{\mathcal{B}}_2\\ \left[\hat{\mathcal{B}}_1\,,\hat{\mathcal{B}}_2\right] &= i\hat{\mathcal{R}}_3\,.\end{aligned}\tag{2.34}$$

The full algebra can be summarised as

$$\begin{aligned}\left[\hat{\mathcal{R}}_i\,,\hat{\mathcal{R}}_j\right] &= i\epsilon_{ijk}\hat{\mathcal{R}}_k\\ \left[\hat{\mathcal{R}}_i\,,\hat{\mathcal{B}}_j\right] &= i\epsilon_{ijk}\hat{\mathcal{B}}_k\\ \left[\hat{\mathcal{B}}_i\,,\hat{\mathcal{B}}_j\right] &= -i\epsilon_{ijk}\hat{\mathcal{R}}_k\,.\end{aligned}\tag{2.35}$$

Taken together it can be stated that the generators of rotations close on themselves, the generators of boosts behave like vectors under rotations, and that the boosts do not close on themselves, as the last equation shows. These generators are identical to the ones in Eq.(2.26) and can be identified as

$$\hat{\mathcal{R}}_i = \epsilon_{ijk}M_{jk}\,,\quad \hat{\mathcal{B}}_i = M_{i0}\,.\tag{2.36}$$

The algebraic structure of Eq.(2.35)can be greatly simplified by considering the following operators:

$$\hat{\mathcal{X}}_i^{\pm} = \frac{1}{2}\left[\hat{\mathcal{R}}_i \pm i\hat{\mathcal{B}}_i\right]\,.\tag{2.37}$$

It is easy to check that in terms of these operators the commutation relations of Eq.(2.35) are

$$\begin{aligned}\left[\hat{\mathcal{X}}_i^{+}\,,\hat{\mathcal{X}}_j^{+}\right] &= i\epsilon_{ijk}\hat{\mathcal{X}}_k^{+}\\ \left[\hat{\mathcal{X}}_i^{-}\,,\hat{\mathcal{X}}_j^{-}\right] &= i\epsilon_{ijk}\hat{\mathcal{X}}_k^{-}\\ \left[\hat{\mathcal{X}}_i^{+}\,,\hat{\mathcal{X}}_j^{-}\right] &= 0\,.\end{aligned}\tag{2.38}$$

Obviously, the effect of the transformation Eq.(2.37) is to split the algebra of Eq.(2.38) into two independent $SU(2)$ algebras [1]. Having at hand this

[1] A similar thing happens in fact when considering $SO(4)$ instead of $SO(3,\,1)$, which also splits into two $SU(2)$ algebras. The crucial difference, however, lies in the fact that in the case of the $SO(4)$ there is no imaginary unit i involved in the transformation analogous to Eq.(2.37). That's why $SO(4)$ is isomorphic to $SU(2)\otimes SU(2)$, whereas $SO(3,\,1)$ is isomorphic to $SL(2,\,C)$

decomposition, the representation of $SO(3, 1)$ is quite straightforward. Irreducible representations are connected to the irreducible representations of the two $SU(2)$'s. Each of the two $SU(2)$ has one Casimir operator, namely

$$\mathcal{X}_i^+\mathcal{X}_i^+ \quad \text{with eigenvalues } j_1(j_1+1)$$

and

$$\mathcal{X}_i^-\mathcal{X}_i^- \quad \text{with eigenvalues } j_2(j_2+1)\,, \tag{2.39}$$

where, in both cases $j_{1,2} = 0, 1/2, 1, \ldots$. Using, as usual, the eigenvalues of the Casimir operators to label representations, the representations of $SO(3, 1)$ can be identified through

$$(2j_1+1,\, 2j_2+1)\,. \tag{2.40}$$

Apart from the trivial representation $(0, 0)$ we will encounter in the following the two Weyl-representations $(0, \frac{1}{2})$ and $(\frac{1}{2}, 0)$, which in particle physics are used to describe the neutrino, the Dirac-representation $(\frac{1}{2}, 0)\oplus(0, \frac{1}{2})$, which is used for, e.g., the electron, and the defining representation $(\frac{1}{2}, \frac{1}{2})$ providing the transformations of four-vectors.

The Poincare group

Apart from Lorentz invariance, there is another symmetry of space-time that is realized for many systems, and, especially, for the field theories that will be studied during this lecture. This is the symmetry under space-time translations,

$$x^\mu \longrightarrow x'^\mu = x^\mu + a^\mu\,. \tag{2.41}$$

Therefore the general space-time invariance group is the ten-parameter (six from the Lorentz piece, four from translations) Poincare group, under which

$$x^\mu \longrightarrow x'^\mu = \Lambda^\mu_\nu x^\mu + a^\mu\,. \tag{2.42}$$

Simple considerations show that the translations do not commute with the Lorentz transformations - just imagine what happens to a translated system being rotated and translated again and interchange the order of the two last transformations. The generators for the translations are the momenta, just consider small changes

$$\delta x^\mu := \varepsilon^\rho g^\mu_\rho = \varepsilon^\rho \partial_\rho x^\mu = i\varepsilon^\rho \hat{p}_\rho x^\mu\,, \tag{2.43}$$

where the $\hat{p}_\mu$ are the momentum operators which in fact act as the generators of translation. They commute with each other,

$$[\hat{p}_\mu,\, \hat{p}_\nu] = 0\,, \tag{2.44}$$

but, of course, not with the generators of the Lorentz group,

$$\left[\hat{M}_{\mu\nu}\,, \hat{p}_\rho\right] = -ig_{\mu\rho}\hat{p}_\nu + ig_{\nu\rho}\hat{p}_\mu\,. \tag{2.45}$$

To construct Casimir operators of this group one has, again, to look for operators that commute with all generators. Such an operator is given by the length of the four-momentum squared, $p_\rho p^\rho$ that is obviously invariant under Lorentz transformations (and therefore commutes with their generators) and that commutes with $\hat{p}_\mu$. The other Casimir operator is constructed by the Pauli-Lubanski four-vector,

$$\hat{W}^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\,\hat{p}_\nu\hat{M}_{\rho\sigma}\,. \tag{2.46}$$

It is easy to check that it commutes with $\hat{p}_\kappa$ and that it transforms like a four-vector under Lorentz transformations. The latter property means that it's length is therefore invariant, leaving the operator $W^\mu W_\mu$ a Casimir operator. The representations of the Poincare group fall into three classes:

1. $P^\rho P_\rho \equiv m^2 > 0$ is a real number. Then, eigenvalues of $W^\mu W_\mu$ are given by $-m^2\, s(s+1)$, where s is the spin of the particle and has discrete values 0, 1/2, 1, ... States within this representation are then characterised by the continuous value of the momentum (three degrees of freedom, since $P^\rho P_\rho \equiv m^2$) and by the third component s_3 of the spin with $s_3 = -s,\, -s+1,\, \ldots,\, s$. Massive particles have thus $2s+1$ spin degrees of freedom.

2. $P^\rho P_\rho = 0$, and, therefore, also $W^\mu W_\mu = 0$. Since $P_\rho W^\rho = 0$, both vectors are proportional, the constant of proportionality is the helicity which is equal to $\pm s$. Thus, massless particles with spin $s > 0$ have two spin degrees of freedom.

3. A case seemingly not realized in nature: $P^\rho P_\rho = 0$ but continuous spin.

2.1.2 Fields and representations of the Lorentz group

Behaviour of fields under coordinate changes

As stated before, Lorentz transformations connect two systems S and S' with each other that have constant relative velocity. In other words, two observers O and O' in the two systems use the coordinates x and x' to characterise events. The connection of these coordinates is given by Eq.(2.7),

$$x^\mu \;\longrightarrow\; x'^\mu = \Lambda^\mu_\nu x^\nu \,.$$

For scalar function, both observers should measure the same value at points connected by the Lorentz transformation,

$$\phi'(x') = \phi(x)\,. \tag{2.47}$$

and for vector fields, being equipped with Lorentz indices, they have

$$A'_\mu(x') = \Lambda_\mu^{\;\nu}\, A_\nu(x)\,. \tag{2.48}$$

In general, arbitrary N-dimensional fields $\Phi_i^{(N)}$ transform like

$$\Phi'^{(N)}_i(x') = \mathcal{D}^{(N)}(\Lambda)_i^j\, \Phi_j^{(N)}(x)\,, \tag{2.49}$$

where $\mathcal{D}^{(N)}(\Lambda)$ is an $N \times N$ matrix. It should be stressed that this matrix acts on both the coordinates and the functional dependence of the field on them. In fact, as will be discussed more explicitly for the individual fields, this matrix is connected with the operator $\hat{\mathcal{M}}$ of Eq.(2.30).
Of course, this game of transforming systems can be played with three systems as well. Then coordinates are connected through

$$x''_\mu = \Lambda_{2\mu}^{\;\;\nu}\, x'_\nu = \Lambda_{2\mu}^{\;\;\nu}\, \Lambda_{1\nu}^{\;\;\rho}\, x_\rho = [\Lambda_2\Lambda_1]_\mu^{\;\rho}\, x_\rho\,, \tag{2.50}$$

and, similarly, for scalar fields

$$\phi''(x'') = \mathcal{D}^{(0)}(\Lambda_2)\, \phi'(x') = \mathcal{D}^{(0)}(\Lambda_2)\, \mathcal{D}^{(0)}(\Lambda_1)\, \phi(x) = \mathcal{D}^{(0)}(\Lambda_2\Lambda_1)\, \phi(x)\,. \tag{2.51}$$

Those more familiar with group theory will immediately recognise the matrices $\mathcal{D}$ to be representations of the Lorentz group. This establishes an intimate connection of all representations of a symmetry group on the fields with all kinds of fields that are allowed in the corresponding Lagrangian of the theory.

Returning to the considerations that lead to the explicit construction of the generators of the Lorentz group and their naming, Eq.(2.40), the dimension of the representation can be read off to be

$$\dim\left(\mathcal{D}^{(N)}\right) = (2j_1+1)\cdot(2j_2+1)\,, \tag{2.52}$$

and, accordingly, fields $\phi^{(j_1,j_2)}$ in this representation have to have the same dimension. Recalling the representation of Λ with help of the generators of the Lorentz group, Eq.(2.28), it becomes obvious that any finite Lorentz transformation

$$\hat{\Lambda} = \exp\left(-\frac{i}{2}\varepsilon^{\mu\nu}\hat{\mathcal{M}}_{\mu\nu}\right)$$

can be represented through the corresponding matrix representation of $\hat{\mathcal{M}}_{\mu\nu}$

$$\mathcal{D}^{(N)}(\Lambda) = \exp\left[-\frac{i}{2}\varepsilon^{\mu\nu}\mathcal{D}^{(N)}\left(\hat{\mathcal{M}}_{\mu\nu}\right)\right]\,. \tag{2.53}$$

Of course, similar statements hold true for infinitesimal Lorentz transformations.

$SL(2,C)$ spinors

As is well known, spins being representations of $SU(2)$ can be only half-integer or integer. Therefore, as already mentioned above, the simplest non-trivial representations of the Lorentz group, i.e. those with minimal dimension, are the two Weyl representations, either $(0,\,\frac{1}{2})$ or $(\frac{1}{2},\,0)$.
For the former case, $(0,\,\frac{1}{2})$, the field ξ_α has two components labelled by $\alpha = 1,\,2$. Being of the form $(0,\,\frac{1}{2})$ we can read off the representations of the generators $\hat{\mathcal{X}}^\pm$, namely

$$\mathcal{D}(\hat{\mathcal{X}}^+) = 0\,,\quad \mathcal{D}(\hat{\mathcal{X}}^-) = \frac{1}{2}\vec{\sigma}\,, \tag{2.54}$$

where the σ_i are the Pauli-matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,,\quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\,,\quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,. \tag{2.55}$$

Returning to these generators being written by the generators of rotations and boosts, this representation reads

$$\mathcal{D}(\hat{\mathcal{R}}) = \frac{1}{2}\,\vec{\sigma}\,,\quad \mathcal{D}(\hat{\mathcal{B}}) = \frac{i}{2}\,\vec{\sigma}\,. \tag{2.56}$$

Therefore, a finite Lorentz transformation parametrised by $\varepsilon_{\mu\nu}$, cf. Eq.(2.28), reads

$$\hat{\Lambda} = \exp\left(i\vec{\vartheta}\cdot\hat{\mathcal{R}} + i\vec{\omega}\cdot\hat{\mathcal{B}}\right) , \tag{2.57}$$

where

$$\vec{\vartheta} = -\left(\varepsilon_{23},\, \varepsilon_{31},\, \varepsilon_{12}\right)^T , \quad \vec{\omega} = -\left(\varepsilon_{01},\, \varepsilon_{02},\, \varepsilon_{03}\right)^T . \tag{2.58}$$

Hence, for two-component Spin-1/2 fields belonging to the representation $(0,\, \frac{1}{2})$ the representation $\mathcal{D}(\Lambda)$ has two indices for the two components α and β and reads

$$a_\alpha^{\ \beta} \equiv \mathcal{D}^{(0,\,1/2)}(\Lambda)_\alpha^{\ \beta} = \left[\exp\left(\frac{i}{2}\vec{\vartheta}\cdot\vec{\sigma} + \frac{i}{2}\vec{\omega}\cdot\vec{\sigma}\right)\right]_\alpha^{\ \beta} . \tag{2.59}$$

It acts on the spinors like

$$\xi'_\alpha(x') = \mathcal{D}^{(0,\,1/2)}(\Lambda)_\alpha^{\ \beta}\xi_\beta(x) . \tag{2.60}$$

Spinors of this representation are also known as left-handed two-component Weyl-spinors.
To construct the two-component spinors in the $(\frac{1}{2},\, 0)$ representation, the spinor of the $(0,\, \frac{1}{2})$ representation have to be complex conjugated. Such complex conjugated spinors are written with a dot as

$$(\xi_\alpha)^* := \eta_{\dot{\alpha}} \tag{2.61}$$

and transform according to

$$\eta'_{\dot{\alpha}}(x') = \left(\mathcal{D}^{(0,\,1/2)}(\Lambda)^*\right)_{\dot{\alpha}}^{\ \dot{\beta}}\ \eta_{\dot{\beta}}(x) . \tag{2.62}$$

These spinors are called right-handed two-component Weyl-spinors. It should be stressed that in the complex conjugation of the transformation matrix only its individual components are complex conjugated.

$$a^{*\ \dot{\beta}}_{\ \dot{\alpha}} = \left(\mathcal{D}^{(1/2,\,0)}(\Lambda)^*\right)_{\dot{\alpha}}^{\ \dot{\beta}} = \left[\exp\left(-\frac{i}{2}\vec{\vartheta}\cdot\vec{\sigma} + \frac{i}{2}\vec{\omega}\cdot\vec{\sigma}\right)\right]_{\dot{\alpha}}^{\ \dot{\beta}} . \tag{2.63}$$

Finally, the representation in terms of the $\hat{\mathcal{X}}^\pm$ operators is given by

$$\mathcal{D}(\hat{\mathcal{X}}^+) = -\frac{1}{2}\vec{\sigma}^* , \quad \mathcal{D}(\hat{\mathcal{X}}^-) = 0 , \tag{2.64}$$

Calculating with $SL(2, C)$ spinors

To calculate with the spinors introduced above, first of all ways have to be discussed for indices to be lowered or raised. For the two component spinors this happens through the $SL(2, C)$ invariant antisymmetric tensor

$$\epsilon^{\alpha\beta} = \epsilon_{\alpha\beta} = \epsilon^{\dot\alpha\dot\beta} = \epsilon_{\dot\alpha\dot\beta} = i\sigma_2\,, \tag{2.65}$$

the analogon to $g_{\mu\nu}$ with

$$\epsilon_{\alpha\beta}\epsilon^{\alpha\beta} = \epsilon_{\dot\alpha\dot\beta}\epsilon^{\dot\alpha\dot\beta} = -2 \tag{2.66}$$

For the Pauli matrices we have

$$\sigma_2\sigma_i\sigma_2 = -\sigma_i^* \tag{2.67}$$

Due to its antisymmetry the following relations hold true

$$\xi_\alpha = -\epsilon_{\alpha\beta}\xi^\beta = \xi^\beta\epsilon_{\beta\alpha}\,, \quad \eta_{\dot\alpha} = -\epsilon_{\dot\alpha\dot\beta}\eta^{\dot\beta} = \eta^{\dot\beta}\epsilon_{\dot\beta\dot\alpha}\,. \tag{2.68}$$

Scalars can be constructed through

$$\xi^\alpha\Xi_\alpha = \epsilon^{\alpha\beta}\xi_\beta\Xi_\alpha = -\xi_\beta\Xi^\beta\,, \quad \eta^{\dot\alpha}H_{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\eta_{\dot\beta}H_{\dot\alpha} = -\eta_{\dot\beta}H^{\dot\beta}\,. \tag{2.69}$$

That these terms are in fact scalars can be tested by using the following property of ϵ when applied on general 2×2 matrices M:

$$\epsilon M^T\epsilon^T = \det M \cdot M^{-1} \tag{2.70}$$

Consider, for instance, the first product,

$$\epsilon^{\alpha\beta}\xi_\beta\Xi_\alpha = \xi^T\epsilon^T\Xi \tag{2.71}$$

and apply a Lorentz transformation, here again abbreviated as a,

$$\begin{aligned} (\xi^T\epsilon^T\Xi)' &= \xi^Ta^T\epsilon^Ta\Xi = \xi^T\epsilon^T\epsilon a^T\epsilon^Ta\Xi = \xi^T\epsilon^T\left[\det(a)a^{-1}\right]a\Xi \\ &= \det(a)\,\xi^T\epsilon^T\Xi = \xi^T\epsilon^T\Xi\,. \end{aligned} \tag{2.72}$$

Four-vectors and vector fields

Four-vectors and corresponding fields are in the $(\frac{1}{2}, \frac{1}{2})$ representation which can be constructed from the Weyl representations through

$$\left(0, \frac{1}{2}\right) \times \left(\frac{1}{2}, 0\right) = \left(\frac{1}{2}, \frac{1}{2}\right) . \tag{2.73}$$

To reconstruct the connection of the indices α and $\dot\beta$ of the two Weyl factors with the index μ of the four-vector the following vectors of Pauli matrices are employed

$$\left(\sigma_\mu\right)_{\alpha\dot\beta} = (\mathbf{1}, \vec\sigma) , \quad \left(\bar\sigma_\mu\right)^{\dot\alpha\beta} = \left(\sigma_\mu\right)_{\alpha\dot\beta}\epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta} = (\mathbf{1}, -\vec\sigma) . \tag{2.74}$$

Again, to lower and raise indices $\epsilon_{\alpha\beta}$ from Eq.(2.65) is employed. Simple consideration show that

$$\left(\bar\sigma_\mu\right)_{\dot\alpha\beta} = \epsilon_{\dot\alpha\dot\gamma}\epsilon_{\beta\delta}\left(\bar\sigma_\mu\right)^{\dot\gamma\delta} \tag{2.75}$$

and that

$$\left(\bar\sigma_\mu\right)_{\dot\alpha\beta} = \left[\left(\sigma_\mu\right)_{\alpha\dot\beta}\right]^* , \tag{2.76}$$

the complex conjugated of σ_μ. Furthermore, the following relations hold

$$\mathrm{Tr}\left[\bar\sigma^\mu\sigma_\nu\right] = 2\delta^\mu_\nu , \quad \left(\sigma_\mu\right)_{\alpha\dot\beta}\left(\bar\sigma^\mu\right)^{\dot\gamma\delta} = 2\delta^\delta_\alpha\delta^{\dot\gamma}_{\dot\beta} . \tag{2.77}$$

Also,

$$\begin{aligned}\left(\sigma_\mu\bar\sigma_\nu + \sigma_\nu\bar\sigma_\mu\right)_\alpha^{\ \beta} &= 2g_{\mu\nu}\delta_\alpha^{\ \beta}\\ \left(\bar\sigma_\mu\sigma_\nu + \bar\sigma_\nu\sigma_\mu\right)^{\dot\alpha}_{\ \dot\beta} &= 2g_{\mu\nu}\delta^{\dot\alpha}_{\ \dot\beta} .\end{aligned} \tag{2.78}$$

With help of Eq.(2.59) the behaviour of the σ_μ under Lorentz transformations can be determined, namely

$$\sigma_\mu\Lambda^\mu_{\ \nu} = \left[\mathcal{D}^{(0,1/2)}(\Lambda)\right]\cdot\sigma_\nu\cdot\left[\mathcal{D}^{(0,1/2)}(\Lambda)\right]^\dagger = a\cdot\sigma_\nu\cdot a^\dagger . \tag{2.79}$$

Therefore, for

$$\mathbf{x} := x_\mu\sigma^\mu = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3\end{pmatrix} \tag{2.80}$$

$$\mathbf{x}' = \left[\mathcal{D}^{(0,1/2)}(\Lambda)\right]\cdot\mathbf{x}\cdot\left[\mathcal{D}^{(0,1/2)}(\Lambda)\right]^\dagger . \tag{2.81}$$

For this representation of four-vectors, we have also the following relations

$$x^\dagger = x \quad \text{and} \quad \det x = x^2 = x^\mu x^\nu g_{\mu\nu}\,. \tag{2.82}$$

Having this in mind covariant four vectors V_μ can be constructed out of two spinors ξ_α and $\eta_{\dot\beta}$,

$$V_\mu = \xi^\alpha (\sigma_\mu)_{\alpha\dot\beta}\eta^{\dot\beta} = \xi_\alpha(\epsilon^T\sigma_\mu\epsilon)^{\alpha\dot\beta}\eta_{\dot\beta} = \xi_\alpha(\bar\sigma_\mu^T)^{\alpha\dot\beta}\eta_{\dot\beta}\,. \tag{2.83}$$

Employing the form of Lorentz transformations of the spinors in Eqs.(2.60) and (2.62) and using the identity of Eq.(2.70) it is easy to check that the V_μ constructed in Eq.(2.83) in fact has the correct transformation behaviour, in other words that it is covariant. To see this consider

$$V'_\mu = \xi^T a^T \epsilon^T \sigma_\mu \epsilon a^* \eta = \xi^T \epsilon^T a^{-1}\sigma_\mu {a^\dagger}^{-1}\epsilon\,\eta\,, \tag{2.84}$$

where

$$\epsilon a^T \epsilon^T = a^{-1}\,, \quad \text{and} \quad \epsilon a^* \epsilon^T = {a^\dagger}^{-1}$$

has been used in accordance with Eq.(2.70). Recalling Eq.(2.79) in the form

$$a^{-1}\sigma_\mu {a^\dagger}^{-1} = \Lambda_\mu^{\ \nu}\sigma_\nu \tag{2.85}$$

finally

$$V'_\mu = \xi^T\epsilon^T\Lambda_\mu^{\ \nu}\sigma_\nu\epsilon\,\eta = \xi_\alpha\left(\epsilon^T\Lambda_\mu^{\ \nu}\sigma_\nu\epsilon\right)^{\alpha\dot\beta}\eta_{\dot\beta}\,, \tag{2.86}$$

i.e. V_μ in fact is a covariant entity with the correct transformation behaviour. This allows to translate Lorentz indices into spinor indices and vice versa. In doing so, a general rule emerges: Each Lorentz index is transformed into two spinor indices through a product with appropriate combinations of σ^μ. According to Eq.(2.80), the four vector V^μ and its mixed spinor representation are connected through

$$V_{\alpha\dot\beta} = V^\mu\,(\sigma_\mu)_{\alpha\dot\beta}\,, \quad V^\mu = \frac{1}{2}\,(\bar\sigma^\mu)^{\dot\beta\alpha}\,V_{\alpha\dot\beta}\,. \tag{2.87}$$

The explicit form of the V^μ, and especially the factor 1/2, can be understood by considering

$$V^\mu = \frac{1}{2}\,(\bar\sigma^\mu)^{\dot\beta\alpha}\,V_{\alpha\dot\beta} = \frac{1}{2}\,(\bar\sigma^\mu)^{\dot\beta\alpha}\,V^\nu\,(\sigma_\nu)_{\alpha\dot\beta} = \frac{1}{2}\mathrm{Tr}\,(\bar\sigma^\mu\sigma_\nu)\,V^\nu = \delta^\mu_\nu V^\nu = V^\mu\,. \tag{2.88}$$

Antisymmetric tensors of rank two

As shown in the previous section, any Lorentz index can be translated into two spinor indices. Therefore, for any tensor of rank two,

$$X^{\mu\nu} \longrightarrow X_{\alpha\beta\dot{\alpha}\dot{\beta}} = X^{\mu\nu}\left(\sigma_\mu\right)_{\alpha\dot{\alpha}}\left(\sigma_\nu\right)_{\beta\dot{\beta}} . \tag{2.89}$$

Denoting symmetrisation and anti-symmetrisation in spinor indices by () and by [], respectively, the tensor $X_{\alpha\beta\dot{\alpha}\dot{\beta}}$ can be decomposed as

$$\begin{aligned} X_{\alpha\beta\dot{\alpha}\dot{\beta}} &= X_{(\alpha\beta)(\dot{\alpha}\dot{\beta})} + X_{[\alpha\beta][\dot{\alpha}\dot{\beta}]} + X_{(\alpha\beta)[\dot{\alpha}\dot{\beta}]} + X_{[\alpha\beta](\dot{\alpha}\dot{\beta})} \\ &= X_{(\alpha\beta)(\dot{\alpha}\dot{\beta})} + \epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}X + \epsilon_{\dot{\alpha}\dot{\beta}}X_{(\alpha\beta)} + \epsilon_{\alpha\beta}X_{(\dot{\alpha}\dot{\beta})} . \end{aligned} \tag{2.90}$$

The irreducible components of $X_{\alpha\beta\dot{\alpha}\dot{\beta}}$ are given by

$$\begin{aligned} X &= \frac{1}{4}\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}X_{\alpha\beta\dot{\alpha}\dot{\beta}} \\ X_{(\alpha\beta)} &= -\frac{1}{2}\epsilon^{\dot{\alpha}\dot{\beta}}X_{(\alpha\beta)\dot{\alpha}\dot{\beta}} \\ X_{(\dot{\alpha}\dot{\beta})} &= -\frac{1}{2}\epsilon^{\alpha\beta}X_{\alpha\beta(\dot{\alpha}\dot{\beta})} . \end{aligned} \tag{2.91}$$

The pre-factors of $1/4$ and $-1/2$ can be deduced by applying Eq.(2.66). If $X^{\mu\nu}$ is a symmetric tensor, i.e. if

$$X^{\mu\nu} = X^{\nu\mu} \tag{2.92}$$

then, obviously, the last two terms of Eq.(2.90) vanish. If $X^{\mu\nu}$ is also traceless, i.e. if $\mathrm{Tr}(X) = X^\mu_\mu = 0$, then also the second term of this decomposition is zero. In contrast, for any antisymmetric tensor

$$X^{\mu\nu} = -X^{\nu\mu} \tag{2.93}$$

only the last two terms yield a non-vanishing result. Phrased in other words, any antisymmetric tensor is equivalent to pairs of symmetric spinors. To construct a mapping similar to the one encountered for four vectors the following matrices will be introduced:

$$\begin{aligned} \left(\sigma^{\mu\nu}\right)_\alpha^{\ \beta} &= \frac{i}{2}\left(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu\right)_\alpha^{\ \beta} \\ \left(\bar{\sigma}^{\mu\nu}\right)^{\dot{\alpha}}_{\ \dot{\beta}} &= \frac{i}{2}\left(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu\right)^{\dot{\alpha}}_{\ \dot{\beta}} , \end{aligned} \tag{2.94}$$

where the form Eq.(2.74) for the σ and $\bar{\sigma}$ has been used. Again, to raise and lower spinor indices, ϵ is used, yielding

$$\begin{aligned}
\left(\sigma^{\mu\nu}\right)_{\alpha\beta} &= \epsilon_{\beta\gamma}\left(\sigma^{\mu\nu}\right)_{\alpha}^{\ \gamma} = \left(\sigma^{\mu\nu}\right)_{\beta\alpha} \\
\left(\bar{\sigma}^{\mu\nu}\right)_{\dot{\alpha}\dot{\beta}} &= \epsilon_{\dot{\alpha}\dot{\gamma}}\left(\bar{\sigma}^{\mu\nu}\right)^{\dot{\gamma}}_{\ \dot{\beta}} = \left(\bar{\sigma}^{\mu\nu}\right)_{\dot{\beta}\dot{\alpha}} \,.
\end{aligned} \tag{2.95}$$

Furthermore,

$$\sigma^{\mu\nu} = +\frac{i}{2}\epsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma}\,, \quad \bar{\sigma}^{\mu\nu} = -\frac{i}{2}\epsilon^{\mu\nu\rho\sigma}\bar{\sigma}_{\rho\sigma}\,, \tag{2.96}$$

$$\left(\sigma^{\mu\nu}\epsilon\right)_{\alpha\beta} = \left[\left(\epsilon^T\bar{\sigma}^{\mu\nu}\epsilon\right)_{\dot{\alpha}\dot{\beta}}\right]^* \tag{2.97}$$

and

$$\begin{aligned}
\frac{1}{2}\mathrm{Tr}\left[\sigma^{\mu\nu}\sigma_{\rho\sigma}\right] &= \delta^{\mu\nu}_{\rho\sigma} + i\epsilon^{\mu\nu}_{\rho\sigma} \\
\frac{1}{2}\mathrm{Tr}\left[\bar{\sigma}^{\mu\nu}\bar{\sigma}_{\rho\sigma}\right] &= \delta^{\mu\nu}_{\rho\sigma} - i\epsilon^{\mu\nu}_{\rho\sigma}
\end{aligned} \tag{2.98}$$

hold true. The tensors of the last identities are given by

$$\delta^{\mu\nu}_{\rho\sigma} = \delta^{\mu}_{\rho}\delta^{\nu}_{\sigma} - \delta^{\mu}_{\sigma}\delta^{\nu}_{\rho}\,, \quad \text{and} \quad \epsilon^{\mu\nu}_{\rho\sigma} = g_{\rho\kappa}g_{\sigma\lambda}\epsilon^{\mu\nu\kappa\lambda}\,. \tag{2.99}$$

Then, for antisymmetric tensors, the spinor decomposition of Eq.(2.90) reads

$$X_{\alpha\beta\dot{\alpha}\dot{\beta}} = X^{\mu\nu}\left(\sigma_\mu\right)_{\alpha\dot{\alpha}}\left(\sigma_\nu\right)_{\beta\dot{\beta}} = \epsilon_{\alpha\beta}X_{(\dot{\alpha}\dot{\beta})} + \epsilon_{\dot{\alpha}\dot{\beta}}X_{(\alpha\beta)}\,. \tag{2.100}$$

Multiplying the second part of this equation with $\epsilon^{\alpha\beta}$, and using Eq.(2.66) and the fact that, due to symmetry properties in the spinor indices, $\epsilon^{\alpha\beta}X_{(\alpha\beta)} = 0$, yields

$$-2X_{(\dot{\alpha}\dot{\beta})} = X^{\mu\nu}\left(\sigma_\mu\right)_{\alpha\dot{\alpha}}\left(\sigma_\nu\right)_{\beta\dot{\beta}}\epsilon^{\alpha\beta}\,. \tag{2.101}$$

Multiplying both sides with $\epsilon^{(\dot{\gamma}\dot{\alpha})}$, and transforming σ_μ into $\bar{\sigma}_\mu$ results in

$$\begin{aligned}
-2X_{(\dot{\alpha}\dot{\beta})}\epsilon^{\dot{\gamma}\dot{\alpha}} &= X^{\mu\nu}\left(\bar{\sigma}_\mu\right)^{\dot{\gamma}\beta}\left(\sigma_\nu\right)_{\beta\dot{\beta}} \\
&= \frac{1}{2}X^{\mu\nu}\left[\left(\bar{\sigma}_\mu\sigma_\nu - \bar{\sigma}_\nu\sigma_\mu\right) + \left(\bar{\sigma}_\mu\sigma_\nu + \bar{\sigma}_\nu\sigma_\mu\right)\right]^{\dot{\gamma}}_{\ \dot{\beta}} \\
&= -iX^{\mu\nu}\left(\bar{\sigma}_{\mu\nu}\right)^{\dot{\gamma}}_{\ \dot{\beta}}\,,
\end{aligned} \tag{2.102}$$

where the second term in the square brackets vanishes after applying Eq.(2.78) due to its symmetry in the Lorentz indices. For the other component similar considerations apply and finally the two spinor tensors $X_{(\dot\alpha\dot\beta)}$ and $X_{(\alpha\beta)}$ are given by

$$X_{(\alpha\beta)} \;=\; \tfrac{i}{2}\left(\sigma_{\mu\nu}\epsilon\right)_{\alpha\beta}X^{\mu\nu} \quad X_{(\dot\alpha\dot\beta)} \;=\; -\tfrac{i}{2}\left(\epsilon^T\bar\sigma_{\mu\nu}\right)_{\dot\alpha\dot\beta}X^{\mu\nu}$$

$$X^{\mu\nu} = \frac{i}{4}\left(\epsilon\sigma^{\mu\nu}\right)^{\alpha\beta}X_{(\alpha\beta)} - \frac{i}{4}\left(\bar\sigma^{\mu\nu}\epsilon^T\right)^{\dot\alpha\dot\beta}X_{(\dot\alpha\dot\beta)}\,. \tag{2.103}$$

Introducing the notion of (anti-) self-dual tensors

$$F^{\pm\mu\nu} = \pm i\epsilon^{\mu\nu\rho\sigma}F^{\pm}_{\rho\sigma}\,, \tag{2.104}$$

where the minus-sign corresponds to anti-self-dual tensors, it is easy to check that self-dual tensors correspond to the un-dotted spinor representation whereas the dotted spinor representation is for the anti-self-dual tensors. It is worth noting that real anti-symmetric tensors of rank two, like for instance the electromagnetic field strength tensor, can always be decomposed into a self-dual and an anti-self-dual component. Defining

$$\tilde F^{\mu\nu} = i\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} \tag{2.105}$$

this decomposition reads

$$F^{\pm}_{\mu\nu} = \frac{1}{2}\left(F_{\mu\nu} \pm \tilde F_{\mu\nu}\right)\,. \tag{2.106}$$

It follows that the corresponding spinor representations are mutually complex conjugated,

$$\bar F_{\dot\alpha\dot\beta} = \left(F_{\alpha\beta}\right)^* \tag{2.107}$$

and have also six parameters. Going back to the $\sigma_{\mu\nu}$, it can be checked explicitly that

$$\left(\sigma_{23},\,\sigma_{31},\,\sigma_{12}\right)^T = \vec\sigma \quad\text{and}\quad \left(\sigma_{10},\,\sigma_{20},\,\sigma_{30}\right)^T = \vec\sigma\,. \tag{2.108}$$

Therefore, identifying entries in the (0, 1/2) and (1/2, 0) representations of the Lorentz algebra for dotted and un-dotted spinors, Eqs.(2.59, 2.63), respectively, Eq.(2.53) reads for spinors

$$\begin{aligned} a_\alpha^{\ \beta} &= \mathcal{D}^{(0,\,1/2)}(\Lambda)_\alpha^{\ \beta} = \left[\exp\left(\frac{i}{4}\varepsilon^{\mu\nu}\sigma_{\mu\nu}\right)\right]_\alpha^{\ \beta}\,, \\ a^{*\ \dot\beta}_{\ \dot\alpha} &= \left(\mathcal{D}^{(1/2,\,0)}(\Lambda)^*\right)_{\dot\alpha}^{\ \dot\beta} = \left[\exp\left(-\frac{i}{4}\varepsilon^{\mu\nu}\bar\sigma_{\mu\nu}\right)\right]_{\dot\alpha}^{\ \dot\beta}\,. \end{aligned} \tag{2.109}$$

2.2 Noether's theorem

2.2.1 Stating the theorem

So far, Lorentz transformation and their properties have been discussed. Lorentz invariance as well as other symmetries are guiding principles for the construction of theories and their Lagrange densities (also dubbed Lagrangians). In general, symmetries are connected to invariance of the action integral. Noether's theorem states that each symmetry in the action is connected to a conserved quantity. In the following action integrals of fields are considered. The corresponding Lagrangians can be understood as functions of these fields ϕ_i and their first derivative,

$$S[\phi] = \int d^4x \mathcal{L}(\phi_i(x), \partial_\mu \phi_i(x)) \,, \tag{2.110}$$

where $d^4x = dx^0 dx^1 dx^2 dx^3$ and $\partial_\mu = \partial/\partial^\mu$. In classical theory the action S has definite units of angular momentum, thus, in quantum theory it is in corresponding units of $\hbar$, in the notation of this lecture, $\hbar \equiv 1$. In other words, in the notation employed here, the action is "dimensionless", and, therefore, the Lagrangian has dimension of inverse length to the fourth.
As usual the equations of motion can be obtained through the extremal principle as Euler-Lagrange equations,

$$\begin{aligned}
\delta S &= \int d^4x \delta\mathcal{L} = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi_i(x)} \delta\phi_i(x) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i(x))} \delta(\partial_\mu \phi_i(x))\right] \\
&= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi_i(x)} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i(x))}\right)\right] \delta\phi_i(x) + \\
&\quad \int d^4x \left[\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i(x))}\right) \delta\phi_i(x)\right] \,,
\end{aligned} \tag{2.111}$$

where the transition from the first to the second line holds true only, if x does not change as well in the variation. Then

$$\delta\left(\partial_\mu \phi_i(x)\right) = \partial_\mu \left(\delta\phi_i(x)\right) \,. \tag{2.112}$$

The last term of the result of Eq.(2.111) is a surface term that vanishes when demanding that the variation of the fields vanishes on the surface of the four-dimensional integration region. By requiring S to be stationary under such variations of the field(s) $\phi_i(x)$ the Euler-Lagrange equations for $\phi_i(x)$ can be obtained,

$$\frac{\partial \mathcal{L}}{\partial \phi_i(x)} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i(x))} \right) = 0\,, \tag{2.113}$$

the classical field equations. They can be identified as the functional derivative of S with respect to $\phi_i(x)$. Note that dropping the surface terms results in an invariance of these equations under the transformation

$$\mathcal{L}' = \mathcal{L} + \partial_\mu L^\mu\,, \tag{2.114}$$

i.e. under adding a total derivative of a vector.
Consider now invariance of the action under infinitesimal transformations

$$\phi_i(x) \longrightarrow \phi_i'(x) = \phi_i(x) + \varepsilon G_i\left[\phi(x)\right]\,, \tag{2.115}$$

where $\phi(x)$ denotes all fields. In this case, the variation of the Lagrangian is

$$\delta\mathcal{L} = \mathcal{L}(\phi_i'(x), \partial_\mu \phi_i'(x)) - \mathcal{L}(\phi_i(x), \partial_\mu \phi_i(x)) = \varepsilon \partial_\mu X^\mu(\phi_i(x))\,. \tag{2.116}$$

On the other hand, using Eq.(2.115), the Lagrangian changes like

$$\begin{aligned} \delta\mathcal{L} &= \varepsilon \frac{\partial \mathcal{L}}{\partial \phi_i} G_i(\phi) + \varepsilon \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \partial_\mu G_i(\phi) \\ &= \varepsilon \left[G_i(\phi)\, \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\, \partial_\mu G_i(\phi) \right]\,, \end{aligned} \tag{2.117}$$

where the Euler-Lagrange equation, Eq.(2.113), and the independence of the variation parameter from space-time have been used. Equating both equations,

$$\partial_\mu X^\mu(\phi_i(x)) = \left[G_i(\phi)\, \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\, \partial_\mu G_i(\phi) \right] \tag{2.118}$$

the conserved current

$$j^\mu(x) = G_i(\phi) \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right) + X^\mu(\phi)\,, \tag{2.119}$$

fulfilling

$$\partial_\mu j^\mu(\phi) = 0 \tag{2.120}$$

can be read-off. This entity is called the Noether-current. Integrating over the spatial components, it leads to the conserved charge corresponding to it,

$$Q = \int d^3\vec{x} j^{\mu=0}(x) \; , \quad \frac{dQ}{dt} = 0 \, . \tag{2.121}$$

The important thing about this charge generates the infinitesimal transformations above, Eq.(2.115),

$$[iQ, \, \phi_i(x)] = G_i[\phi(x)] \, . \tag{2.122}$$

2.2.2 First consequences of Noether's theorem

Energy-momentum tensor

Consider linear translations under which the fields transform as

$$\phi'_i(x') = \phi_i(x) \, . \tag{2.123}$$

Demanding that the fields stay unchanged at the same points,

$$\delta_L \phi_i(x) = \phi'_i(x) - \phi_i(x) = \varepsilon^\rho \partial_\rho \phi_i(x) \tag{2.124}$$

for infinitesimal translations. If the Lagrangian does not depend explicitly on space-time,

$$\partial_L \mathcal{L} = \mathcal{L}(\phi'_i(x), \partial_\mu \phi'_i(x)) - \mathcal{L}(\phi_i(x), \partial_\mu \phi_i(x)) = \varepsilon^\rho \partial_\rho \mathcal{L} \, . \tag{2.125}$$

Hence,

$$G_{\rho,\, i}(\phi(x)) = \partial_\rho \phi_i(x) \, , \quad X^\mu_\rho = \delta^\mu_\rho \mathcal{L}(\phi, \partial_\mu \phi) \tag{2.126}$$

and the Noether-current reads

$$j_\rho^{\ \mu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \partial_\rho \phi_i - \delta^\mu_\rho \mathcal{L}(\phi, \partial_\mu \phi) = T_\rho^{\ \mu} \, . \tag{2.127}$$

This is the energy-momentum tensor.

Angular momentum tensor

Turning to infinitesimal Lorentz transformations applied on the fields, and reconciling Eqs.(2.29,2.30),

$$\phi_i'(x') = \left[1 - \frac{i}{2}\varepsilon^{\rho\sigma}\hat{S}_{\rho\sigma}\right]_i^{\ j} \phi_j(x)\,. \tag{2.128}$$

But, after Taylor expansion, also

$$\phi_i'(x') = \left[1 + \frac{1}{2}\varepsilon^{\rho\sigma}\left(x_\sigma\partial_\rho - x_\rho\partial_\sigma\right)\right]\phi_i'(x) = \left[1 + \frac{i}{2}\varepsilon^{\rho\sigma}\hat{L}_{\sigma\rho}\right]\phi_i'(x)\,. \tag{2.129}$$

Therefore, up to first order in the small parameter ε,

$$\delta_L\phi_i(x) = \phi_i'(x) - \phi_i(x) = -\frac{i}{2}\varepsilon^{\rho\sigma}\left[\left(\hat{L}_{\rho\sigma}\right)\delta_i^{\ j} + \left(\hat{S}_{\rho\sigma}\right)_i^{\ j}\right]\phi_j(x) \tag{2.130}$$

But since in a lorentz invariant theory Lagrangians are scalars, its variation reads

$$\delta_L\mathcal{L} = \frac{i}{2}\varepsilon^{\rho\sigma}\hat{L}_{\sigma\rho}\mathcal{L}\,. \tag{2.131}$$

Therefore, the entities constituting the Noether current are

$$\begin{aligned} G_{\rho\sigma,\,i}(\phi) &= -i\left[\left(\hat{L}_{\rho\sigma}\right)\delta_i^{\ j}\left(\hat{S}_{\rho\sigma}\right)_i^{\ j}\right]\phi_j \\ X^\mu_{\rho\sigma}(\phi) &= \left(x_\rho\delta^\mu_\sigma - x_\sigma\delta^\mu_\rho\right)\mathcal{L} \end{aligned} \tag{2.132}$$

and the current reads

$$j_{\rho\sigma}{}^{\mu}(\phi) = x_\rho T_\sigma{}^\mu - x_\sigma T_\rho{}^\mu - i\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\left(\hat{S}_{\rho\sigma}\right)_i^{\ j}\phi_j \equiv \mathcal{M}_{\rho\sigma}{}^{\mu}\,, \tag{2.133}$$

the angular momentum density.
Taken together, the conserved currents that correspond to the two conserved currents of Eqs.(2.127) and (2.133) are the generators of the Poincare group, P_ρ and $M_{\rho\sigma}$,

$$P_\rho = \int d^3x T_\rho{}^0 \quad \text{and} \quad M_{\rho\sigma} = \int d^3x M_{\rho\sigma}{}^0\,. \tag{2.134}$$

Non-uniqueness of the Noether current

The demand to have a conserved current according to

$$\partial_\mu j^\mu = 0 \tag{2.135}$$

does not fix the current completely. In fact, if $f^{\mu\nu}$ is a totally antisymmetric tensor of rank two, then, also

$$\tilde{j}^\mu = j^\mu + \partial_\nu f^{\mu\nu} \tag{2.136}$$

is a conserved current. This is important when considering massless theories leading notoriously to massless one-particle states resulting in non-convergence of the spatial integral leading to the conserved charge. Then, by choosing and appropriate tensor, one can make these contributions vanish.

2.3 Actions for field theories

2.3.1 General considerations

When discussing Noether's theorem the general form of the action has been given as, c.f. Eq.(2.110),

$$S[\phi] = \int d^4x \mathcal{L}(\phi_i(x), \partial_\mu \phi_i(x)) \,. \tag{2.137}$$

This integral has the following properties:

1. Locality:
   This is obvious when realizing that only one space-time coordinate appears and is integrated over. This immediately leads to local interactions.

2. $\mathcal{L}$ is real (in field theory: Hermite-an):
   In classical mechanics this means that the energy is a real quantity, in field theory, this boils down to having a stable and "meaningful" time evolution of probabilities.

3. $\mathcal{L}$ depends only on fields and their derivatives (with up to second derivative w.r.t. time):

In field theory, higher derivatives would lead either to states with negative norm (i.e. negative probability, not an appealing concept), or to particles moving with superluminal velocities (i.e. tachyons, usually considered to be a sign of an unhealthy theory).

4. Poincare invariance:
   In other words, $\mathcal{L}$ must not depend explicitly on x.

5. Renormalizability:
   In theories living in four dimensional this means that the Lagrangian must have dimension four or less in every single term.

2.3.2 Scalar theories

Neutral scalar fields

Starting with a single real field, the most general Lagrangian is of the form

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - V(\phi)\,, \tag{2.138}$$

where the first time is the kinetic term and the second term is the potential term, giving rise to interactions etc.. The plus sign in front of the kinetic term reflects the demand to have positive energies. Expanding the potential in powers of ϕ is is obvious that any term linear in ϕ can be absorbed in a redefinition $\phi(x) \to \phi(x) + a$. Thus,

$$V(\phi) = \frac{1}{2}\mu^2\phi^2 + \frac{g}{3!}\phi^3 + \frac{\lambda}{4!}\phi^4 \dots\,. \tag{2.139}$$

Considering physical units in field theory, space and time both have dimensions -1, mass and energy have dimension $+1$ (in units of energy). Since the action, in the units used, has dimension equal to zero, the Lagrangian has the inverse dimension of $d^n x$ in an n-dimensional space, i.e. $+n$. By inspection of the kinetic term, it can be seen that the real scalar field has dimension

$$\dim[\phi(x)] = \frac{n-2}{2} \stackrel{n\to 4}{\longrightarrow} 1\,. \tag{2.140}$$

Renormalizability disallows mass dimensions larger than n for the individual terms in the Lagrangian, therefore, in four dimensions, the expansion of the potential stops at the quartic term. Demanding symmetry under reflections of ϕ, $\phi \to -\phi$, such a potential in four dimensions is of the form

$$V(\phi) = \frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4!}\phi^4 \,. \tag{2.141}$$

This potential, together with the kinetic term for the ϕ defines a very simple field theory, called the $\lambda\phi^4$-theory. This theory plays a significant role in the mechanism of spontaneous symmetry breaking. Its Euler-Lagrange equation reads

$$\left(\partial_\mu\partial^\mu + \mu^2\right)\phi = -\frac{\lambda}{3!}\phi^3 \,. \tag{2.142}$$

In field theory, the term $\sim \mu^2$ in the potential of Eq.(2.141) is called the mass term, the other term constitutes the interaction term.

Charged scalar fields

The $\lambda\phi^4$ theory can also be formulated as a theory for a complex scalar field. Then its Lagrangian is given by

$$\mathcal{L} = (\partial_\mu\phi)(\partial^\mu\phi^*) - \mu^2\phi\phi^* - \frac{\lambda}{2}(\phi\phi^*)^2 \,. \tag{2.143}$$

Apart from some simple factors this theory has another difference with respect to the neutral (real) field theory: It also enjoys a $U(1)$ symmetry. In other words, under transformations

$$\begin{array}{rcl} \phi(x) & \longrightarrow & \phi'(x) \;\;= e^{i\theta}\,\phi(x) \\ \phi^*(x) & \longrightarrow & \phi^{*\prime}(x) = e^{-i\theta}\,\phi^*(x) \end{array} \tag{2.144}$$

the Lagrangian remains invariant. This symmetry, basically invariance under re-phasing of the fields, has a continuous group structure that can be easily identified as the simple $U(1)$ with θ being the parameter of the phase transformation. The infinitesimal version of this is

$$\delta\phi = \phi'(x) - \phi(x) = i\theta\phi(x) \quad \text{and} \quad \delta\phi^* = \phi^{*\prime}(x) - \phi^*(x) = -i\theta\phi^*(x) \tag{2.145}$$

leading to the Noether current

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}i\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}(-i\phi^*) = -i\left(\phi^*\partial_\mu\phi - \partial_\mu\phi^*\cdot\phi\right) = -i\phi^*\overset{\leftrightarrow}{\partial}\phi \,. \tag{2.146}$$

The corresponding charge is

$$Q = \int d^3x j^0(x) = i \int d^3x \left(\dot{\phi}^* \phi - \phi^* \dot{\phi}\right) . \tag{2.147}$$

As mentioned already before, the commutators of this charge with the fields generates the infinitesimal transformation. Thus,

$$[iQ,\, \phi(x)] = i\phi(x) \qquad \text{and} \quad [iQ,\, \phi^*(x)] = i\phi^*(x)\,. \tag{2.148}$$

The commutators above immediately show that the fields ϕ and ϕ^* carry conserved charges ± 1, respectively. This is in striking contrast to real fields (it is impossible to apply a phase transition of this kind to a single real field) and, therefore, real fields are neutral.

$O(N)$-scalar fields

Consider now an N-dimensional scalar field $\Phi = (\phi_1,\, \phi_2,\, \dots,\, \phi_n)^T$. Its Lagrangian in a $\lambda\phi^4$ type of theory reads

$$\mathcal{L} = \frac{1}{2}\sum_{i=1}^{n}\left[(\partial_\mu\phi_i)(\partial^\mu\phi_i) - \mu^2\phi_i^2\right] - \left[\frac{\lambda}{8}\sum_{i=1}^{n}\phi_i^2\right]^2 . \tag{2.149}$$

The pictorial interpretation of this model is of the fields ϕ_i being an N-dimensional vector Φ, and the theory then enjoys an $O(N)$-symmetry under rotations of Φ. The case $N = 2$ can be simply connected to the charged fields discussed before. Just rewrite

$$\phi = \frac{\phi_1 + i\phi_2}{\sqrt{2}} \quad \text{and} \quad \phi^* = \frac{\phi_1 - i\phi_2}{\sqrt{2}}\,. \tag{2.150}$$

In other words, any scalar charged (complex) field can be represented by two scalar neutral (real) fields that enjoy an $O(2)$ symmetry. Re-phasing of such a model amounts to rotations in the ϕ_1-ϕ_2 plane in which the charged field ϕ and its complex conjugate ϕ^* are constructed.

2.3.3 Theories with spinors

$SL(2,\, C)$ spinors (two-component spinors)

Having discussed scalar fields a Lagrange density will be constructed for free two-component spinors which represent the simplest possible non-trivial fields with respect to the Lorentz-transformation. Such a Lagrangian consists of two parts, a kinetic and a mass term. To build the kinetic term, the connection between four-vectors and spinor indices has to be employed. For the derivative it reads, cf. Eq.(2.87)

$$\partial_\mu \longrightarrow \partial_{\alpha\dot{\beta}} = (\sigma_\mu)_{\alpha\dot{\beta}}\, \partial_\mu \,. \tag{2.151}$$

Therefore, the simplest Lorentz scalar containing at least one derivative and two spinors reads

$$\eta^{*\alpha} \left(\sigma_\mu\right)_{\alpha\dot{\beta}} \partial_\mu \eta^{\dot{\beta}} = \eta^\dagger \sigma^\mu \partial_\mu \eta \,, \tag{2.152}$$

where the definition of the dotted spinor by means of the complex conjugate one has been plugged in,

$$\left(\eta^{\dot{\alpha}}\right)^* = (\eta^*)^{\dot{\alpha}} \quad \text{and} \quad (\xi_\alpha)^* = (\xi^*)_\alpha \,. \tag{2.153}$$

To guaranteed that the first guess, Eq.(2.152), becomes a real quantity, its Hermite-an conjugate

$$\left(\partial_\mu \eta^\dagger\right) \sigma^\mu \eta \tag{2.154}$$

has to come into the game. But the sum of both terms yields a total derivative which, of course, vanishes such that the action is zero. But subtracting and multiplying with i provides a meaningful result,

$$\mathcal{L}_L^{\text{kin.}} = \frac{i}{2}\left[\eta^\dagger \sigma^\mu \partial_\mu \eta - \left(\partial_\mu \eta^\dagger\right) \sigma^\mu \eta\right] = \frac{i}{2} \eta^\dagger \sigma^\mu \overleftrightarrow{\partial_\mu}\, \eta \,, \tag{2.155}$$

giving rise to the most simple kinetic term for dotted spinors η. In the same fashion, a kinetic term for un-dotted spinors can be constructed and reads

$$\mathcal{L}_R^{\text{kin.}} = \frac{i}{2}\left[\xi^*_{\dot{\beta}} \left(\bar{\sigma}^\mu\right)^{\dot{\beta}\alpha} \overleftrightarrow{\partial_\mu}\, \xi_\alpha\right] = \frac{i}{2} \xi^\dagger \bar{\sigma}^\mu \overleftrightarrow{\partial_\mu}\, \xi \,. \tag{2.156}$$

Both kinetic terms can be added, if the two spinorial fields are independent from each other, then the combined Lagrangian $\mathcal{L}^{\text{kin.}} = \mathcal{L}_L^{\text{kin.}} + \mathcal{L}_R^{\text{kin.}}$ obeys a symmetry of the form

$$\begin{array}{rclrcl} \xi'(x) & = & e^{i\theta_R}\, \xi(x) & \xi^{*\prime}(x) & = & e^{-i\theta_R}\, \xi^*(x) \\ \eta'(x) & = & e^{i\theta_L}\, \eta(x) & \eta^{*\prime}(x) & = & e^{-i\theta_L}\, \eta^*(x) \end{array} \,. \tag{2.157}$$

Such a symmetry is called a chiral symmetry, it is symbolised by $U(1)_R \times U(1)_L$, since it has two parameters, the two phase angles. This symmetry again leads to Noether currents and conserved charges[2]. Since again the action is dimensionless, the Lagrangian has mass dimension n, the number of dimensions. Therefore, each spinor field has mass dimension

$$\dim[\xi] = \dim[\eta] = \frac{n-1}{2} \stackrel{n\to 4}{\longrightarrow} \frac{3}{2}\,. \tag{2.158}$$

As a next step, mass terms will be constructed. Obviously, the simplest Lorentz-scalars that can be made up for two spinors are

$$\eta^{\dot\alpha}\epsilon_{\dot\alpha\dot\beta}\eta^{\dot\beta} = \eta^T\epsilon\eta \quad\text{and}\quad \xi^{\alpha}\epsilon_{\alpha\beta}\eta^{\beta} = \xi^T\epsilon\xi\,. \tag{2.159}$$

The only possibility to keep such terms different from zero is to assume anti-commutating numbers - which, of course, is more than reasonable, since the spinors are meant to describe particles with half-integer spin, i.e. fermions obeying Fermi statistic. Only then the result of the terms above,

$$\eta_1\eta_2 - \eta_2\eta_1 \quad\text{and}\quad \xi_1\xi_2 - \xi_2\xi_1 \tag{2.160}$$

yield finite results. Therefore, introducing complex masses, suitable mass terms read

$$\begin{aligned}
\mathcal{L}_L^{\rm m} &= -\frac{1}{2}\left(m_L\eta^T\epsilon\eta - m_L^*\eta^\dagger\epsilon\eta^*\right)\\
\mathcal{L}_R^{\rm m} &= -\frac{1}{2}\left(m_R^*\xi^\dagger\epsilon\xi^* - m_R\xi^T\epsilon\xi\right)\,,
\end{aligned} \tag{2.161}$$

where

$$\epsilon^\dagger = \epsilon^T = -\epsilon \tag{2.162}$$

and

$$(\theta\xi)^* = \xi^*\theta^* \tag{2.163}$$

for two Grassmann-numbers θ and ξ has been used. A mass term of the kind above is called a Majorana mass term. It breaks the chiral symmetry. To have a mass term that respects a somewhat limited version of chiral symmetry, consider the Dirac mass term

$$\mathcal{L}_D^{\rm m} = -\left(m\xi^\dagger\eta + m^*\eta^\dagger\xi\right) \tag{2.164}$$

[2]It should be stressed here that it is absolutely possible to construct kinetic terms with more factors of derivatives. Such terms, however, would lead to tachyonic states

that connects the two kinds of spinors and thus establishes a connection. It obeys a $U(1)_V$ symmetry in which both angles of the transformation above, Eq.(2.157), are set equal,

$$\begin{pmatrix} \xi'(x) \\ \eta'(x) \end{pmatrix} = e^{i\theta} \begin{pmatrix} \xi(x) \\ \eta(x) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \xi^{*\prime}(x) \\ \eta^{*\prime}(x) \end{pmatrix} = e^{-i\theta} \begin{pmatrix} \xi^*(x) \\ \eta^*(x) \end{pmatrix} . \tag{2.165}$$

This transformation leaves an $U(1)_V$ charge Q_V conserved, particles are thus carrying a charge $Q_V = +1$, anti-particles have a charge $Q_V = -1$. Note that so far three complex masses have been introduced, m_L, m_R, and m, By re-phasing of the fields two of them can be set real.

Dirac spinors

Apart from the Dirac mass term, the spinors so far have been completely independent from each other. Only the introduction of the Dirac mass term, maybe motivated by the wish to have the symmetry $U(1)_V$, couples the two kinds of spinors. In such a case the introduction of bi-spinors is of great help to analyse the structure of the theory. In a first step a bi-spinorial field

$$\psi(x) = \begin{pmatrix} \xi_\alpha(x) \\ \eta^{\dot\alpha}(x) \end{pmatrix} \tag{2.166}$$

is introduced. The four-dimensional analogon to the two-dimensional Pauli matrices are the Dirac matrices γ^μ, which, in chiral representation, read

$$\begin{aligned} \gamma^\mu &= \begin{pmatrix} 0 & (\sigma^\mu)_{\alpha\dot\beta} \\ (\bar\sigma^\mu)^{\dot\alpha\beta} & 0 \end{pmatrix} \\ \gamma^0 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \vec\gamma = \begin{pmatrix} 0 & -\vec\sigma \\ \vec\sigma & 0 \end{pmatrix} . \end{aligned} \tag{2.167}$$

Similar to the Pauli matrices, Eq.(2.78), the Dirac matrices anti-commute,

$$\{\gamma^\mu,\, \gamma^\nu\} \equiv \gamma^\mu,\, \gamma^\nu + \gamma^\nu,\, \gamma^\mu = 2g^{\mu\nu} . \tag{2.168}$$

The free Lagrangian for massive Dirac spinors with real mass m is then

$$\mathcal{L}_D = \mathcal{L}_D^{\text{kin.}} + \mathcal{L}_D^{\text{m}} = \frac{i}{2}\bar\psi\gamma^\mu \overleftrightarrow{\partial}_\mu \psi - m\bar\psi\psi = \frac{i}{2}\bar\psi \overleftrightarrow{\not\partial} \psi - m\bar\psi\psi\,, \tag{2.169}$$

where the Dirac-conjugated (barred) bi-spinor is given by

$$\bar{\psi} = \psi^\dagger \gamma^0 \,, \tag{2.170}$$

and the Feynman dagger is defined as

$$\partial\!\!\!/ \equiv \partial^\mu \gamma_\mu = \partial_\mu \gamma^\mu \,. \tag{2.171}$$

Furthermore, a matrix γ_5 is defined, which in chiral representation is

$$\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \,. \tag{2.172}$$

Obviously, the two spinors are eigenvectors of the matrix γ_5 with eigenvalues ± 1. This finding can be used to decompose the Dirac spinor into two chiral spinors with chirality ± 1 through

$$\psi_R = P_R \psi \equiv \frac{1+\gamma_5}{2}\psi = \begin{pmatrix} \xi_\alpha \\ 0 \end{pmatrix} \quad \text{and} \quad \psi_L = P_L \psi \equiv \frac{1-\gamma_5}{2}\psi = \begin{pmatrix} 0 \\ \eta^{\dot{\alpha}} \end{pmatrix} \,. \tag{2.173}$$

To reconstruct the representation of the Lorentz transformation for Dirac spinors consider the tensor corresponding to Eq.(2.94). It reads

$$\sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^\mu,\, \gamma^\nu\right] = \begin{pmatrix} (\sigma^{\mu\nu})_\alpha^{\ \beta} & 0 \\ 0 & (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}_{\ \dot{\beta}} \end{pmatrix} \,, \tag{2.174}$$

and, therefore, the Lorentz-transformed Dirac spinor is

$$\psi'(x') = \exp\left(-\frac{i}{4}\varepsilon^{\mu\nu}\sigma_{\mu\nu}\right)\psi(x) = A\psi(x) \,. \tag{2.175}$$

The transformation matrix A satisfies

$$\bar{A}\gamma^\mu A = \Lambda^\mu_{\ \nu}\gamma^\nu \,, \text{where} \quad \bar{A} = \gamma_0 A^\dagger \gamma_0 \,. \tag{2.176}$$

Obviously, under Lorentz transformations the terms

$$\bar{\psi}\psi \,, \quad \bar{\psi}\gamma_5\psi \,, \quad \bar{\psi}\gamma_\mu\psi \,, \quad \bar{\psi}\gamma_\mu\gamma_5\psi \,, \quad \bar{\psi}\sigma_{\mu\nu}\psi \tag{2.177}$$

transform like scalars, pseudo-scalars, vectors, axial vectors, and tensors, respectively. Consider now, once more, the chiral transformations belonging to the group $U(1)_R \times U(1)_L$ acting on the spinors, but with opposite phases, $\theta = \theta_R = -\theta_L$,

$$\begin{array}{rclrcl} \xi'(x) &=& e^{i\theta}\,\xi(x) & \xi^{*\prime}(x) &=& e^{-i\theta}\,\xi^*(x) \\ \eta'(x) &=& e^{-i\theta}\,\eta(x) & \eta^{*\prime}(x) &=& e^{i\theta}\,\eta^*(x) \end{array} .$$

Plugging this into the definition of the Dirac spinor, two transformations emerge, namely, cf. Eq.(2.165),

$$\psi'(x) = e^{i\theta}\psi(x)\,, \quad \psi^{*\prime}(x) = e^{-i\theta}\psi^*(x) \tag{2.178}$$

and

$$\psi'(x) = e^{i\gamma_5\theta}\psi(x)\,, \quad \psi^{*\prime}(x) = e^{-i\gamma_5\theta}\psi^*(x)\,. \tag{2.179}$$

The latter transformation is called the chiral transformation, both lead to Noether currents, namely

$$j^\mu = -\bar\psi\gamma^\mu\psi \quad \text{and} \quad j_5^\mu = -\bar\psi\gamma^\mu\gamma^5\psi\,. \tag{2.180}$$

However, as mentioned before, the transformations $U(1)_V$ lead to conserved charges $Q = \pm 1$ associated with the particles $\psi = (\xi,\, \eta)T$ and $\psi^* = (\xi^*,\, \eta^*)^T$, respectively. A transformation connecting these two particles types ψ and ψ^* is dubbed charge conjugation. But just applying complex conjugation on the spinors would not do the trick, because $(\xi_\alpha)^* = \xi^*_{\dot\alpha}$ connecting dotted and un-dotted spinors. This is bad news, since the resulting bi-spinor after charge conjugation should still have a proper behaviour under Lorentz transformations. This requires to have upper components of the bi-spinor being an un-dotted spinor. Therefore, upper and lower components of the bi-spinors have to be interchanged, and in each component indices have to be raised or lowered. This is realized by a transformation

$$\psi = \left(\begin{array}{c} \xi_\alpha \\ \eta^{\dot\beta} \end{array}\right) \xrightarrow{C} \psi^{\mathcal{C}} = \left(\begin{array}{c} \eta^{*\beta}\epsilon_{\beta\alpha} \\ \epsilon^{\dot\alpha\dot\beta}\xi^*_{\dot\beta} \end{array}\right) = i\gamma^2\psi^* \equiv \mathcal{C}\bar\psi^T\,. \tag{2.181}$$

Therefore, the 4×4 matrix $\mathcal{C}$ reads

$$\mathcal{C} = i\gamma^2\gamma^0 = -\mathcal{C}^{-1} = -\mathcal{C}^\dagger\,. \tag{2.182}$$

It obeys

$$\mathcal{C}^{-1}\gamma^\mu\mathcal{C} = -\gamma^{\mu T}\,. \tag{2.183}$$

This immediately results in the fact that the free Lagrangian for a massive Dirac spinor is invariant under charge conjugation, leaving the question of which operator is constructor or destructor a matter of convention.

Majorana spinors

Assume a Lagrangian for only one $SL(2,\,C)$ spinor, say η, with real mass m

$$\mathcal{L} = \frac{i}{2}\eta^\dagger\sigma^\mu \overleftrightarrow{\partial_\mu}\,\eta - \frac{m}{2}\left(\eta^T\epsilon\eta - \eta^\dagger\epsilon\eta^*\right)\,. \tag{2.184}$$

This can be cast into a form with bi-spinors by defining the Majorana spinor

$$\psi_M \equiv \begin{pmatrix} \eta^{*\beta}\epsilon_{\alpha\beta} \\ \eta^{\dot\alpha} \end{pmatrix} = \begin{pmatrix} -i\sigma_2\eta^* \\ \eta \end{pmatrix}\,. \tag{2.185}$$

Obviously, this spinor is its own charge conjugate,

$$\psi_M^{\mathcal{C}} = \psi_M \tag{2.186}$$

and in this sense a Majorana spinor is a real Dirac spinor, which can be used to describe neutral particles. In terms of this spinor the Lagrangian reads

$$\mathcal{L}_M = \frac{i}{4}\bar\psi_M\gamma^\mu \overleftrightarrow{\partial_\mu}\,\psi_M - \frac{m}{2}\bar\psi_M\psi_M\,. \tag{2.187}$$

2.4 Gauge symmetries

2.4.1 $U(1)$ gauge fields and electromagnetism

Local gauge transformations

Consider, once again, the Lagrangians for free charged massive scalar or Dirac fields,

$$\begin{aligned} \mathcal{L}_{\rm scalar} &= (\partial_\mu\phi^*)(\partial^\mu\phi) - \mu^2\phi^*\phi - \frac{\lambda}{2}\left(\phi^*\phi\right)^2 \\ \mathcal{L}_{\rm Dirac} &= \bar\psi\left(i\gamma^\mu\partial_\mu - m\right)\psi\,. \end{aligned} \tag{2.188}$$

As has been discussed, these Lagrangians are invariant under global $U(1)$ transformations acting on the fields like

$$\begin{aligned} \phi'(x) &= e^{i\theta}\phi(x)\,, & \phi^{*\prime}(x) &= e^{-i\theta}\phi^*(x) \\ \psi'(x) &= e^{i\theta}\psi(x)\,, & \bar\psi'(x) &= e^{-i\theta}\bar\psi(x)\,, \end{aligned} \tag{2.189}$$

where the phase parameter θ is obviously independent of space-time, i.e. global. But, in principle, at different points in space-time, different observers should be allowed to have their own, private phase convention, and this convention should not affect the measurement. In other words, it seems to be more physical to ask for the invariance of the Lagrangians above under local $U(1)$ phase transformations

$$\theta = \theta(x)\,. \tag{2.190}$$

Such local phase transformations are called gauge transformations, invariance under them defines a gauge theory. But obviously, the kinetic terms of the Lagrangians above are not invariant under gauge transformations. For instance,

$$\begin{aligned}\partial_\mu\phi(x) \longrightarrow \partial_\mu\phi'(x) = \partial_\mu\left[e^{i\theta(x)}\phi(x)\right] &= i\left[(\partial_\mu\theta(x))\,e^{i\theta(x)}\phi(x) + e^{i\theta(x)}\partial_\mu\phi(x)\right]\\ &\neq ie^{i\theta(x)}\partial_\mu\phi(x)\,,\end{aligned} \tag{2.191}$$

and terms proportional to the derivative of the phase emerge. To restore the invariance under such local transformations, such extra terms have to be compensated for. This can be done by modifying the derivatives accordingly, in the simplest case by "subtracting" the phase. Such a subtraction is achieved by defining the covariant derivative, which, acting on a scalar field, reads

$$D_\mu\phi(x) \equiv \left[\partial_\mu - ieA_\mu(x)\right]\phi(x)\,, \tag{2.192}$$

where the gauge field A_μ has been introduced and the particles have been given a charge e. Invariance under local phase transformations is then guaranteed by demanding that the gauge fields transforms like

$$A_\mu(x) \longrightarrow A'_\mu(x) = A_\mu(x) + \frac{1}{e}\partial_\mu\theta(x)\,. \tag{2.193}$$

Then Eq.(2.191) with the covariant derivative replacing the ordinary one reads

$$\begin{aligned}(D_\mu\phi(x))' &= \left[\partial_\mu - ieA'_\mu(x)\right]\phi'(x)\\ &= i\left[(\partial_\mu\theta(x))\,e^{i\theta(x)}\phi(x) + e^{i\theta(x)}\partial_\mu\phi(x)\right] - \qquad (2.194)\\ &\qquad i\left[eA_\mu(x) + \partial_\mu\theta(x)\right]e^{i\theta(x)}\phi(x)\\ &= e^{i\theta(x)}\left[\partial_\mu - ieA_\mu(x)\right]\phi(x) = D_\mu\phi(x)\,. \qquad (2.195)\end{aligned}$$

The construction principle of the covariant derivative acting on an arbitrary field Φ with charge q that transforms like

$$\Phi'(x) = e^{iq\theta(x)}\Phi(x) \tag{2.196}$$

is such that

$$(D_\mu \Phi(x))' = e^{iq\theta(x)} D_\mu \Phi(x)\,, \tag{2.197}$$

i.e.

$$D_\mu \Phi(x) = \left[\partial_\mu - iqeA_\mu(x)\right] \Phi(x)\,. \tag{2.198}$$

Also,

$$D_\mu \Phi^*(x) = \left[D_\mu \Phi(x)\right]^*\,. \tag{2.199}$$

This fixes the Lagrangians above and it introduces an interaction of the gauge field with the matter fields. Taken together, Eq.(2.188) now reads

$$\begin{aligned}
\mathcal{L}_{\rm scalar} &= (D_\mu\phi)^*(D^\mu\phi) - \mu^2\phi^*\phi - \frac{\lambda}{2}\left(\phi^*\phi\right)^2 \\
\mathcal{L}_{\rm Dirac} &= \bar\psi\left(i\gamma^\mu D_\mu - m\right)\psi\,.
\end{aligned} \tag{2.200}$$

From these equations, also the dimension of the gauge fields can be read off:

$$\dim\left[A_\mu\right] = \dim\left[\partial_\mu\right] = 1\,. \tag{2.201}$$

Kinetic terms for the gauge fields

But theories described by the Lagrangians of Eq.(2.200) are pretty boring, because the gauge fields are not dynamical. To cure this and to add some spice to life, one would like to add kinetic terms for the gauge fields as well. Again, the demand to be gauge invariant provides a guiding principle. The most naive idea for a Lorentz scalar with two gauge fields, mass dimension of four, would be a term like

$$(\partial_\mu A^\mu(x))^2 \tag{2.202}$$

but it is quite obvious that such a term violates gauge invariance and is therefore not acceptable. A quantity that consists of only one derivative and is gauge invariant is the field strength tensor

$$F^{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,. \tag{2.203}$$

With its help, a suitable kinetic term can be found as

$$\mathcal{L}^{\rm kin.}_{\rm gauge} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}\,. \tag{2.204}$$

This term also has a number of nice properties. Plugging in the form of the field strength tensor,

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix} \tag{2.205}$$

it can be immediately seen that this term is equal to

$$\mathcal{L}^{\text{kin.}}_{\text{gauge}} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} = \frac{1}{2}(\vec{E}^2 - \vec{B}^2)\,, \tag{2.206}$$

the Lagrangian for the electromagnetic field. Then, using the tensor

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} = \begin{pmatrix} 0 & -B^1 & -B^2 & -B^3 \\ B^1 & 0 & E^3 & -E^2 \\ B^2 & -E^3 & 0 & E^1 \\ B^3 & E^2 & -E^1 & 0 \end{pmatrix} \tag{2.207}$$

the Euler-Lagrange equations can be written in compact form as

$$\partial_\mu \tilde{F}^\mu\nu = 0 \quad \text{and} \quad \partial_\mu F^\mu\nu = j^\nu \tag{2.208}$$

and it turns out that they are identical to the homogenous and inhomogeneous Maxwell equations. Taken together, Lagrangians for scalar electrodynamics or spinor electrodynamics are given by

$$\begin{aligned} \mathcal{L}_{\text{scalar}} &= -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + (D_\mu\phi)^*(D^\mu\phi) - \mu^2\phi^*\phi - \frac{\lambda}{2}\left(\phi^*\phi\right)^2 \\ \mathcal{L}_{\text{Dirac}} &= -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi\,. \end{aligned} \tag{2.209}$$

2.4.2 Extension to $SU(N)$

Matter fields

To extent the findings of the previous section from the electromagnetic $U(1)$ symmetry to the more general case of an $SU(N)$ symmetry one first has to find an analogon to the charged scalar and the Dirac fields of Eq.(2.188).

Concentrating in the following on Dirac fields (for scalars, the line of reasoning is identical), first of all, the bi-spinors ψ have to be replaced by an N-component vector of bi-spinors

$$\psi = (\psi_1,\, \psi_2,\, \ldots,\, psi_N)^T \,. \tag{2.210}$$

Invariance of the Lagrangian under $SU(N)$ gauge transformations then means that the Lagrangian remains invariant under

$$\psi_i'(x) = [U(x)]_i^{\,j}\, \psi_j(x) = \left[e^{ig\theta_a(x)T_a}\right]_i^{\,j} \psi_j(x)\,, \tag{2.211}$$

where the T_a are the generators of the group G and g is the corresponding coupling constant. Defining now the gauge fields (for mathematicians: connection fields)

$$[A_\mu(x)]_i^{\,j} = \sum_{a=1}^{\dim(G)} A_\mu^a(x)\, [T_a]_i^{\,j} \tag{2.212}$$

the covariant derivative can be defined as

$$[D_\mu]_i^{\,j} = \partial_\mu \delta_i^{\,j} - ig\, [A_\mu(x)]_i^{\,j} \,. \tag{2.213}$$

The field A_μ transforms under the local gauge transformation $U(x)$ as

$$A_\mu(x) \longrightarrow A'_\mu(x) = U(x)A_\mu(x)U^\dagger(x) + \frac{i}{g}\left(\partial_\mu U(x)\right) U^\dagger(x)\,, \tag{2.214}$$

and, correspondingly, the covariant derivative transforms like

$$D_\mu(x) \longrightarrow D'_\mu(x) = U(x)D_\mu(x)U^\dagger(x)\,. \tag{2.215}$$

Under these transformations the Lagrangian of Eq.(2.209) remains invariant.

Gauge fields

Again, the gauge fields defined above have to be made dynamic. To do so, an analogon to the field strength tensor has to be constructed. It can be proven that this analogon reads

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig\,[A_\mu,\, A_\nu] = \frac{i}{g}\,[D_\mu,\, D_\nu] \,. \tag{2.216}$$

The commutator does not vanish, because the fields A_μ are products of individual gauge (vector) fields with the non-commuting generators of the gauge group. Under gauge transformations, the field strength tensor behaves as

$$F_{\mu\nu} \longrightarrow F'_{\mu\nu} = U F_{\mu\nu} U^\dagger \,, \tag{2.217}$$

and only the term

$$\mathrm{Tr}\left[F_{\mu\nu}F^{\mu\nu}\right] = F^a_{\mu\nu}F^{a\mu\nu} \,, \tag{2.218}$$

where the trace is over the generator indices of the group, is gauge invariant. Taken together, the Lagrange density for an $SU(N)$ gauge group with spinors reads

$$\mathcal{L} = -\frac{1}{4}F^{a\mu\nu}F^a_{\mu\nu} + \bar{\psi}^i \left[i\gamma^\mu \left(D_\mu\right)_i^{\ j} - m\delta_i^{\ j}\right]\psi_j \,. \tag{2.219}$$

Chapter 3

Free field theories and their quantization

The starting point for the construction of gauge field theories are free quantum field theories. In this chapter, their canonical quantization will be discussed. After quantizing both real and complex scalar fields, representing neutral and charged fields, the attention is turned onto the quantization of the free Dirac (bi-spinor) field. Finally, vector fields coming into play in gauge field theories, will be dealt with. The procedure there will be exemplified for the case of the simplest gauge theory, i.e. with the quantization of the free electromagnetic field.

3.1 Quantization of the free scalar field

3.1.1 Canonical quantization

Lagrangian, Hamiltonian, and commutators

As a starting point, consider a real scalar field theory, like, for instance, the $\lambda\phi^4$ theory defined through its Lagrangian by

$$\mathcal{L} = \frac{1}{2}\left[\partial_\mu\phi\,\partial^\mu\phi - \mu^2\phi^2\right] - \frac{\lambda}{4!}\,\phi^4\,. \tag{3.1}$$

For canonical quantization of this theory, the Lagrange function is needed which emerges from the Lagrangian above by integration over the three-dimensional space at a given time t,

$$L(t) = \int \mathrm{d}^3\vec{x}\mathcal{L}\left(\phi(t,\,\vec{x}),\,\dot{\phi}(t,\,\vec{x})\right)\,, \tag{3.2}$$

where

$$\dot{\phi}(t,\,\vec{x}) := \frac{\partial}{\partial_t}\phi(t,\,\vec{x})\,. \tag{3.3}$$

The conjugate momentum is defined as

$$\pi(\vec{x}\,,t) := \frac{\delta\mathcal{L}}{\delta\dot{\phi}(\vec{x}\,,t)} = \frac{\partial\mathcal{L}}{\partial\dot{\phi}(\vec{x}\,,t)} = \dot{\phi}(\vec{x}\,,t)\,. \tag{3.4}$$

Canonical quantization is performed by going from fields to field operators that fulfill the following canonical equal-time commutation relations

$$\begin{aligned} [\phi(\vec{x}\,,t)\,,\pi(t,\,\vec{y})] &= i\delta^3(\vec{x}-\vec{y}) \\ [\phi(\vec{x}\,,t)\,,\phi(\vec{y}\,,t)] &= [\pi(\vec{x}\,,t)\,,\pi(\vec{y}\,,t)] = 0\,. \end{aligned} \tag{3.5}$$

The Hamilton function is

$$H = \int \mathrm{d}^3x\left(\pi\dot{\phi} - \mathcal{L}\right) = \int \mathrm{d}^3x\left[\frac{1}{2}\left(\pi^2 + (\vec{\nabla}\phi)^2 + \mu^2\phi^2\right) + \frac{\lambda}{4!}\,\phi^4\right] \tag{3.6}$$

leading to the equations of motion for the operators

$$\begin{aligned} i\dot{\phi} &= [\phi,\,H] = i\pi \\ i\dot{\pi} &= [\pi,\,H] = -i\left[\left(-(\vec{\nabla}\phi)^2 + \mu^2\right)\phi + \frac{\lambda}{3!}\,\phi^3\right]\,. \end{aligned} \tag{3.7}$$

This is identical to the Euler-Lagrange equation that can be obtained from the Lagrangian and reads

$$\left(\partial_\mu^2 + \mu^2\right)\phi + \frac{\lambda}{3!}\,\phi^3 = 0\,. \tag{3.8}$$

In the following the potential will be omitted, the resulting equation of motion is the Klein-Gordon equation.

Field operators

For the solution of the Klein-Gordon equation the field ϕ is Fourier-transformed in three dimension withy fixed time to read

$$\phi(\vec{x}\,,t) = \int \frac{d^3\vec{k}}{(2\pi)^{3/2}} q_{\vec{k}}(t)\, e^{i\vec{k}\cdot\vec{x}} \tag{3.9}$$

transforming the Klein-Gordon equation into the equation for a harmonical oscillator,

$$\ddot{q}_{\vec{k}} + \left(\vec{k}^2 + \mu^2\right)\, q_{\vec{k}} = 0\,. \tag{3.10}$$

The solution of this is, as usual,

$$q_{\vec{k}}(t) = q_1(\vec{k})\, e^{-ik_0 t} + q_2(\vec{k})\, e^{ik_0 t}\,, \tag{3.11}$$

where

$$k_0 := \sqrt{\vec{k}^2 + \mu^2} > 0\,. \tag{3.12}$$

Since the field is real, the corresponding field operator of Eq.(3.9) is Hermitian and the q_1 and q_2 are thus connected by

$$q_2(\vec{k}) = q_1^\dagger(-\vec{k})\,. \tag{3.13}$$

Defining

$$q_1(\vec{k}) := \frac{1}{\sqrt{2k_0}}\, a(\vec{k})\,, \tag{3.14}$$

Eq.(3.9) can be written as

$$\phi(\vec{x}\,,t) = \int \frac{d^3\vec{k}}{(2\pi)^3 k_0} \left[a(\vec{k})\, e^{-ik\cdot x} + a^\dagger(\vec{k})\, e^{ik\cdot x}\right]\,. \tag{3.15}$$

The conjugate momentum reads

$$\pi(\vec{x}\,,t) = \int \frac{d^3\vec{k}}{\sqrt{(2\pi)^3 2k_0}} \left[-ik_0\, a(\vec{k})\, e^{-ik\cdot x} + ik_0\, a^\dagger(\vec{k})\, e^{ik\cdot x}\right]\,. \tag{3.16}$$

The operators $a(\vec{k})$ can be calculated as

$$a(\vec{k}) = i \int \mathrm{d}^3x \left[\frac{e^{-ik\cdot x}}{\sqrt{(2\pi)^3 2k_0}}\, \overleftrightarrow{\partial_0}\, \phi(x)\right] \tag{3.17}$$

where the arrows on top of the derivative symbolize how this acts on the neighbors. Therefore, in momentum space the commutators are

$$\begin{aligned}
\left[a(\vec{k})\,, a^\dagger(\vec{q})\right] &= \delta^3(\vec{k}-\vec{q}) \\
\left[a(\vec{k})\,, a(\vec{q})\right] &= \left[a^\dagger(\vec{k})\,, a^\dagger(\vec{q})\right] = 0\,.
\end{aligned} \tag{3.18}$$

Momentum operator and vacuum energy

The total four momentum of the field is given along the lines of Eq.(2.134) by the integral of the energy-momentum tensor over the full space,

$$\begin{aligned}
P^\mu &= \int \mathrm{d}^3x \left(\dot{\phi}\partial^\mu\phi - g^{\mu 0}\mathcal{L}\right) \\
&= \int \mathrm{d}^3x\, \frac{1}{2}k^\mu \left[a^\dagger(\vec{k})a(\vec{k}) + a(\vec{k})a^\dagger(\vec{k})\right]\,.
\end{aligned} \tag{3.19}$$

Comparing this with the commutator from Eq.(3.18) for equal momenta,

$$\left[a(\vec{k}),\, a^\dagger(\vec{k})\right] = \delta^3(\vec{p}=\vec{0}) = \int \frac{d^3\vec{x}}{(2\pi)^3}\, e^{i\vec{p}\vec{x}}\Bigg|_{\vec{p}=\vec{0}} = \frac{1}{(2\pi)^3}\int \mathrm{d}^3\vec{x}\,, \tag{3.20}$$

and defining the number operator

$$n(\vec{k}) = a^\dagger(\vec{k})a(\vec{k})\,, \tag{3.21}$$

the total four momentum is

$$P^\mu = \int \mathrm{d}^3\vec{k}\, k^\mu\, n(\vec{k}) + \frac{1}{2}\int \frac{d^3\vec{k}d^3\vec{x}}{(2\pi)^3}\, k^\mu\,. \tag{3.22}$$

For all space components of P^μ, i.e. for $\mu = 1,\, 2,\, 3$ this is given by the sum of momenta of all quanta, i.e. the first term. In this case the second term vanishes due to symmetry (integral of an odd function over the whole space). For the energy component, however, the second term contributes. It can be interpreted as the integral over all degrees of freedom in the whole space, each occupying a small box with size $2\pi\hbar$ and filled with one harmonic oscillator that contributes its vacuum energy. Since this vacuum energy is given by $(\hbar/2)k_0$ for every momentum, this amounts to an infinite sum of oscillators of frequency k_0, each of which with energies rising with the absolute value of the unconstrained momentum. To ensure the four momentum operator to be well-defined one therefore shifts the energy by this term. Hence, the four-momentum operator is given by

$$P^\mu = \int \mathrm{d}^3\vec{k}\, k^\mu\, n(\vec{k})\,. \tag{3.23}$$

This operator generates linear translations of the fields and therefore

$$[iP_\mu,\, \phi(x)] = \partial_\mu \phi(x)\,. \tag{3.24}$$

It should be stressed here that the problems related to the vacuum energy would not have occurred if from the beginning normal ordering would have been used for all quantities, i.e. an ordering in which all creation operators $\dagger a$ are to the left of all annihilation operators a. The vacuum state $|0\rangle$ is defined through

$$a(\vec{k})|0\rangle = 0\,, \tag{3.25}$$

i.e. by its property to be annihilated by the annihilation operators. Therefore, applying the momentum operator on the vacuum yields

$$P^\mu|0\rangle = \int \mathrm{d}^3\vec{k}\, k^\mu\, n(\vec{k})|0\rangle = 0\,. \tag{3.26}$$

In general, states composed of n particles with momenta $\vec{k}_i$ (the states of the Fock space) have the form

$$\left|\vec{k}_1,\, \vec{k}_2,\, \ldots,\, \vec{k}_n\right\rangle = a^\dagger(\vec{k}_1),\, a^\dagger(\vec{k}_2),\, \ldots,\, a^\dagger(\vec{k}_n)\left|0\right\rangle\,. \tag{3.27}$$

Due to the commutator relation

$$\begin{aligned}\left[P^\mu,\, a^\dagger(\vec{q})\right] &= \left[\int \mathrm{d}^3\vec{k}\, k^\mu\, n(\vec{k}),\, a^\dagger(\vec{q})\right] = \int \mathrm{d}^3\vec{k}\, k^\mu a^\dagger(\vec{k})\left[a(\vec{k}),\, a^\dagger(\vec{q})\right] \\ &= \int \mathrm{d}^3\vec{k}\, k^\mu a^\dagger(\vec{k})\delta^3(\vec{k}-\vec{q}) = q^\mu a^\dagger(\vec{q}) \end{aligned} \tag{3.28}$$

applying the momentum operator on such states yields

$$P^\mu\left|\vec{k}_1,\, \vec{k}_2,\, \ldots,\, \vec{k}_n\right\rangle = (k_1^\mu + k_2^\mu + \ldots + k_n^\mu)\left|\vec{k}_1,\, \vec{k}_2,\, \ldots,\, \vec{k}_n\right\rangle\,. \tag{3.29}$$

Therefore, the operators $a^\dagger(\vec{k})$ are correctly interpreted as creators of (on-shell) particles with momentum $\vec{k}$. Of course, particles quantized using the commutators follow the Bose statistics and are therefore dubbed bosons. In contrast, quantizing a theory with anti-commutators, signified by curly ($\{,\}$) instead of square brackets ($[,]$), leads to particles obeying Fermi statistics which are called fermions. To understand why the real scalar theory has to be quantized with commutators take another look at the momentum operator, Eq.(3.19). If the theory had been quantized with anti-commutators, the energy operator P^0 would have been a simple c-number following a trivial algebra.

3.1.2 Propagators

As a next step propagators for the scalar fields will be constructed. Using the commutators for the creation and annihilation operators, Eq.(3.18), commutators for arbitrary space-times can be evaluated,

$$
\begin{aligned}
&[\phi(x)\,,\phi(y)] \\
&= \int \frac{d^3\vec{k}d^3\vec{q}}{(2\pi)^3\sqrt{4k_0q_0}}\left(\left[a(\vec{k})\,,a^\dagger(\vec{q})\right]e^{i(q\cdot y-k\cdot x)} + \left[a^\dagger(\vec{k})\,,a(\vec{q})\right]e^{i(k\cdot x-q\cdot y)}\right) \\
&= \int \frac{d^3\vec{k}}{(2\pi)^3 2k_0}\left(e^{-ik\cdot(x-y)} - e^{ik\cdot(x-y)}\right) \\
&= \int \frac{d^4k}{(2\pi)^3}\epsilon(k_0)\delta(k^2-\mu^2)e^{-ik\cdot(x-y)} =: i\Delta(x-y)\,,
\end{aligned}
\tag{3.30}
$$

where the function $\epsilon(x)$ has been introduced. It changes with the sign of x,

$$
\epsilon(x) = \begin{cases} +1 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -1 & \text{for } x < 0 \end{cases} \; . \tag{3.31}
$$

The commutator $\Delta(x)$ has the following properties:

1. Lorentz invariance $\Delta(x) = \Delta(x')$.
2. It is an even function of x, $\Delta(x) = -\Delta(-x)$.
3. Micro-causality, i.e. for space-like distance $x^2 < 0$ it is zero, $\Delta(x) = 0$.
4. For equal times:

$$
\text{for } x_0 = 0 \quad \left\{ \begin{array}{lcl} \Delta(x) & = & 0 \\ \dot{\Delta}(x) & = & -\delta^3(\vec{x}) \end{array} \right. \tag{3.32}
$$

As a next step towards the Feynman propagator the time-ordered product is defined. It reads

$$T\phi(x)\phi(y) = \vartheta(x^0 - y^0)\phi(x)\phi(y) \pm \vartheta(y^0 - x^0)\phi(y)\phi(x)\,, \tag{3.33}$$

where the plus sign refers to bosons and the minus is for fermions. The ϑ function is given by

$$\vartheta(x) = \begin{cases} +1 & \text{for } x > 0 \\ \frac{1}{2} & \text{for } x = 0 \\ 0 & \text{for } x < 0 \end{cases}\,. \tag{3.34}$$

The vacuum expectation value for this time ordered product is, for the real scalar field,

$$\begin{aligned}
&\langle 0\,|T\phi(x)\phi(y)|\,0\rangle \qquad\qquad (3.35)\\
&= \int \frac{\mathrm{d}^3\vec{k}\mathrm{d}^3\vec{q}}{(2\pi)^3\sqrt{4k_0q_0}}\Big\{\theta(x_0 - y_0)\left\langle 0\left|a(\vec{k})\,a^\dagger(\vec{q})\right|0\right\rangle e^{i(q\cdot y - k\cdot x)} + \\
&\qquad\qquad \theta(y_0 - x_0)\left\langle 0\left|a(\vec{q})\,a^\dagger(\vec{k})\right|0\right\rangle e^{i(k\cdot x - q\cdot y)}\Big\}\\
&= \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3 2k_0}\Big\{\theta(x_0 - y_0)e^{-ik\cdot(x-y)} + \theta(y_0 - x_0)e^{ik\cdot(x-y)}\Big\}\\
&= \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{ie^{-ik\cdot(x-y)}}{k^2 - \mu^2 - i\epsilon} =: \Delta_F(x - y)\,. \qquad (3.36)
\end{aligned}$$

The ϵ that has been introduced in the imaginary part is understood in the limit $\epsilon \to 0$. It is used to shift the poles from the real axis a little bit into the complex plane, namely to

$$\omega_{i=1,2} : k_0 = \pm\sqrt{k^2 + \mu^2 - i\epsilon} = \pm\sqrt{\vec{k}^2 + m^2} \mp i\epsilon = \pm\omega \mp i\epsilon\,. \tag{3.37}$$

To go from the third to the last line of Eq.(3.35) the two terms have been identified as residues,

$$\begin{aligned}
\int\limits_{-\infty}^{\infty} \frac{\mathrm{d}k_0}{2\pi}\frac{e^{-ik\cdot(x-y)}}{k^2 - \mu^2} &= \oint \frac{\mathrm{d}k_0}{2\pi}\frac{e^{-i\left[k_0(x_0-y_0) - \vec{k}\cdot(\vec{x}-\vec{y})\right]}}{k_0^2 - \vec{k}^2 - \mu^2 - i\epsilon}\\
&= 2\pi i\sum_{i=1}^{N}\mathrm{Res}\left[\frac{e^{-i\left[k_0(x_0-y_0) - \vec{k}\cdot(\vec{x}-\vec{y})\right]}}{k_0^2 - \vec{k}^2 - \mu^2 - i\epsilon},\ \omega_i\right]
\end{aligned}$$

$$
\begin{aligned}
&= 2\pi i \left\{ \lim_{k_0 \to \omega} \frac{(k_0-\omega)e^{-i\left[k_0(x_0-y_0)-\vec{k}\cdot(\vec{x}-\vec{y})\right]}}{k_0^2-\vec{k}^2-\mu^2} + \right. \\
&\qquad \left. \lim_{k_0 \to -\omega} \frac{(k_0+\omega)e^{-i\left[k_0(y_0-x_0)-\vec{k}\cdot(\vec{y}-\vec{x})\right]}}{k_0^2-\vec{k}^2-\mu^2} \right\} \\
&= 2\pi i \left\{ \frac{1}{2\omega} \left[\theta(x_0-y_0)e^{-ik\cdot(x-y)} + \theta(y_0-x_0)e^{-ik\cdot(y-x)} \right] \right\}
\end{aligned}
\tag{3.38}
$$

To close the integration region (i.e. to go from $\int$ to $\oint$) there are two options: either the lower or the upper half-plane. If $x_0 - y_0 \geq 0$ one takes the lower half-plane. This ensures that for complex $k_0 \to \infty$ the integral over the half circle yields zero and that one picks up the pole at $+\omega$. In contrast, for $x_0 - y_0 \leq 0$ one takes the upper half-plane, finally picking up the pole at $-\omega$. It can be shown that the Feynman propagator is the inverse operator of the Klein-Gordon operator, i.e. that

$$
\left(\partial^2 + \mu^2\right) \Delta_F(x) = -i\delta^4(x)\,. \tag{3.39}
$$

In other words, the Feynman propagator is a Green function for the Klein-Gordon equation.

3.1.3 Complex scalar field

For a complex scalar field, similar considerations hold true. The main difference, of course, is that the field is not self-adjoint any longer, instead it appears in two types, each of which equipped with its own creation and annihilation operators. Then the field operators for ϕ and ϕ^* are connected by hermitian conjugation and read

$$
\begin{aligned}
\phi(x) &= \int \frac{\mathrm{d}^3\vec{k}}{\sqrt{(2\pi)^3 2k_0}} \left[a_-(\vec{k})e^{-ik\cdot x} + a_+^\dagger(\vec{k})e^{ik\cdot x}\right] \\
\phi^*(x) &= \int \frac{\mathrm{d}^3\vec{k}}{\sqrt{(2\pi)^3 2k_0}} \left[a_+(\vec{k})e^{-ik\cdot x} + a_-^\dagger(\vec{k})e^{ik\cdot x}\right]\,.
\end{aligned}
\tag{3.40}
$$

The creation and annihilation operators satisfy

$$
\begin{aligned}
\left[a_+(\vec{k}),\, a_+^\dagger(\vec{q})\right] &= \left[a_-(\vec{k}),\, a_-^\dagger(\vec{q})\right] = \delta^3(\vec{k}-\vec{q}) \\
\left[a_+(\vec{k}),\, a_-^\dagger(\vec{q})\right] &= \left[a_-(\vec{k}),\, a_+^\dagger(\vec{q})\right] = 0
\end{aligned}
\tag{3.41}
$$

and number operators for the particles and anti-particles can be defined as

$$n_\pm(\vec{k}) = a^\dagger_\pm(\vec{k})a_\pm(\vec{k})\,. \tag{3.42}$$

The energy-momentum operator is, in full analogy to the real scalar field, given by

$$P^\mu = \int \mathrm{d}^3\vec{k}k^\mu \left[n_+(\vec{k}) + n_-(\vec{k})\right]\,, \tag{3.43}$$

and the overall charge is, according to Eq.(2.147),

$$Q = \int \mathrm{d}^3\vec{x}\left[\dot{\phi}(x)\phi^*(x) - \phi^*(x)\dot{\phi}(x)\right] = \int \mathrm{d}^3\vec{k}\left[n_+(\vec{k}) - n_-(\vec{k})\right]\,, \tag{3.44}$$

This shows that both the particles and the anti-particles have the same mass but opposite charge. Examples for such particles are the charged pions or kaons ($pi^\pm$ or $K^\pm$), or the pair of electrical neutral kaons (K^0 and $\bar{K}^0$), where the charge is not the electrical one, but strangeness.
In any case, the commutator of the field operators reads

$$[\phi(x),\, \phi^*(y)] = i\Delta(x-y) \tag{3.45}$$

and the Feynman propagator is given by

$$\langle 0\,|T\phi(x)\,\phi^*(y)|\,0\rangle = \Delta_F(x-y)\,. \tag{3.46}$$

Replacing the hermitian conjugate fields ϕ^* by the original ones, ϕ would yield zero for both the commutator and the propagator. It should be noted that identical results can be obtained by using two real fields $\phi_{1,\,2}$ instead of the complex field ϕ and its conjugate ϕ^*.

3.2 Quantization of the free Dirac field

3.2.1 Canonical quantization

Lagrangian, anti-commutators, and Hamiltonian

According to Eq.(2.169) the Lagrangian of the free Dirac field is

$$\mathcal{L}_D = \frac{i}{2}\bar{\psi}\overset{\leftrightarrow}{\not\partial}\psi - m\bar{\psi}\psi = \bar{\psi}\left(i\not\partial - m\right)\psi\,. \tag{3.47}$$

As already stated, the field ψ in classical theory is a Grassmann number. The canonical conjugate momentum π for the field ψ is, written in spinor components labeled by α,

$$\pi_\psi^\alpha(x) = \frac{\partial \mathcal{L}}{\partial \dot{\psi}_\alpha(x)} = i\bar{\psi}^\beta(x)\left(\gamma^0\right)_\beta^{\ \alpha} = i(\psi^*)^\alpha(x)\,. \tag{3.48}$$

In other words the canonical conjugate momentum to the spinor fields ψ is its complex conjugate ψ^*, quite a surprising result. The momentum is defined through the "right-hand" derivative. Since the Dirac fields are Grassmann numbers the sequence matters and one can distinguish "right-hand" and "left-hand" derivatives,

$$\begin{aligned} \text{“left-hand” derivative} \quad &: \quad \frac{\partial}{\partial\theta}AB &=& \left[\frac{\partial}{\partial\theta}A\right]B + (-)^{|A|}A\left[\frac{\partial}{\partial\theta}B\right] \\ \text{“right-hand” derivative} \quad &: \quad \frac{\partial AB}{\partial\theta} &=& A\left[\frac{\partial B}{\partial\theta}\right] + (-)^{|B|}\left[\frac{\partial A}{\partial\theta}\right]B\,. \end{aligned} \tag{3.49}$$

For Grassmann numbers the exponent $|A|$ equals one, for c-numbers it is zero.
Since the spinor describe fermions, the commutators in Eqs.(3.5) are replaced by anti-commutators,

$$\begin{aligned} \left\{\psi_\alpha(t,\,\vec{x})\,,\pi_\psi^\beta(t,\,\vec{y})\right\} &= i\delta_\alpha^\beta\delta^3(\vec{x}-\vec{y}) \\ \left\{\psi_\alpha(t,\,\vec{x})\,,\psi_\beta(t,\,\vec{y})\right\} &= \left\{\psi^{*\alpha}(t,\,\vec{x})\,,\psi^{*\beta}(t,\,\vec{y})\right\} = 0\,. \end{aligned} \tag{3.50}$$

Using Eq.(3.48) this can be cast into

$$\left\{\psi(t,\,\vec{x})\,,\bar{\psi}(t,\,\vec{y})\right\} = \gamma^0\delta^3(\vec{x}-\vec{y})\,. \tag{3.51}$$

The Hamilton function then reads

$$\begin{aligned} H &= \int \mathrm{d}^3\vec{x}\left(\pi_\psi\dot{\psi} - \mathcal{L}\right) = \int \mathrm{d}^3\vec{x}\bar{\psi}\left(-i\vec{\gamma}\cdot\vec{\partial} + m\right)\psi \\ &= \int \mathrm{d}^3\vec{x}\psi^\dagger\left(-i\vec{\alpha}\cdot\vec{\nabla} + m\beta\right)\psi\,, \end{aligned} \tag{3.52}$$

where the definitions

$$\vec{\alpha} := \gamma^0\vec{\gamma} \quad \text{and} \quad \beta := \gamma^0 \tag{3.53}$$

have been used[1]. The Euler-Lagrange equation of the Dirac Lagrangian is the Dirac equation of motion for spinors,

$$(i\gamma^\mu \partial_\mu - m)\,\psi = 0\,. \tag{3.54}$$

It could be also obtained from the Hamilton function above through the Heisenberg operator equation $\dot{\psi} = -i\,[\psi,\, H]$ and the anti-commutators.

Solutions of the Dirac equation and field operators

To construct solutions of the Dirac equation, Eq.(3.54), the spinors are expanded in terms of plane waves,

$$\psi_\alpha(x) = e^{-ip\cdot x} u_\alpha(p)\,, \tag{3.55}$$

where, obviously, the u_α have four components and satisfy

$$(\not{p} - m) u_\alpha(p) = 0\,. \tag{3.56}$$

This is a linear equation, it has non-trivial solutions only, if

$$\det(\not{p} - m) = \left(p^2 - m^2\right)^2 = 0\,, \tag{3.57}$$

hence, the two solutions can be distinguished according to the sign of the energy,

$$p_0 = \pm\sqrt{\vec{p}^2 + m^2}\,. \tag{3.58}$$

The solutions with positive and negative energies correspond to particles and anti-particles and they are denoted by u and v, respectively. They fulfill

$$(\not{p} - m) u^{(s)}(\vec{p}) = 0 \quad \text{and} \quad -(\not{p} + m) v^{(s)}(\vec{p}) = 0\,, \tag{3.59}$$

where the (spin-) index s labels the two possible solutions. Usually, in the Dirac representation for the gamma matrices the eigenvalues of the spin-operator are taken for s. In the chiral base for spinors and the corresponding representation of the gamma matrices, the eigenvalues of helicity, ± 1, are used to label the solutions.

[1] If the canonical conjugate momentum would have been defined through a "left-hand" derivative instead of being defined by a "right-hand" derivative, the Hamiltonian would read $\mathcal{H} = \dot{\psi}\pi - \mathcal{L}$ instead of the current form $\mathcal{H} = \pi\dot{\psi} - \mathcal{L}$. The sequence is due to the fact that $\delta\mathcal{H}$ must not depend on $\delta\psi$.

That Eq.(3.59) has two solutions can be seen quite easily by reconsidering it in the rest frame of the particle or anti-particle. Then Eq.(3.59) becomes

$$m(\gamma^0 - 1)u^{(s)}(m,\, \vec{p} = \vec{0}) = 0 \quad \text{and} \quad - m(\gamma^0 + 1)v^{(s)}(m,\, \vec{p} = \vec{0}) = 0\,. \tag{3.60}$$

In the usual (Dirac-) representation, the gamma matrices read

$$\gamma^0 \;=\; \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \qquad \gamma^{i=1,\,2,\,3} \;=\; \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{3.61}$$

showing that in this basis solutions for the u and v in their rest frame read

$$\begin{aligned} u^{(1)}(m\,,\vec{0}) \;&=\; \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad & u^{(-1)}(m\,,\vec{0}) \;&=\; \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \\ v^{(1)}(m\,,\vec{0}) \;&=\; \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad & v^{(-1)}(m\,,\vec{0}) \;&=\; \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \end{aligned} \tag{3.62}$$

To go to systems other than the rest frame, simple boosts could be employed. Instead, it is simpler to observe that for on-shell particles with momentum p

$$(\not{p} - m)(\not{p} + m) = p^2 - m^2 = 0 \tag{3.63}$$

and to identify

$$u^{(i)}(p) = (\not{p} + m)u^{(i)}(m,\, \vec{0}) \quad \text{and} \quad v^{(i)}(p) = -(\not{p} - m)v^{(i)}(m,\, \vec{0})\,. \tag{3.64}$$

Similarly, the conjugate spinors are defined by

$$\bar{u}^{(i)}(p) = \bar{u}^{(i)}(m,\, \vec{0})(\not{p} + m) \quad \text{and} \quad \bar{v}^{(i)}(p) = \bar{v}^{(i)}(m,\, \vec{0})(-\not{p} + m)\,., \tag{3.65}$$

where “barred” spinors are connected to the “un-barred” ones through

$$\bar{\psi} = \psi^\dagger\gamma^0\,, \tag{3.66}$$

cf. Eq.(2.170) independent of representation. Recalling Eq. (2.181), describing charge conjugation for spinors, i.e. the connection of particles and anti-particles,

$$\psi \xrightarrow{C} \psi^{\mathcal{C}} = i\gamma^2\psi^* \equiv \mathcal{C}\bar{\psi}^T\,, \tag{3.67}$$

one immediately understands the connection of the u and v, namely

$$\begin{aligned} v^{(s)}(p) &= i\gamma^2 u^{*(s)}(p) = C\bar{u}^{(s)T}(p) \\ u^{(s)}(p) &= i\gamma^2 v^{*(s)}(p) = C\bar{v}^{(s)T}(p)\,. \end{aligned} \tag{3.68}$$

It is easy to check that the u and v defined in this manner fulfill the same equations of motion, Eqs.(3.59). Furthermore, the u and v can be normalized such that

$$\begin{aligned} \bar{u}^{(\alpha)}(k)u^{(\beta)}(k) &= +2m\delta_{\alpha\beta} \qquad & \bar{u}^{(\alpha)}(k)v^{(\beta)}(k) &= 0 \\ \bar{v}^{(\alpha)}(k)v^{(\beta)}(k) &= -2m\delta_{\alpha\beta} \qquad & \bar{v}^{(\alpha)}(k)u^{(\beta)}(k) &= 0\,. \end{aligned} \tag{3.69}$$

Also, it can be shown that

$$\bar{u}^{(\alpha)}(k)\gamma^\mu u^{(\beta)}(k) = \bar{v}^{(\alpha)}(k)\gamma^\mu v^{(\beta)}(k) = 2k^\mu\delta_{\alpha\beta}\,. \tag{3.70}$$

Going back to the equations of motion for the u and v it can be stated that they span all solutions for which $-\not{p} = m$ and $\not{p} = m$, respectively. This allows to construct projectors on the corresponding subspaces, namely $(\pm\not{p}+m)/2m$. Therefore, using the normalization of spinors, Eqs.(3.69), the sums over spins read

$$\begin{aligned} \sum_{s=\pm} u^{(s)}(p)\bar{u}^{(s)}(p) &= \not{p}+m \\ \sum_{s=\pm} v^{(s)}(p)\bar{v}^{(s)}(p) &= \not{p}-m\,. \end{aligned} \tag{3.71}$$

The spin s can be replaced by helicities, labeling spinor states. To this end the helicity operator is defined by

$$\frac{1}{|\vec{p}|}(\vec{p}\cdot\vec{\sigma})u^{(s)}(p) = su^{(s)}(p) \tag{3.72}$$

with eigenvalues $s = \pm 1$. It is easy to check that indeed $(1 \pm \vec{n}\cdot\vec{\sigma})$ with $\vec{n} = \vec{p}/|\vec{p}|$ acts as a projector on the different helicity states. The 4×4 matrix $\vec{\sigma}$ is defined by

$$\vec{\sigma} := \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} = (\sigma^{23},\,\sigma^{31},\,\sigma^{12})\,, \tag{3.73}$$

cf. Eq.(2.108). Then, without performing the spin sum, the spinor products in Eq.(3.71) yield

$$
\begin{aligned}
u^{(s)}(p)\bar{u}^{(s)}(p) &= (\not{p}+m)\frac{1+s\vec{n}\cdot\vec{\sigma}}{2} \\
v^{(s)}(p)\bar{v}^{(s)}(p) &= (\not{p}-m)\frac{1-s\vec{n}\cdot\vec{\sigma}}{2}\,.
\end{aligned}
\tag{3.74}
$$

This allows to construct field operators for particles and anti-particles,

$$
\begin{aligned}
\psi_{\mathrm{particle}}(x) &= \int\frac{\mathrm{d}^3\vec{p}}{\sqrt{(2\pi)^3 2p_0}}\sum_{s=\pm}\Big\{b(p,\,s)u^{(s)}(p)e^{-ip\cdot x}+ \\
&\qquad d^\dagger(p,\,s)v^{(s)}(p)e^{ip\cdot x}\Big\} \\
\psi_{\mathrm{anti-particle}}(x) &= \int\frac{\mathrm{d}^3\vec{p}}{\sqrt{(2\pi)^3 2p_0}}\sum_{s=\pm}\Big\{d(p,\,s)\left[i\gamma^2 v^{*(s)}(p)\right]e^{-ip\cdot x}+ \\
&\qquad b^\dagger(p,\,s)\left[i\gamma^2 u^{*(s)}(p)\right]e^{ip\cdot x}\Big\}\,,
\end{aligned}
\tag{3.75}
$$

where

$$
\psi_{\mathrm{anti-particle}}(x) = i\gamma^2\psi^*_{\mathrm{particle}}(x)\,. \tag{3.76}
$$

The creation and annihilation operators $b(p,\,s)$ and $d^\dagger(d,\,s)$ can be obtained by projection on definite spin states. Employing the scalar product $\otimes$

$$
U\otimes\psi := \int\mathrm{d}^3x\bar{U}(x)\gamma_0\psi(x) \tag{3.77}
$$

and introducing the projectors U and V on u- and v-states, respectively,

$$
\begin{aligned}
U_{p,\,s} &:= u^{(s)}(p)\,\frac{e^{-ip\cdot x}}{\sqrt{(2\pi)^3 2p_0}} \\
V_{p,\,s} &:= v^{(s)}(p)\,\frac{e^{ip\cdot x}}{\sqrt{(2\pi)^3 2p_0}}
\end{aligned}
\tag{3.78}
$$

the operators read

$$
b(p,\,s) = U_{p,\,s}\otimes\psi_{\mathrm{particle}} \quad\text{and}\quad d^\dagger(p,\,s) = V_{p,\,s}\otimes\psi_{\mathrm{particle}}\,. \tag{3.79}
$$

This gives rise to the anti-commutators

$$
\Big\{b(p,\,s),\,b^\dagger(q,\,s')\Big\} = \Big\{d(p,\,s),\,d^\dagger(q,\,s')\Big\} = \delta_{ss'}\delta^3(\vec{p}-\vec{q})\,. \tag{3.80}
$$

Physical operators

Recall the energy-momentum tensor, cf. Eq.(2.127),

$$T^{\mu\rho} = \frac{\partial \mathcal{L}}{\partial(\partial_\rho \phi_i)} \partial^\mu \phi_i - g^{\mu\rho} \mathcal{L} \,, \tag{3.81}$$

and the fact that the momentum operator is obtained through spatial integration over the energy-momentum tensor, cf. Eq.(2.134),

$$P^\mu = \int \mathrm{d}^3 x T^{\mu 0} \,. \tag{3.82}$$

Hence

$$\begin{aligned} P^\mu &= \int \mathrm{d}^3 x \left[i\psi^\dagger \partial^\mu \psi - g^{\mu 0} \mathcal{L} \right] \\ &= \int \mathrm{d}^3 p \sum_{s=\pm 1} p^\mu \left[b^\dagger(p,\, s) b(p,\, s) - d(p,\, s) d^\dagger(p,\, s) \right] . \end{aligned} \tag{3.83}$$

Applying the anti-commutators of Eq.(3.80) on the product $dd^\dagger$ the momentum operator can also be written as

$$\begin{aligned} P^\mu &= \int \mathrm{d}^3 p \sum_{s=\pm 1} p^\mu \left[b^\dagger(p,\, s) b(p,\, s) + d^\dagger(p,\, s) d(p,\, s) \right] + \\ & \quad \int \frac{\mathrm{d}^3 p \mathrm{d}^3 x}{(2\pi)^3} \sum_{s=\pm 1} (-p^\mu) \,. \end{aligned} \tag{3.84}$$

This transformation finally motivates also Diracs idea of the particle-hole interpretation. In the expansion of the field operator for particles, Eq.(3.75) the operator $b(p,\, s)$ plays the role as annihilation operator of states with positive energy and $d^\dagger(p,\, s)$ is the creation operator for states with negative energy. Using this interpretation motivates the form of the four-momentum operator above. One could have reached this form immediately, i.e. without using he anti-commutator, by identifying $\tilde{d} = d^\dagger$ as annihilation operator for some yet unspecified states. This gives rise to

$$P^\mu = \int \mathrm{d}^3 p \sum_{s=\pm 1} p^\mu \left[b^\dagger(p,\, s) b(p,\, s) - \tilde{d}^\dagger(p,\, s) \tilde{d}(p,\, s) \right] . \tag{3.85}$$

Hence, the operator $\tilde{d}^\dagger$ becomes a creation operator for a particle with negative four-momentum $-p^\mu = (-\sqrt{\vec{p}^2 + m^2}, -\vec{p})^T$. The tricky question to answer now is how such operators act on a vacuum. To overcome these difficulties the empty vacuum $|\tilde{0}\rangle$ containing no particle or anti-particle states at all is replaced by a vacuum $|0\rangle$ that is filled with all states with negative energies, i.e.

$$|0\rangle = \left(\prod_{\text{all}p,\,s} \tilde{d}^\dagger(p,\,s\right)|\tilde{0}\rangle\,. \tag{3.86}$$

Any creation operator of a negative energy state $\tilde{d}^\dagger$ will annihilate this vacuum, in contrast any annihilation operator $\tilde{d}$ acting on this vacuum will result in a state

$$\tilde{d}(p,\,s)|0\rangle \neq 0\,. \tag{3.87}$$

On this new vacuum obviously the roles of $\tilde{d}^\dagger(p,\,s)$ and $\tilde{d}(p,\,s)$ as creation and annihilation operators are interchanged. This allows to go back to the original operators, $\tilde{d}^\dagger \to d$ and $\tilde{d} \to d^\dagger$, which are then interpreted as annihilation and creation operators, respectively. In other words, $d^\dagger(p,\,s)$ eliminates a state with negative four-momentum in the vacuum, $-p^\mu\rangle$ leading to the creation of a "hole" with positive four-momentum. In this sense, $d^\dagger$ creates holes and d annihilates them. In the framework of this interpretation, the c-number in the last line of Eq.(3.84) reflects the energy of the filled "sea". It is worth mentioning that for (super-symmetric) theories with equal numbers of bosonic and fermionic states these zero-point energies, i.e. the corresponding c-numbers, cancel each other.
The $U(1)_V$ symmetry again gives rise to a Noether current and a corresponding conserved charge, which can be obtained through

$$\begin{aligned}
Q &= \frac{q_{\text{particle}}}{2}\int \mathrm{d}^3\vec{x}\left[\psi^\dagger,\,\psi\right]\\
&= q_{\text{particle}}\int \mathrm{d}^3\vec{p}\sum_{s=\pm1}\left[b^\dagger(p,\,s)b(p,\,s) - d^\dagger(p,\,s)d(p,\,s)\right]\\
&= q_{\text{particle}}\int \mathrm{d}^3\vec{p}\sum_{s=\pm1}\left[N_s^+(p) - N_s^-(p)\right]\,,
\end{aligned} \tag{3.88}$$

where the particle's charge is q_{particle} and two number operators have been introduced. To avoid problems with c-numbers, an anti-symmetrized version of the current has been used. Instead of using such an ansatz one could have employed normal ordering, denoted by two colons ::, instead.[2] If, by definition, $b^\dagger$ creates an electron with charge $q_{\text{electron}} = -1$, then $d^\dagger$ creates

[2]The basic idea of normal ordering is the following. Using the ordinary form of the

a positron with opposite charge. Applying charge conjugation, which, as an unitary operation, translates into

$$\mathcal{C}\psi(x)\mathcal{C}^{-1} = \psi^C(x) = \mathcal{C}\bar{\psi}^T(x) \tag{3.92}$$

for operators, results in a connection of the b's and d's, namely

$$\mathcal{C}b(p,\, s)\mathcal{C}^{-1} = d(p,\, s) \quad \text{and} \quad \mathcal{C}d^\dagger(p,\, s)\mathcal{C}^{-1} = b^\dagger(p,\, s)\,. \tag{3.93}$$

Using the following properties of the charge conjugation operation,

$$\mathcal{C}\mathcal{C}^\dagger = \mathcal{C}^\dagger\mathcal{C} = 1 \quad \text{and} \quad \mathcal{C}|0\rangle = \mathcal{C}^\dagger|0\rangle = |0\rangle \tag{3.94}$$

---

charge

$$Q = \frac{q_{\text{particle}}}{2}\int \mathrm{d}^3\vec{x}\psi^\dagger\psi \tag{3.89}$$

and plugging in the b's and d's one ends up with

$$Q = q_{\text{particle}}\int \mathrm{d}^3\vec{p}\sum_{s=\pm1}\left[b^\dagger(p,\, s)b(p,\, s) + d(p,\, s)d^\dagger(p,\, s)\right]\,, \tag{3.90}$$

which leads to a physical vacuum with infinite charge due to the c-number already encountered in the momentum operator. To subtract away its contribution normal ordering is applied leading to

$$\begin{aligned}
:Q: \quad &= \quad \frac{q_{\text{particle}}}{2}\int \mathrm{d}^3\vec{x} : \psi^\dagger\psi : \\
&= \quad q_{\text{particle}}\int \mathrm{d}^3\vec{p}\sum_{s=\pm1} : \left[b^\dagger(p,\, s)b(p,\, s) + d(p,\, s)d^\dagger(p,\, s)\right] : \\
&= \quad q_{\text{particle}}\int \mathrm{d}^3\vec{p}\sum_{s=\pm1}\left[b^\dagger(p,\, s)b(p,\, s) - d^\dagger(p,\, s)d(p,\, s)\right]\,.
\end{aligned}$$

As can be read off, this normal ordering boils down to having the positive energy parts to the right of negative energy parts. This is actually the definition of this ordering prescription. The minus sign then stems from the anti-commuting property of the b's and d's. Similarly, the momentum operator can be written as

$$:P^\mu: = \int \mathrm{d}^3p\sum_{s=\pm1} p^\mu\left[b^\dagger(p,\, s)b(p,\, s) + d^\dagger(p,\, s)d(p,\, s)\right]\,,$$

leading to a positive definite (normal-ordered) Hamiltonian

$$:\mathcal{H}: = \int \mathrm{d}^3p\sum_{s=\pm1} p^0\left[b^\dagger(p,\, s)b(p,\, s) + d^\dagger(p,\, s)d(p,\, s)\right]\,. \tag{3.91}$$

yields

$$\mathcal{C}b(p,\, s)|0\rangle = d(p,\, s)|0\rangle \quad \text{and} \quad \mathcal{C}d^\dagger(p,\, s)|0\rangle = b^\dagger(p,\, s)|0\rangle\,. \tag{3.95}$$

To explicitly check the connection of spin and helicity the angular momentum density has to be employed. This will be the subject of a problem in the tutorials.

Propagators

Using the expansions for the field operators, Eq.(3.75) and the anti-commutators for the b's and d's, Eqs.(3.80) one ends up with the anti-commutators for Dirac fields

$$\left\{\psi_\alpha(x),\, \bar\psi_\beta(y)\right\} = (i\partial\!\!\!/ + m)_{\alpha\beta}\, i\Delta(x-y) =: iS_{\alpha\beta}(x-y) \tag{3.96}$$

Also, in full analogy to the scalar fields, the Dirac field and its conjugate anti-commute with themselves,

$$\{\psi_\alpha(x),\, \psi_\beta(y)\} = \left\{\bar\psi_\alpha(x),\, \bar\psi_\beta(y)\right\} = 0\,. \tag{3.97}$$

Again, the Feynman propagator for the Dirac field is defined through the vacuum expectation value of the time-ordered product,

$$\begin{aligned} S_F(x-y) &:= \langle 0|T\psi(x)\bar\psi(y)|0\rangle \\ &= \left\langle 0\left|\left[\theta(x^0-y^0)\psi(x)\bar\psi(y) - \theta(y^0-x^0)\bar\psi(y)\psi(x)\right]\right|0\right\rangle \\ &= (i\partial\!\!\!/ + m)_{\alpha\beta}\, i\Delta_F(x-y)\,. \end{aligned} \tag{3.98}$$

This leads to

$$S_F(x-y) = i\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{p\!\!\!/ + m}{p^2-m^2+i\epsilon}e^{-ip\cdot(x-y)} = i\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{e^{-ip\cdot(x-y)}}{p\!\!\!/ - m + i\epsilon}\,, \tag{3.99}$$

and again the Feynman propagator is the inverse operator of the kernel of the equation of motion,

$$(i\partial\!\!\!/ - m)S_F(x) = i\delta^4(x)\,. \tag{3.100}$$

3.2.2 Parity transformations

Parity

Consider parity transformations $\mathcal{P}$, i.e. inversion of space

$$x^\mu = (t,\,\vec{x}) \to x'^\mu := x^\mu_{\mathcal{P}} = \mathcal{P}x^\mu = (t,\,-\vec{x})\,. \tag{3.101}$$

To discuss their effect on Dirac spinors, remember that Dirac spinors are built from an un-dotted spinor in the (0, 1/2) representation of the Lorentz group and a dotted spinor in the (1/2, 0) representation. This immediately shows that the interaction of $\mathcal{P}$ with the generators of the Lorentz group has to be evaluated. In the simple notion of rotations and boosts generated by the angular momentum operator $\vec{J}$ and the boost operator $\vec{K}$ this is an easy task. Since the former carries indices for two space components, $\vec{J} = (M_{23},\,M_{31},\,M_{12})^T$, whereas the latter has only one spatial index, $\vec{K} = (M_{10},\,M_{20},\,M_{30})^T$, cf. Eq.(2.58), the former is even under parity transformations and therefore invariant whereas the latter is odd under parity. In other words,

$$\mathcal{P}\vec{J}\mathcal{P}^{-1} = \vec{J} \quad \text{and} \quad \mathcal{P}\vec{K}\mathcal{P}^{-1} = -\vec{K}\,. \tag{3.102}$$

The operators $\hat{\mathcal{X}}_i^\pm = \frac{1}{2}\left[\hat{\mathcal{R}}_i \pm i\hat{\mathcal{B}}_i\right]$ of Eq.(2.37) can be rewritten as

$$\hat{\vec{\mathcal{X}}}^\pm = \frac{1}{2}\left[\vec{J} \pm i\vec{K}\right]\,, \tag{3.103}$$

under parity transformations they behave like

$$\mathcal{P}\hat{\vec{\mathcal{X}}}^\pm\mathcal{P}^{-1} = \hat{\vec{\mathcal{X}}}^\mp\,. \tag{3.104}$$

Therefore, under parity the two representations (1/2, 0) and (0, 1/2) are swapped, resulting in

$$\mathcal{P}\xi_\alpha(x)\mathcal{P}^{-1} = \eta^{\dot{\alpha}}(x_{\mathcal{P}}) \quad \text{and} \quad \mathcal{P}\eta^{\dot{\alpha}}(x)\mathcal{P}^{-1} = \xi_\alpha(x_{\mathcal{P}})\,. \tag{3.105}$$

Returning to the constructive definition of the Dirac bi-spinor in Eq.(2.166),

$$\psi(x) = \begin{pmatrix} \xi_\alpha(x) \\ \eta^{\dot{\alpha}}(x) \end{pmatrix}\,, \tag{3.106}$$

applying the parity operation and using

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.107}$$

leads to

$$\mathcal{P}\psi(x)\mathcal{P}^{-1} = \begin{pmatrix} \eta^{\dot{\alpha}}(x_{\mathcal{P}}) \\ \xi_{\alpha}(x_{\mathcal{P}}) \end{pmatrix} = \gamma_0\psi(x_{\mathcal{P}})\,. \tag{3.108}$$

Similarly, for anti-particles,

$$\mathcal{P}\psi_{\text{anti-particle}}(x)\mathcal{P}^{-1} = -\gamma_0\psi_{\text{anti-particle}}(x_{\mathcal{P}})\,. \tag{3.109}$$

In view of the further discussion it should be stressed that Dirac-particles and their anti-particles have opposite parity, i.e. they describe states with negative parity[3]. This can be used in the following way. Applying the expansions of Eqs.(3.75) for the field operators in the equations above, the parity transformation can also be written for the individual parts. For spinors $u(p,\, s)$ with eigenvalue s of the helicity operator the parity-conjugated spinor is given by

$$\mathcal{P}u(p,\, s)\mathcal{P}^{-1} = \gamma_0 u^{(s)}(p_{\mathcal{P}})\,. \tag{3.110}$$

It is straightforward to check that it satisfies

$$(\not{p} - m)\gamma_0 u^{(s)}(p_{\mathcal{P}}) = 0 \quad\text{and}\quad \frac{\vec{p}\cdot\vec{\sigma}}{|\vec{p}|}\gamma_0 u^{(s)}(p_{\mathcal{P}}) = -s\gamma_0 u^{(s)}(p_{\mathcal{P}})\,. \tag{3.111}$$

Since the latter equation is also satisfied by $u(p,\, -s)$, and due to freedom in the choice of phase one can define

$$u(p,\, -1) = \gamma_0 u(p_{\mathcal{P}},\, +1) \tag{3.112}$$

leading to

$$\gamma_0 u(p_{\mathcal{P}},\, -1) = +u(p,\, +1)\,. \tag{3.113}$$

Employing the definition $v(p,\, s) := i\gamma_2 u^*(p,\, s)$ a similar identity can be deduced for the v-spinors,

$$\gamma_0 v(p_{\mathcal{P}},\, -1) = -v(p,\, +1)\,. \tag{3.114}$$

Hence, for the creation and annihilation operators

$$\mathcal{P}b(p,\, s)\mathcal{P}^{-1} = +b(p_{\mathcal{P}},\, -s) \quad\text{and}\quad \mathcal{P}d(p,\, s)\mathcal{P}^{-1} = -d(p_{\mathcal{P}},\, -s)\,. \tag{3.115}$$

[3]This immediately leads to some important physical consequences for bound states consisting of a Dirac-particle and an anti-Dirac-particle. The lowest s-wave states have relative orbital angular momentum of $l = 0$ and total angular momentum (total spin) of either $J = 0$ or $J = 1$. Furthermore, such two component bound states have parity $(-1)^l = 0$ multiplied with the inner parity of the particles, -1. Altogether this leads to bound states that can be classified as $J^P = 0^-$ or $J^P = 1^-$, respectively, i.e. either a pseudo-scalar or a vector boson.

The Weyl field

Free massless Dirac fields have no term in the Lagrangian, Eq.(2.169), which mixes the dotted and un-dotted spinor components. Only the Dirac mass-term $m\bar{\psi}\psi = m(\xi^\dagger\eta + \eta^\dagger\xi)$ does this job.
Therefore, one could imagine that massless fermionic fields (like, e.g. the neutrinos in the Standard Model) exist which contain only one active spinor component. Such fields are dubbed Weyl fields. Naively, their free Lagrangian would then be given by, say,

$$\mathcal{L}_W = \frac{i}{2}\eta^\dagger\sigma^\mu \overleftrightarrow{\partial_\mu}\, \eta = \frac{1}{2}\bar{\psi}_L\gamma^\mu \overleftrightarrow{\partial_\mu}\, \psi\,, \tag{3.116}$$

cf. Eq.(2.173), where the chiral spinor is given by

$$\psi_L = \frac{1-\gamma_5}{2}\psi = \left(\begin{array}{c} 0 \\ \eta^{\dot{\alpha}} \end{array}\right). \tag{3.117}$$

In contrast to the Dirac-Lagrangian, Eq.(2.169), the Weyl-Lagrangian in Eq.(3.116) is not invariant under parity. This is because the ξ-field is lacking. In addition, the Weyl-Lagrangian is also not invariant under charge-conjugation, since this operation also mixes left- and right-handed spinors. However, the combined action of $\mathcal{C}$ and $\mathcal{P}$ leaves the Weyl-Lagrangian invariant.
For Dirac spinors, their action is given by

$$\mathcal{CP}\psi(x)(\mathcal{CP})^{-1} = \mathcal{CP}\psi(x)\mathcal{P}^{-1}\mathcal{C}^{-1} = \gamma_0\mathcal{C}\bar{\psi}^T(x_\mathcal{P})\,. \tag{3.118}$$

Therefore, for each single spinor component the combined parity and charge conjugation is defined to result in

$$\begin{aligned} \mathcal{CP}\xi_\alpha(x)\mathcal{P}^{-1}\mathcal{C}^{-1} &= \epsilon^{\dot{\alpha}\dot{\beta}}\xi^*_{\dot{\beta}}(x_\mathcal{P}) \\ \mathcal{CP}\eta^{\dot{\alpha}}(x)\mathcal{P}^{-1}\mathcal{C}^{-1} &= \eta^{*\beta}(x_\mathcal{P})\epsilon_{\beta\alpha}\,. \end{aligned} \tag{3.119}$$

This shows that the Weyl-Lagrangian is $\mathcal{CP}$-invariant. The equations of motion and the anti-commutator of the Weyl-field are given by

$$i(\sigma^\mu)_{\alpha\dot{\beta}}\partial_\mu\eta^{\dot{\beta}} = 0 \quad \text{and} \quad \left\{\eta(x),\, \eta^\dagger(y)\right\} = \delta^3(\vec{x}-\vec{y})\,. \tag{3.120}$$

The base vectors $u_W(p)$ can be defined through

$$(p_0 + \vec{p}\cdot\vec{\sigma})u_W(p) = 0\,. \tag{3.121}$$

The corresponding 2×2-matrix has rank 1, the only solution obeys

$$\frac{\vec{p}\cdot\vec{\sigma}}{|\vec{p}|} u_W(p) = -u_W(p)\,. \tag{3.122}$$

This shows that this solution has helicity -1, i.e. it is a left-handed particle. In other words, a massless, chiral spinor with chirality -1 is the left-handed Weyl spinor. Its anti-particle is defined through the combined action of $\mathcal{CP}$ (remember, charge conjugation alone is not a valid option, since it does not exit for this particle),

$$\eta_{\rm anti}(x) = \mathcal{CP}\eta(x_{\mathcal{P}})\mathcal{P}^{-1}\mathcal{C}^{-1} = i\sigma^2\eta^*(x)\,. \tag{3.123}$$

This anti-particle fulfills the complex conjugate of the equation of motion, Eq.(3.120),

$$(\partial_0 + \vec{\sigma}\cdot\vec{\nabla})\eta = 0 \longrightarrow (\partial_0 - \vec{\sigma}^*\cdot\vec{\nabla})\eta^* = 0 \tag{3.124}$$

leading to, cf. Eq.(3.121)

$$(p_0 - \vec{p}\cdot\vec{\sigma})u_{W,\rm anti}(p) = 0\,. \tag{3.125}$$

and

$$\frac{\vec{p}\cdot\vec{\sigma}}{|\vec{p}|} u_{W,\rm anti}(p) = u_{W,\rm anti}(p)\,. \tag{3.126}$$

In other words, the anti-particle has helicity $+1$. Solutions with negative energy can be constructed as well, the corresponding v-spinors are obtained again through a $\mathcal{CP}$-transformation,

$$v_W(p) = i\sigma_2 u^*_{W,\rm anti}(p) \quad \text{and} \quad v_{W,\rm anti}(p) = -i\sigma_2 u^*_W(p) \tag{3.127}$$

Normalizing by $u^\dagger_W(p)u_W(p) = 2p_0$ leads to

$$u_W(p)u^\dagger_W(p) = v_W(p)v^\dagger_W(p) = p^\mu\sigma_\mu = p_0 - \vec{p}\vec{\sigma} \tag{3.128}$$

and

$$u_{W,\rm anti}(p)u^\dagger_{W,\rm anti}(p) = v_{W,\rm anti}(p)v^\dagger_{W,\rm anti}(p) = p^\mu\sigma^*_\mu = p_0 + \vec{p}\vec{\sigma}\,. \tag{3.129}$$

Applying also $v^\dagger_W(p_{\mathcal{P}})u_W(p) = 0$ the field operators can be expanded as

$$\begin{aligned}
\eta(x) &= \int \frac{{\rm d}^3p}{\sqrt{(2\pi)^3 2p_0}} \left\{b(p)u_W(p)e^{-ip\cdot x} + d^\dagger(p)v_W(p)e^{ip\cdot x}\right\} \\
\eta_{\rm anti}(x) &= \int \frac{{\rm d}^3p}{\sqrt{(2\pi)^3 2p_0}} \left\{d(p)u_{W,\rm anti}(p)e^{-ip\cdot x} + b^\dagger(p)v_{W,\rm anti}(p)e^{ip\cdot x}\right\}\,,
\end{aligned} \tag{3.130}$$

where the creation and annihilation operators have the only non-vanishing equal-time anti-commutators

$$\left\{b(p),\, b^\dagger(q)\right\} = \left\{d(p),\, d^\dagger(q)\right\} = \delta^3(\vec{p}-\vec{q})\,. \tag{3.131}$$

Since also $u_W(p_\mathcal{P})$ is a solution of the equation of motion for positive energies one might use the arbitrariness of phases and define

$$u_{W,\mathrm{anti}}(p) = u_W(p_\mathcal{P})\,. \tag{3.132}$$

Then the $\mathcal{CP}$-transformation applied on the creation and annihilation operators leads to

$$\mathcal{CP}b_W^{(\dagger)}(p)\mathcal{P}^{-1}\mathcal{C}^{-1} = d^{(\dagger)}(p_\mathcal{P}) \quad \text{and} \quad \mathcal{CP}d_W^{(\dagger)}(p)\mathcal{P}^{-1}\mathcal{C}^{-1} = b^{(\dagger)}(p_\mathcal{P})\,. \tag{3.133}$$

3.3 Quantization of the electromagnetic field

In previous sections the quantization of free scalar or Dirac fields has been discussed. The starting point in both cases were relativistic generalizations of the Schrödinger equation, namely either the Klein-Gordon equation for scalars or the Dirac equation for the fermions. Historically, these equations were basically the result of some guesswork. This is due to the fact that in both cases there was no measurable classical limit at hand, i.e. no classical field theory to give any form of guidance was available. For the electromagnetic fields, however, the situation was completely different, since the Maxwell equations were known for quite some time. Nevertheless, in spite of the existence of a well-defined and well-known classical limit, the quantization of the electromagnetic field (and any other gauge theory with spin-1 bosons) proves to be a delicate task. For the quantization procedure the vector potential is chosen, since the previous discussion of classical gauge theory already established its important role: The field strength tensor $F^{\mu\nu}$ is defined through derivatives of this potential and therefore makes a perfect candidate for a kinetic term, and also, the potential is the entity that couples with the matter fields. On the other hand, when using the electric and magnetic field $\vec{E}$ and $\vec{B}$ would give rise to derivatives of the δ-function in the equal-time commutators triggering serious doubts about the locality of the theory. However, even using the vector potential for quantizing the electromagnetic field does not cure all problems. For vector fields a new problem emerges and cries for new solutions, namely the following: The vector potential A^μ naively has four degrees of freedom, the particles described by it are the photons, which have two polarizations or two degrees of freedom only. This is an obvious mismatch and one has to convince oneself that by performing the quantization procedure one does not over-quantize the theory, i.e. that one does not quantize non-dynamical degrees of freedom. It should be stressed that this mismatch of degrees of freedom is due to an internal symmetry - as is often the case in such situations. This internal symmetry is the gauge invariance which guarantees that under transformations

$$A_\mu(x) \longrightarrow A'_\mu(x) = A_\mu(x) + \frac{1}{e}\partial_\mu\theta(x)\,, \tag{3.134}$$

the field strength tensor

$$F^{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,. \tag{3.135}$$

is invariant. In other words, whereas A^μ is gauge dependent, $F_{\mu\nu}$ is not. This freedom can be used to enforce certain relations on A^μ. For instance, by choosing

$$\Lambda(x) = -\frac{1}{e}\theta(x) = \int\limits_{-\infty}^{t} \mathrm{d}t' A^0(\vec{x},\, t') \tag{3.136}$$

the transformed potential

$$A'^{\mu}_{\Lambda}(x) = A^\mu(x) - \partial^\mu\Lambda(x) \tag{3.137}$$

has vanishing time-component, i.e. it satisfies

$$A'^{0}_{\Lambda}(x) = 0\,. \tag{3.138}$$

This choice of gauge is called the "temporal gauge". In fact, this choice does not fully fix the vector potential, since yet another gauge transformation not involving the time t could be applied, such that the spatial components change but the temporal one stays fixed. Once in the temporal gauge one could also demand that

$$\vec{\nabla}\cdot\vec{A} - \nabla^2\Lambda = 0\,. \tag{3.139}$$

The potential in this case would finally satisfy

$$A^0 = 0 \quad \text{and} \quad \vec{\nabla}\cdot\vec{A} = 0\,. \tag{3.140}$$

This is called the Coulomb gauge, which is manifestly non-covariant, i.e. Lorentz-invariance is not trivially satisfied[4]. A manifestly covariant choice is the Lorentz-gauge, given by

$$\partial_\mu A^\mu = 0\,. \tag{3.141}$$

Another popular choice are the axial choices, where

$$A_z = 0\,. \tag{3.142}$$

It should be stressed here that neither the Lorentz- nor the axial gauges fully fix the vector potential.
The dynamical degrees of freedom that have to be quantized are those that are independent of the choice of gauge, i.e. gauge invariant. Therefore one has to pick these degrees of freedom, clearly those which finally constitute $F^{\mu\nu}$. For instance, without charges one can easily set $A^0 = 0$. Since $F^{\mu\nu}$ is related to the curl of $\vec{A}$, $\vec{\nabla}\times\vec{A}$, and because the photons' polarizations are transverse (classically: in electromagnetic waves the fields are perpendicular to the direction of the field) the longitudinal component of the vector potential has vanishing curl, $\vec{\nabla}\times\vec{A}_L = 0$. Then, one is left with the two transverse degrees of freedom, exactly as demanded. This is well-known from classical theory, and has to be translated now into the quantum world. In other words, the theory has to be quantized after a gauge has been fixed, this amounts to quantizing a constrained system, which is a new feature encountered here. To highlight this, in the following the free electromagnetic fields will be quantized both in the Coulomb- and in the Lorentz-gauge.

3.3.1 Canonical quantization in the Coulomb gauge

Lagrangian, commutators, and Hamiltonian

Consider first the canonical quantization in the Coulomb gauge. Again, the quantization procedure is started by the Lagrangian for the free fields, in this case Eq.(2.206),

$$\mathcal{L}^{\text{kin.}}_{\text{gauge}} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} = \frac{1}{2}(\vec{E}^2 - \vec{B}^2)\,. \tag{3.143}$$

[4]In fact, to check Lorentz-invariance of the resulting theory is quite a task which will not be tackled in the framework of this lecture.

The conjugate momentum to the vector potential is obtained as derivative of the Lagrangian w.r.t. the time derivative of A^μ,

$$\pi_\mu(x) \;=\; \frac{\partial \mathcal{L}}{\partial(\partial_0 A^\mu)}\,. \tag{3.144}$$

Employing the antisymmetry of $F^{\mu\nu}$ one immediately realizes that $F^{00} \sim \partial_0 A^0 = 0$ and that the individual components are therefore

$$\begin{aligned}\pi_0(x) &= \frac{\partial \mathcal{L}}{\partial(\partial_0 A^0)} = 0\\ \pi_i(x) &= \frac{\partial \mathcal{L}}{\partial(\partial_0 A^i)} = -E_i\,.\end{aligned} \tag{3.145}$$

For the equal-time commutators this is bad news, dumping the commutator $[A^0,\, \pi^0]$ the remaining ones are

$$\begin{aligned}\left[A_i(t,\,\vec{x}),\, \pi_j(t,\,\vec{y})\right] &= -\left[A_i(t,\,\vec{x}),\, E_j(t,\,\vec{y})\right] = i\delta_{ij}\delta^3(\vec{x}-\vec{y})\\ \left[A_\mu(t,\,\vec{x}),\, A_\nu(t,\,\vec{y})\right] &= \left[\pi_i(t,\,\vec{x}),\, \pi_j(t,\,\vec{y})\right] = \left[E_i(t,\,\vec{x}),\, E_j(t,\,\vec{y})\right] = 0\\ \left[A_0(t,\,\vec{x}),\, \pi_i(t,\,\vec{y})\right] &= -\left[A_0(t,\,\vec{x}),\, E_i(t,\,\vec{y})\right] = 0\,.\end{aligned} \tag{3.146}$$

Of course, this way of dumping a problem looks like a cheap sidestep strategy, on top of this, by ignoring the π_0 commutators manifest Lorentz covariance is violated. The latter point translates into the need for a careful check of Lorentz invariance after the full quantization procedure has been accomplished. Because $\pi_0 = 0$, A_0 commutes with everything, just as any ordinary number does. Thus, there is no need to keep it a operator, and since it is not dynamical it can be eliminated from the Hamiltonian altogether by solving for it in terms of the other, dynamical variables. From these considerations it follows that

$$\dot{A}_0 = i\,[\mathcal{H},\, A_0] = 0 \tag{3.147}$$

leading to the Gauss' law,

$$\frac{\partial \mathcal{L}}{\partial A_0} = 0 \quad \text{and} \quad \vec{\nabla}\cdot\vec{E} = 0\,. \tag{3.148}$$

Then the Hamiltonian reads

$$\begin{aligned}\mathcal{H} &= \int \mathrm{d}^3x\,(\pi_\mu \partial_0 A^\mu - \mathcal{L})\\ &= \int \mathrm{d}^3x \left[\frac{1}{2}\left(\vec{E}^2 + \vec{B}^2\right) + \vec{E}\cdot\vec{\nabla}A^0\right],\end{aligned} \tag{3.149}$$

where for the transition from the first to the second line the identity

$$\vec{E} = -\vec{\nabla} A^0 - \partial_0 \vec{A} \tag{3.150}$$

has been used.
Imposing that $A^0 = 0$, the Hamiltonian in Coulomb gauge becomes

$$\mathcal{H} = \int \mathrm{d}^3x \left[\frac{1}{2}\left(\vec{E}^2 + \vec{B}^2\right)\right], \tag{3.151}$$

with Gauss' law, Eq.(3.148),imposed as additional constraint.
The actual implementation of this constraint is the tricky point of the procedure. For instance, imposing $\vec{\nabla} \cdot \vec{E} = 0$ as an operator equation has some unwanted effects on the commutators above. In particular,

$$[A_i(t,\, \vec{x}),\, E_j(t,\, \vec{y})] = -i\delta_{ij}\delta^3(\vec{x} - \vec{y}) \tag{3.152}$$

immediately implies that

$$\left[A_i(t,\, \vec{x}),\, \vec{\nabla} \cdot \vec{E}(t,\, \vec{y})\right] = -i\partial_i\delta^3(\vec{x} - \vec{y})\,. \tag{3.153}$$

This is bad news, since the l.h.s. of this equation would vanish if $\vec{\nabla} \cdot \vec{E} = 0$ was an operator relation, whereas in any case the r.h.s. would not vanish. There are two ways to deal with this situation. First of all one could start modifying the commutator relations, Eq.(3.146), and in particular Eq.(3.152) to make things consistent.
Instead, one could abandon the idea of $\vec{\nabla} \cdot \vec{E} = 0$ as an operator relation and impose a somewhat weaker constraint, namely

$$\vec{\nabla} \cdot \vec{E}\, |\psi\rangle = 0 \tag{3.154}$$

for all states in the Fock space of the theory. In other words, Gauss' law is imposed on the states, dividing the multitude of all states into a "physical" Fock space of the theory that satisfy Eq.(3.154) and into "unphysical" states that do not satisfy the constraint and are ignored. This clearly makes the commutator relations consistent with Gauss' law. In this way, one ends up in the temporal gauge.
Alternatively one could impose the same constraint in an even weaker version, namely as

$$\langle\psi|\, \vec{\nabla} \cdot \vec{E}\, |\psi\rangle = 0\,. \tag{3.155}$$

In this fashion the classical Maxwell equations would be only realized through averages in quantum theory. These ideas will be used later on, when the electromagnetic field is quantized in the Lorentz gauge.
After this little detour, the focus will shift back on maintaining $\vec{\nabla} \cdot \vec{E} = 0$ as an operator relation and on modifying the commutator relations accordingly. From Eq.(3.153) it is clear that the product of δ_{ij} and the δ function has to be replaced by something guaranteeing that the r.h.s. of this equation also vanishes. This is achieved by a "transverse" δ-function, i.e. by the transition

$$\delta_{ij}\delta^3(\vec{x} - \vec{y}) \longrightarrow \delta_{ij}^{\text{tr.}}(\vec{x} - \vec{y}) \,, \tag{3.156}$$

where $\partial_i \delta_{ij}^{\text{tr.}}(\vec{x} - \vec{y}) = 0$. This function reads

$$\delta_{ij}^{\text{tr.}}(\vec{x} - \vec{y}) := \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \, e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \left(\delta_{ij} - \frac{k_i k_j}{\vec{k}^2}\right) \,, \tag{3.157}$$

which clearly has the desired property. In a next step, the commutator relation is modified such that it satisfies

$$[A_i(t,\, \vec{x}),\, E_j(t,\, \vec{y})] = -i\delta_{ij}^{\text{tr.}}(\vec{x} - \vec{y}) \,. \tag{3.158}$$

It follows immediately that the operator $\vec{\nabla} \cdot \vec{E}$ also commutes with everything (it is just a number!), hence it can be set consistently to zero. In addition, Eq.(3.158) implies that

$$\left[\vec{\nabla} \cdot \vec{A}(t,\, \vec{x}),\, E_j(t,\, \vec{y})\right] = 0 \,. \tag{3.159}$$

Therefore, also $\vec{\nabla} \cdot \vec{A} = 0$ as an operator equation, so finally the Coulomb gauge condition is recovered on the level of operators.
A comment is in order here. Starting from Eq.(3.158) one might argue that the vector potential and the electric field do not commute at space-like separations, which might lead to a violation of causality. But then, the vector potential is gauge dependent, therefore it cannot be measured and one does not have to care at all about this issue. What matters is that the electric and magnetic field components do commute at space-like separations thus respecting causality. This is easy to check by either the original commutation relations, Eq.(3.152), or by the modified ones, Eq.(3.158).

Field operators and the propagator in Coulomb gauge

Coming back to the modified commutation relations to proceed with the quantization of the vector potential, and to the consequences of interpreting $\vec{\nabla} \cdot \vec{A} = 0$ as an operator equation, one immediately understands that due to the two constraints only two dynamical degrees of freedom remain to be dealt with. From $\vec{\nabla} \cdot \vec{A} = 0$ it follows that A is completely transversal, exactly as in the classical case. There, the equations of motion for the free fields read

$$\partial_\mu \partial^\mu \vec{A}(t,\, \vec{x}) = 0 \tag{3.160}$$

in the Coulomb gauge, i.e. the classical solution is provided by plane waves. This holds also true for the quantum case leading to an according expansion of the operator $\vec{A}$ (remember, still $A^0 = 0$ is to be interpreted as an operator equation),

$$\vec{A}(t,\, \vec{x}) = \int \frac{\mathrm{d}^3\vec{k}}{\sqrt{(2\pi)^3 2k_0}} \sum_{\lambda=1,\,2} \vec{\varepsilon}(\vec{k},\, \lambda) \left[a(\vec{k},\, \lambda) e^{-ik\cdot x} + a^\dagger(\vec{k},\, \lambda) e^{ik\cdot x} \right] . \tag{3.161}$$

Here, the $\vec{\varepsilon}(\vec{k},\, \lambda)$ are the two transverse orthonormal polarization vectors, for which

$$\vec{k} \cdot \vec{\varepsilon}(\vec{k},\, \lambda) = 0 \quad \text{and} \quad \vec{\varepsilon}(\vec{k},\, \lambda) \cdot \vec{\varepsilon}(\vec{k},\, \lambda') = \delta_{\lambda\lambda'} \,. \tag{3.162}$$

To ensure that this expansion satisfies the commutation relations above, the creation and annihilation operators have to satisfy the usual commutation relations

$$\begin{aligned} \left[a(\vec{k},\, \lambda)\,, a^\dagger(\vec{q},\, \lambda') \right] &= \delta^3(\vec{k} - \vec{q}) \delta_{\lambda\lambda'} \\ \left[a(\vec{k},\, \lambda)\,, a(\vec{q},\, \lambda') \right] &= \left[a^\dagger(\vec{k},\, \lambda)\,, a^\dagger(\vec{q},\, \lambda') \right] = 0\,. \end{aligned} \tag{3.163}$$

This looks very similar to two replica labeled by the polarizations λ of the scalar case. One can now choose the polarization vectors, for instance for linear polarization one has

$$\vec{\varepsilon}(\vec{k},\, 1) \times \vec{\varepsilon}(\vec{k},\, 2) = \frac{\vec{k}}{k_0 = \sqrt{\vec{k}^2}} \tag{3.164}$$

and

$$\vec{\varepsilon}(-\vec{k},\, 1) = -\vec{\varepsilon}(\vec{k},\, 1) \quad \text{and} \quad \vec{\varepsilon}(-\vec{k},\, 2) = +\vec{\varepsilon}(\vec{k},\, 2)\,. \tag{3.165}$$

From the field operators for linear polarizations one can easily construct field operators for circular polarizations by a simple base transformation,

$$a^\dagger_{R,\,L}(\vec{k}) = \frac{1}{\sqrt{2}}\left[a^\dagger(\vec{k},\,1) \pm a^\dagger(\vec{k},\,2)\right]\,, \tag{3.166}$$

in striking similarity to the the connection of two real (neutral) with one complex (charged) scalar field. In any case, the Hamiltonian in its quantized form reads

$$\mathcal{H} = \frac{\mathrm{d}^3\vec{k}}{\sqrt{(2\pi)^3 2k_0}}\left[k_0\cdot\sum_{\lambda=1,\,2} a^\dagger(\vec{k},\,\lambda)a(\vec{k},\,\lambda)\right] = \frac{\mathrm{d}^3\vec{k}}{\sqrt{(2\pi)^3 2k_0}}\left[k_0\cdot\sum_{\lambda=1,\,2} n(\vec{k},\,\lambda)\right] \tag{3.167}$$

with the number operator

$$n(\vec{k},\,\lambda) = a^\dagger(\vec{k},\,\lambda)a(\vec{k},\,\lambda)\,. \tag{3.168}$$

Furthermore, employing the analogy once more, the photon propagator in this gauge is

$$iD^{\text{tr.}}_{\mu\nu}(x,\,y) = \langle 0\,|TA_\mu(x)A_\nu(y)|\,0\rangle\,, \tag{3.169}$$

where again the time-ordering ensures that positive energies are propagated forward in time and that negative energies travel backward in time. Here, again, the superscript “tr.” indicates that this is the transverse propagator, involving field operators for which the modified commutation relations hold. Since the field operators, i.e. the vector potentials, are gauge dependent, the propagator is a gauge dependent quantity as well. Gauge invariance is then ensured only, if both the field operators, that is the polarization vectors and their sum, and the propagator are used in the same gauge. Keeping this in mind, one can use the formalism already known from the scalar and Dirac fields and ends up with

$$D^{\text{tr.}}_{\mu\nu}(x,\,y) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{e^{-ik\cdot(x-y)}}{k^2+i\epsilon}\sum_{\lambda=1,\,2}\varepsilon_\mu(k,\,\lambda)\varepsilon_\nu(k,\,\lambda)\,, \tag{3.170}$$

where, once again, the $i\epsilon$-prescription takes care of the proper time-ordering. In contrast, the spin information is contained in the polarization vectors which demand the choice of a specific frame making the propagator non-covariant.

This propagator can be cast into a from that separates frame-independent and frame-dependent pieces in a transparent fashion. Assume n^μ to be a time-like vector of unit length. The polarization vectors in contrast are space-like, again with unit length, and are therefore orthogonal to n. In other words, $n^\mu \varepsilon_\mu(k,\,\lambda) = 0$. Also, by construction k^μ is orthogonal to the polarization vectors but of course, this does not have to apply also for the helping vector n. To complete the orthogonal construction therefore the part of k orthogonal to n can be projected out by

$$k_\perp^\mu = \frac{k^\mu - (k\cdot n)n^\mu}{\sqrt{(k\cdot n)^2 - k^2}} \tag{3.171}$$

Then

$$k_\perp \cdot n = k_\perp \cdot \varepsilon(k\,,\lambda) = 0\,. \tag{3.172}$$

In this basis, the polarization part of the propagator reads

$$\sum_{\lambda=1,\,2} \varepsilon_\mu(k,\,\lambda)\varepsilon_\nu(k,\,\lambda) = -g_{\mu\nu} + n_\mu n_\nu - k_{\perp,\,\mu}k_{\perp,\,\nu}\,, \tag{3.173}$$

leading finally to

$$D^{\mathrm{tr.}}_{\mu\nu}(x,\,y) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\left[\frac{e^{-ik\cdot(x-y)}}{k^2+i\epsilon}\left(-g_{\mu\nu} - \frac{k^2 n_\mu n_\nu + (k\cdot n)(k_\mu n_\nu + k_\nu n_\mu)k_\mu k_\nu}{(k\cdot n)^2 - k^2}\right)\right] \tag{3.174}$$

for the photon propagator in Coulomb gauge. The first term above is manifestly covariant. The remaining ugly mess gives a clear idea why in general one prefers to work in gauges that are manifestly covariant.

3.3.2 Canonical quantization in the Lorentz gauge

Fixing the gauge

Having experienced the awful form of the propagator in Coulomb gauge motivates to try to quantize in a manifestly covariant fashion. Then the commutation relation

$$[A_\mu(t,\,\vec{x}),\,\pi_\nu(t,\,\vec{y})] = ig_{\mu\nu}\delta^3(\vec{x}-\vec{y}) \tag{3.175}$$

has to be imposed. This, however, is a tricky point. Remember that the conjugate momenta are given by

$$\pi_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu)} \tag{3.176}$$

which, for the free electromagnetic Lagrangian of Eq.(2.206),

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \tag{3.177}$$

leads to a vanishing temporal component of the conjugate momentum due to the anti-symmetry of the field strength tensor. Hence, in a first step this Lagrangian has to be altered. Adding a Lagrange multiplier to live up for the extra constraint the Lagrangian of the free electromagnetic field reads

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{\alpha}{2} (\partial \cdot A)^2 \,, \tag{3.178}$$

from which the conjugate momenta can be deduced,

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu)} = F^{\mu 0} - \alpha g^{\mu 0} (\partial \cdot A) \,. \tag{3.179}$$

This leaves the spatial components unchanged, i.e. $\pi_i = -E_i$, but in addition the previously zero temporal component now is $\pi_0 = -\alpha(\partial \cdot A)$. Maxwell's equations are

$$\partial_\mu \partial^\mu A_\nu - (1-\alpha)\partial_\nu(\partial \cdot A) = 0 \,. \tag{3.180}$$

With the nice choice $\alpha = 1$ dubbed Feynman gauge the Maxwell equations simplify to plane-wave equations, and this choice is made for the rest of this section[5].

As before in the case of Coulomb gauge one has to tackle the problem of over-quantizing the theory. Now, because of the commutation relations above, there are four degrees of freedom, out of which only two are dynamical. In other words in the following one has to get rid of two degrees of freedom which are not dynamical. In contrast to the case discussed before, however,

[5]Of course, any physical observable better does not depend on this choice, and in principle one would have to quantize now with α unspecified and check for this independence at the end. Only to keep things short and to focus on the underlying concepts rather than lengthy calculations the discussion now sticks to this choice.

the option to modify the commutation relations is not valid any longer, since already the Lagrangian has been changed. The constraint $\partial \cdot A = 0$ can not be imposed as an operator equation, because then the conjugate momentum $\pi_0 = -\partial \cdot A$ would vanish and the full point of adding the extra term in the action was to avoid this. Hence, the only option left is to consider the weaker constraints.
Starting with the analogy of Eq.(3.154),

$$\partial \cdot A \, |0\rangle = 0 \tag{3.181}$$

one immediately realizes that this is in contradiction with the commutators. Consider the case of $\mu = \nu = 0$. Then the expectation value of the commutator of Eq.(3.175) sandwiched between a state $|\psi\rangle$ satisfying Eq.(3.181) reads

$$\left\langle \psi \left| \left[A^0(\vec{x},\, t),\, \pi^0(\vec{y},\, t) \right] \right| \psi \right\rangle = -i\delta^3(\vec{x} - \vec{y}) \, \langle \psi | \psi \rangle \; . \tag{3.182}$$

Because $\pi_0 = -\partial \cdot A$, the l.h.s. of this equation vanishes, whereas the r.h.s. does not. This, again, is bad news.
Therefore one is lead to try an ansatz analogous to Eq.(3.155), i.e. to the satisfaction of the constraint only as an average,

$$\langle 0 \, | \partial \cdot A | \, 0 \rangle = 0 \, . \tag{3.183}$$

To really implement this constraint into the quantization procedure, one can impose a condition to the positive energies only, namely

$$\partial \cdot A^{(+)} \, |0\rangle = 0 \tag{3.184}$$

leading to

$$\begin{aligned} \langle 0 \, | \partial \cdot A | \, 0 \rangle &= \left\langle 0 \left| \left(\partial \cdot A^{(-)} + \partial \cdot A^{(+)} \right) \right| 0 \right\rangle \\ &= \left\langle 0 \left| \left(\partial \cdot A^{(-)} \right) \right| 0 \right\rangle = \left(\partial \cdot A^{(+)} \, |0\rangle \right)^\dagger |0\rangle = 0 \, . \end{aligned} \tag{3.185}$$

Field operators and the propagator in Lorentz gauge

As seen before, for $\alpha = 1$ the individual components A_μ of the vector potential satisfy plane-wave equations, hence they are expanded as plane waves,

$$A_\mu(x) = \int \frac{\mathrm{d}^3\vec{k}}{\sqrt{(2\pi)^3 2k_0}} \sum_{\lambda=0}^{3} \left[a(k,\, \lambda)\varepsilon_\mu(k,\, \lambda)e^{-ik\cdot x} + a^\dagger(k,\, \lambda)\varepsilon^*_\mu(k,\, \lambda)e^{ik\cdot x} \right] \, , \tag{3.186}$$

where, again, the ε are polarization vectors. For $\lambda = 1, 2$ the transverse polarization vectors representing the physical polarization, introduced in the previous section can be used. They are characterized by $\varepsilon_\mu(k,\, 1)k^\mu = \varepsilon_\mu(k,\, 2)k^\mu = 0$ and by $\varepsilon_0(k,\, 1) = \varepsilon_0(k,\, 1) = 0$. The polarization labeled by $\lambda = 0$ is dubbed the scalar polarization and it is conventionally chosen to point along the time axis of the corresponding frame. Finally, $\lambda = 3$ is called the longitudinal polarization, and it is convention to let the corresponding polarization vector point into the same direction as the momentum of the photon. For example, if $\vec{k}$ points along the z-axis, then the (linear) polarization vectors are

$$\varepsilon(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\,, \quad \varepsilon(1) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}\,, \quad \varepsilon(2) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}\,, \quad \varepsilon(3) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\,. \tag{3.187}$$

These polarization vectors satisfy the general relation

$$\varepsilon(k,\, \lambda) \cdot \varepsilon(k,\, \lambda') = g^{\lambda\lambda'}\,. \tag{3.188}$$

Using this relation together with the plane-wave expansion for the field operator it is straightforward to show that the covariant commutation relations of Eq.(3.175) will be satisfied if the creation and annihilation operators follow

$$\begin{aligned} \left[a(k,\, \lambda),\, a^\dagger(q,\, \lambda')\right] &= -\delta^3(\vec{k} - \vec{q})g^{\lambda\lambda'} \\ \left[a(k,\, \lambda),\, a(q,\, \lambda')\right] &= \left[a^\dagger(k,\, \lambda),\, a^\dagger(q,\, \lambda')\right] = 0\,. \end{aligned} \tag{3.189}$$

The Gupta-Bleuler metric

As hindsighted before, the operator $a(k,\, \lambda)$ is interpreted as annihilation operator, which, acting on the bare vacuum $|0\rangle$ leads to

$$a(k,\, \lambda)|0\rangle = 0\,. \tag{3.190}$$

In analogy to the scalar case then the Hamiltonian is given by

$$\mathcal{H} = \int \frac{\mathrm{d}^3\vec{k}}{\sqrt{(2\pi)^3 2k_0}} k_0 \left[\sum_{\lambda=0}^{3} -g^{\lambda\lambda} a^\dagger(k,\, \lambda)a(k,\, \lambda)\right]\,. \tag{3.191}$$

But wait ! This looks like a serious problem. The sign in front of the $\lambda = 0$ polarization piece, $-k_0 a^\dagger(k,\, 0) a(k,\, 0)$, implies that naively the energy spectrum of this Hamiltonian is unbounded from below, i.e. that negative energies are possible, leading to states with a negative norm. Such states are called ghost states. By insisting on manifest Lorentz invariance, the metric tensor $g^{\lambda\lambda'}$ appeared on the r.h.s. of the commutators, which for scalar polarizations, lead to the unwanted extra sign. This sign reverses the role of creation and annihilation operators. To dwell on this a little more, consider a state consisting of one scalar photon,

$$|1_S\rangle = \int \frac{\mathrm{d}^3\vec{k}}{\sqrt{(2\pi)^3 2k_0}} f(k) a^\dagger(k,\, 0) |0\rangle \ . \tag{3.192}$$

The norm of this state is given by

$$\begin{aligned} \langle 1_S | 1_S \rangle &= \int \frac{\mathrm{d}^3\vec{k}}{\sqrt{(2\pi)^3 2k_0}} \int \frac{\mathrm{d}^3\vec{k}'}{\sqrt{(2\pi)^3 2k_0'}} f(k) f(k') \left\langle 0 \left| a(k',\, 0) a^\dagger(k,\, 0) \right| 0 \right\rangle \\ &= -\langle 0 | 0 \rangle \int \frac{\mathrm{d}^3\vec{k}}{\sqrt{(2\pi)^3 2k_0}} \left| f(k) \right|^2 < 0 \,, \end{aligned} \tag{3.193}$$

i.e. it is negative. The preliminary conclusion is that the enforced Lorentz invariance manifest by the occurrence of $g^{\lambda\lambda'}$ leads to a mixture of states with positive or negative norm and thus destroys any probability interpretation. It looks as if quantization has failed miserably. This conclusion, however, is premature, since still the constraint from Eq.(3.184),

$$\partial \cdot A^{(+)} |0\rangle = 0 \,, \tag{3.194}$$

has to be imposed. Placing the positive energy part of the expansion in Eq.(3.186) into this condition one arrives at

$$\begin{aligned} &\partial \cdot A^{(+)} \\ &\quad = -i \int \frac{\mathrm{d}^3\vec{k}}{\sqrt{(2\pi)^3 2k_0}} \left\{ \sum_{\lambda=0}^{3} a(k,\, \lambda) \ [k \cdot \varepsilon(k,\, \lambda)]\, e^{-ik\cdot x} \right\} \\ &\quad = -i \int \frac{\mathrm{d}^3\vec{k}}{\sqrt{(2\pi)^3 2k_0}} \left\{ [a(k,\, 0)\, k \cdot \varepsilon(k,\, 0) + a(k,\, 3)\, k \cdot \varepsilon(k,\, 3)]\, e^{-ik\cdot x} \right\} \,, \end{aligned} \tag{3.195}$$

where for the transition from the first to the second line the transversality of the two physical polarizations, i.e. $\varepsilon_\mu(k,\,1)k^\mu = \varepsilon_\mu(k,\,2)k^\mu = 0$ has been used. Applying this relation on physical photons, one realizes immediately that due to their four-vectors being light-like one has

$$k \cdot \varepsilon(k,\,0) = -k \cdot \varepsilon(k,\,3)\,. \tag{3.196}$$

This leads to the constraint Eq.(3.184) being translated into

$$\left[a(k,\,3) - a(k,\,0)\right]|\psi\rangle = 0 \tag{3.197}$$

for all physical states. In other words, the states in the physical Fock space are always such that the number of photons with scalar and longitudinal polarizations is balanced. This is obviously not true for the state with one scalar photon only, $|1_S\rangle$, and it is thus not too surprising that this unphysical state has a physically meaningless norm. However, one can decompose now any Fock state into a piece with physical and a piece with unphysical polarizations only, $|\psi\rangle \to |\psi_P\rangle|\psi_U\rangle$, and check the effect of the constraint in the version of Eq.(3.197) on these states. It is clear that the states with physical, tranverse polarizations only, $|\psi_P\rangle$ are not affected, and that the unphysical ones get annihilated because they must contain an equal number of scalar and longitudinal photons, $\left[a(k,\,3) - a(k,\,0)\right]|\psi_U\rangle = 0$. Then the expectation value of the Hamiltonian in any such state is

$$\langle\psi\,|\mathcal{H}|\,\psi\rangle = \int \frac{\mathrm{d}^3\vec{k}}{\sqrt{(2\pi)^3 2k_0}} k_0 \left\langle \psi_P \left| \left[\sum_{\lambda=1}^{2} a^\dagger(k,\,\lambda)a(k,\,\lambda)\right] \right| \psi_P \right\rangle\,, \tag{3.198}$$

because the unphysical pieces have been projected out in due course. Obviously now the seemingly misplaced minus sign has dropped out and the theory looks sane again. [6]

[6] It should be noted that the constraint of Eq.(3.197) does not fix the unphysical states completely. Let $|\phi\rangle$ be any state which is given as a linear combination of states with fixed numbers of unphysical photons,

$$|\phi\rangle = C_0|\phi_0\rangle + C_1|\phi_1\rangle + \dots\,,$$

where the C_i are normalizations. The expectation value of the field operator A_μ in this state is

$$\langle\phi|A_\mu(x)|\phi\rangle = C_0^* C_1 \left\langle \phi_0 \left| \int \frac{\mathrm{d}^3\vec{k}}{\sqrt{(2\pi)^3 2k_0}} e^{-ik\cdot x}\left[\varepsilon_\mu(k,\,3)a(k,\,3) + \varepsilon_\mu(k,\,0)a(k,\,0)\right] \right| \phi_1 \right\rangle + \text{c.c.}\,.$$

The propagator in Lorentz gauge

The big benefit of this tedious procedure will become manifest now. The photon propagator is

$$iD_{\mu\nu}(x,\,y) = \langle 0\,|TA_\mu(x)A_\nu(y)|\,0\rangle\ , \tag{3.199}$$

where, again, the time-ordering is responsible for the negative energy particles to travel backward in time. Using the expansion of the field operators in Lorentz gauge, Eq.(3.186) the expression above reads

$$\begin{aligned} D_{\mu\nu}(x-y) &= -g_{\mu\nu}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{e^{-ik\cdot(x-y)}}{k^2+i\epsilon} \\ &= g_{\mu\nu}\frac{1}{4i\pi^2}\frac{1}{|x-y|^2+i\epsilon}\,. \end{aligned} \tag{3.200}$$

This is identical to the manifestly covariant term of the propagator in Coulomb gauge - by implementing Lorentz-invariance one has obtained a manifestly covariant propagator.

Fixing $|\phi_0\rangle = |0\rangle$ implies, due to the constraint, that

$$|\phi_1\rangle = \int\frac{\mathrm{d}^3\vec{k}}{\sqrt{(2\pi)^3 2k_0}}f(k)\left[a^\dagger(k,\,3) - a^\dagger(k,\,0)\right]|0\rangle\ .$$

Choosing real polarization vectors along the construction outlined above, i.e. with n pointing along the time axis

$$\varepsilon_\mu(k,\,0) = n_\mu \quad\text{and}\quad \varepsilon_\mu(k,\,3) = \frac{k_\mu - n_\mu(k\cdot n)}{k\cdot n}$$

then

$$\varepsilon_\mu(k,\,0) + \varepsilon_\mu(k,\,3) = \frac{k_\mu}{k\cdot n}$$

and, provided the expectation value is meaningful at all, it reads,

$$\langle\phi|A_\mu(x)|\phi\rangle = \partial_\mu\int\frac{\mathrm{d}^3\vec{k}}{\sqrt{(2\pi)^3 2k_0}}\left[\frac{iC_o^*C_1e^{-ik\cdot x}f(k) + \text{c.c.}}{k\cdot n}\right] = \partial_\mu\Lambda(x)\,,$$

where $\Lambda(x)$ is a c-number function satisfying $\partial_\mu\partial^\mu\Lambda(x) = 0$ which can be chosen at will thus adjusting the vector $|\phi\rangle$.

Chapter 4

Interacting field theories: $\lambda\phi^4$

So far quantum field theories describing free fields have been considered. Now the focus is shifted to the by far more interesting case of interacting field theories. There, various forces, such as electromagnetic, weak or strong forces, couple different particle species. At the level of Lagrange densities this translates into adding terms that mix different and more than two fields. This mixing, of course, should not happen in a random fashion; instead symmetries and the requirements of Lorentz invariance, locality, etc. define the form of the extra terms. In principle, interacting field theories can be quantized exactly as free field theories. In practice, however, this is not a big help, since due to interactions the simple particle interpretation is lost and it is not always clear, how one can make sense of the result. This unwanted feature is a direct result of the fact that, in contrast to free field theories, exact solutions of these theories in most cases are not available. To obtain any reasonable interpretation of interacting theories, one therefore has to define first what entities should be calculated and how they are to be compared with experiment. Luckily, in particle physics, most of the experiments are accelerator-based. Hence particles can be considered as quasi-free before and after the collision has taken place. Then, one can expand around these asymptotic states.
In this chapter, starting from such asymptotic states, the in- and out-fields of a real scalar field theory and the S-matrix connecting them are constructed. A reduction formula for the S-matrix will be derived, which allows for a perturbative expansion. This expansion can be represented by Feynman graphs. In what follows then, the same program will be repeated for spinor quantum electrodynamics.

4.1 Asymptotic states and the S-matrix

As already stated in the introduction of this chapter, the name of the game in interacting field theories is to provide a particle interpretation of the quanta, and to formulate a scattering theory making use of the so defined particles. In high-energy collisions, such a construction can be obtained in quite a pictorial way. There two particles, for instance at accelerator-based experiments, are approaching each other, acting (asymptotically) like free particles. Then they collide and interact. The result of this interaction is a bunch of particles which leave the interaction region and, again, when they are far enough from each other, act like free particles. Measurements of such processes are based on cross sections, hence, the aim is to calculate them. To do so, it suffices for the moment to state that cross sections are proportional to the probability for the corresponding process to occur, i.e. to the square of the amplitude. An amplitude $\mathcal{A}$ for a process that transforms an incoming state "in" into an outgoing state "out" is clearly given by

$$\mathcal{A}_{\text{in}\to\text{out}} = \langle \text{in}|\text{out}\rangle \ . \tag{4.1}$$

Experimentally in most cases the state $|\text{in}\rangle$ is prepared and the state $|\text{out}\rangle$ is measured. Since, by construction, it was assumed that the incoming and outgoing particles are asymptotically free, it is simple to construct the two states. For situations with no bound states in the initial or final state, the $|\text{in}\rangle$ states belong to the same Fock space as the $|\text{out}\rangle$ states, and, therefore, there is an isomorphic mapping between them. This mapping, of course, depends on the interactions of the theory. It is called S-matrix, defined by

$$S|\text{in}\rangle = |\text{out}\rangle \,, \tag{4.2}$$

leading to

$$\mathcal{A}_{\text{in}\to\text{out}} = \langle \text{in}|S|\text{in}\rangle \ . \tag{4.3}$$

Assume now that the theory is formulated in terms of a field ϕ. The $|\text{in}\rangle$ states can be described by the asymptotic free fields ϕ_{in}. The $|\text{out}\rangle$ states are created by the asymptotic free fields ϕ_{out}. These two fields are connected by the S matrix as

$$\phi_{\text{out}} = S^1\,\phi_{\text{in}}\,S \,. \tag{4.4}$$

The subtlety about the S matrix is that the field that carries the $|\text{in}\rangle$ states over to the $|\text{out}\rangle$ states, maybe creating or annihilating a few fields is neither ϕ_{in} nor ϕ_{out}, but the interacting field ϕ. Unfortunately there still is no particle interpretation for this field. What can be said however, is that the propagator of a particle (or free field) is known to be the vacuum expectation value of the time-ordered product of two fields. Thus, it can be expected that the vacuum expectation values of time-ordered products of $2n$-fields ϕ,

$$\langle 0|T\left[\phi(x_1)\phi(x_2)\ldots\phi(x_{2n})\right]|0\rangle \tag{4.5}$$

will yield the amplitude for the propagation of n fields through the interaction region. This means that the S-matrix can also be expressed in terms of such propagators. The formula relating these objects is called reduction formula. Shifting the tricky points once more: to derive such a formula, a particle interpretation for the ϕ is mandatory - how else can the propagators be interpreted? Under the assumption that no bound states occur, this is not too complicated. It just is assumed that ϕ can be expanded in terms of the ϕ_{in},

$$\phi = f(\phi_{\text{in}}) \tag{4.6}$$

and that the equations of motion for ϕ can be solved perturbatively in terms of the ϕ_{in}. Through this perturbative solution a particle interpretation of ϕ in terms of ϕ_{in} is available. This also means that the time-ordered products of Eq.(4.5) can be expressed perturbatively in terms of vacuum expectation values of products of the free fields ϕ_{in}. These products can be calculated exactly. The details of this program will be given in the next sections, first for real scalar field theory and then for spinor quantum electrodynamics.

The feasibility of this program relies on the interactions to be weak, i.e. of a small coupling constant that can be used as perturbative parameter. If the coupling constant is strong, like, e.g. in QCD, then the particle interpretation breaks down and other assumptions have to be made to render perturbation theory a useful tool.

4.2 Interacting real scalar fields

4.2.1 Asymptotic fields and the S-matrix

In- and out-fields

Consider $\lambda\phi^4$-theory. Its action is given by

$$S[\phi] = \int \mathrm{d}^4x \left[\frac{1}{2}\Big(\partial_\mu\phi(x)\partial^\mu\phi(x) - m_0^2\phi^2(x)\Big) - \frac{\lambda_0}{4!}\phi^4(x)\right] \tag{4.7}$$

and the momentum conjugate to the field reads

$$\pi(x) = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi(x))} = \dot{\phi}(x)\,. \tag{4.8}$$

The Hamiltonian is

$$H = \frac{1}{2}\int \mathrm{d}^3\vec{x}\left[\pi^2(x) - \left|\vec{\nabla}\phi(x)\right|^2 + m_0^2\phi^2(x) + \frac{\lambda_0}{4!}\phi^4(x)\right]\,. \tag{4.9}$$

The only non-vanishing equal-time commutators are, from canonical quantization, given by

$$[\phi(t,\,\vec{x}),\,\pi(t,\,\vec{y})] = i\delta^3(\vec{x}-\vec{y})\,. \tag{4.10}$$

They reproduce, together with the Hamiltonian the classical equations of motion as operator equation of motion, i.e.

$$\partial^2\phi + m_0^2\phi + \frac{\lambda_0}{3!}\phi = 0\,. \tag{4.11}$$

The problem with interacting field theories can be rephrased once more, this time as a problem connected with the time-evolution of the fields ϕ. Basically, $\phi(x)$ just does not create single particles of definite position. This is due to the fact that the equal-time commutators fix the fields at $t = 0$ and allow for an expansion in plane waves. For later or earlier times, however, on has to provide a time evolution for the creation and annihilation operators. In free field theories the time evolution operator can be expressed through $\exp(\pm i\omega_k t$ for the operators $a(k)$ and $a^\dagger(k)$. In interacting field theories, the time evolution is through terms like

$$\exp\left[-iHt\right]a(k)\exp\left[iHt\right]\,. \tag{4.12}$$

There, the Hamiltonian can not be replaced by a sum of harmonic oscillators leading to the simple forms above. Again, there is thus no particle interpretation, and it is not clear how to choose a Fock space.
To achieve any progress in this situation, a number of assumptions are to be made. First of all, a vacuum $|0\rangle$ is defined which is Lorentz-invariant, non-degenerate, translationally invariant and which has zero energy w.r.t. the interaction Hamiltonian. Also, no bound states are allowed in the theory. Then each stable particle can be associated with a distinct dynamical field. In the case considered there is only one particle type and one such field, which is defined to have a measured mass m[1]. Each of the assumptions above is also true for free field theories, making them implicitly translates into the interactions being too weak to produce bound states or to trigger radical changes in the spectrum of states of the theory.
All these assumptions lead to the possibility that a particle interpretation can be borrowed from free field theory and, thus, a Fock space can be built. This is formally accomplished by the collision idea presented in the previous section. Assuming that the collision takes place at $t = 0$ the asymptotic state, asymptotically long, i.e. $t \to -\infty$ before the collision, contains free particles only. If $\phi_{\text{in}}(x)$ is a free scalar field operator, satisfying the Klein-Gordon equation

$$(\partial^2 + m^2)\phi_{\text{in}}(x) = 0 \tag{4.13}$$

then this operator creates free particles of mass m. Thinking of these particles as isolated parts of a larger system, there still remains self-interaction of these particles. Hence, using $\phi_{\text{in}}(x)$ amount to absorbing the self-interactions of free single particles. Since $\phi_{\text{in}}(x)$ is a free field the usual plane wave expansions can be employed leading to a particle interpretation. By assumption, the vacuum of the theory contains no free isolated particles, therefore

$$a_{\text{in}}(k)|0\rangle = 0\,. \tag{4.14}$$

n-body states at $t \to -\infty$ are given by

$$a^\dagger_{\text{in}}(k_1)a^\dagger_{\text{in}}(k_2)\ldots a^\dagger_{\text{in}}(k_n)|0\rangle\,. \tag{4.15}$$

These states together with the vacuum constitute the Fock space of the interacting theory. They are collectively denoted as $|\text{IN}\rangle$. Long after the collision

[1]This mass is different from the one in the Hamiltonian, to live up for shifts in the mass due to the interactions. This point will be elucidated later on.

has taken place, i.e. at $t \to \infty$, again the particles are isolated and have physical mass m. Let these particles be created and destroyed by the free scalar field $\phi_{\text{out}}(x)$. Like $\phi_{\text{in}}(x)$, $\phi_{\text{out}}(x)$ satisfies the Klein-Gordon equation, and in full analogy, the operators $a^d agger_{\text{out}}$ can be used to construct a Fock space from the vacuum.

Spectral density

Having in mind the description above, one might be tempted to write

$$\begin{aligned} \lim_{t\to-\infty} \phi(x) &= \sqrt{Z}\,\phi_{\text{in}}(x) \\ \lim_{t\to\infty} \phi(x) &= \sqrt{Z}\,\phi_{\text{out}}(x)\,, \end{aligned} \tag{4.16}$$

where the factor $\sqrt{z}$ is an a-priori unknown normalization indicating that the fields ϕ do more than just creating or destroying single particles. If $|n\rangle$ denotes an n-particle IN state such as the ones in Eq.(4.15) then

$$\langle n|\phi_{\text{in}}(x)|0\rangle = \delta_{n1}\,. \tag{4.17}$$

Assuming also that

$$\langle n|\phi(x)|0\rangle \neq 0 \quad \text{for} \quad n \neq 1 \tag{4.18}$$

it follows that Z can be determined to be

$$\langle 1|\phi(x)|0\rangle = \sqrt{Z}\,\langle 1|\phi_{\text{in}}(x)|0\rangle\,. \tag{4.19}$$

Hence, Z is expected to range between 0 and 1, and $\sqrt{Z}$ is the amplitude to create a one=-particle state from the vacuum. This looks too good to be true. and in fact the grain of salt is that Eq.(4.16) cannot be implemented as an operator equation. As usual, the underlying reason is that such an interpretation would conflict with the equal-time commutators. To understand the reason consider first the commutator of ϕ at unequal times.
The IN states are eigenstates of energy and momentum. For example, the state

$$|1\rangle = a^\dagger_{\text{in}}(k_1)|0\rangle \tag{4.20}$$

has the four-momentum eigenvalue

$$P^\mu|1\rangle = k_1^\mu|1\rangle\,, \tag{4.21}$$

where $k_1^2 - m^2 = 0$. Since they form a Fock space, the IN states are assumed to be complete,

$$1 = \sum_n |n\rangle\langle n| \,, \tag{4.22}$$

where n labels all quantum numbers, etc., needed to describe the n-body state. In the following the properties of the IN states are used to calculate

$$i\Delta'(x-y) := \langle 0|\,[\phi(x),\,\phi(y)]\,|0\rangle \tag{4.23}$$

in terms of the free scalar propagator $\Delta(x-y)$. Inserting a one into the vacuum expectation value of a product of fields translates into

$$\begin{aligned}
\langle 0|\phi(x)\phi(y)|0\rangle &= \sum_n \langle 0|\phi(x)|n\rangle\,\langle n|\phi(y)|0\rangle \\
&= \sum_n \left\langle 0|e^{iP\cdot x}\phi(0)e^{-iP\cdot x}|n\right\rangle \left\langle n|e^{iP\cdot y}\phi(0)e^{-iP\cdot y}|0\right\rangle \\
&= \sum_n e^{-ip_n\cdot(x-y)}\,\langle 0|\phi(0)|n\rangle\,\langle n|\phi(0)|0\rangle \\
&= \sum_n \left[e^{-ip_n\cdot(x-y)}\,|\langle 0|\phi(0)|n\rangle|^2\,.\right]
\end{aligned} \tag{4.24}$$

In this derivation above, a number of relations have been used. First of all, it is clear that the momentum operator generates infinitesimal translations in space-time. Therefore,

$$\phi(x) = e^{iP\cdot x}\phi(0)e^{-iP\cdot x}\,. \tag{4.25}$$

Using that the eigenvalue of P^μ is the four-momentum p_n corresponding to a state $|n\rangle$ and that the vacuum is translationally invariant, i.e. that it has no four-momentum leads to the desired result. The commutator thus reads

$$\Delta'(x-y) = -i\sum_n \left[\left(e^{-ip_n\cdot(x-y)} - e^{ip_n\cdot(x-y)}\right)|\langle 0|\phi(0)|n\rangle|^2\right]\,. \tag{4.26}$$

Inserting yet another one, this time of the form

$$1 = \int \mathrm{d}^4q\,\delta^4(p-q) \tag{4.27}$$

allows to cast δ' into a more convenient form, namely

$$\begin{aligned}
&i\Delta'(x-y) = \\
&= \sum_n \left\{\int \mathrm{d}^4q\,\left[\delta^4(p-q)\left(e^{-ip_n\cdot(x-y)} - e^{ip_n\cdot(x-y)}\right)|\langle 0|\phi(0)|n\rangle|^2\right]\right\} \\
&=: \int \frac{\mathrm{d}^4q}{(2\pi)^3}\,\rho(q)\left(e^{-iq\cdot(x-y)} - e^{iq\cdot(x-y)}\right)\,,
\end{aligned} \tag{4.28}$$

where the spectral density $\rho(q)$ is defined through

$$\rho(q) := (2\pi)^3 \sum_n \left[\delta^4(q - p_n)\, |\langle 0|\phi(0)|n\rangle|^2\right] . \tag{4.29}$$

The spectral density has a number of properties:

1. By the definition above, $\rho(q) > 0$.
2. Physical states $|n\rangle$ have positive energies, $p_n^0 > 0$, and hence the spectral density vanishes outside the forward light-cone.
3. It is Lorentz-invariant.
4. If $\phi(x)$ were just a free field, such as, e.g., $\phi_{\rm in}(x)$, then only the term for $n = 1$ in the definition would be non-zero. In this case the spectral density would be

   $$\rho(q) := (2\pi)^3 \int \mathrm{d}^4k\, \delta^4(q - k) \tag{4.30}$$

   and Eq.(4.28) would turn into the commutator of a free real scalar field, Eq.(3.30)

To express Eq.(4.28) by Eq.(3.30), the properties of the spectral density can be used to re-write it as

$$\rho(q) = \varrho(q)\Theta(q^0) . \tag{4.31}$$

Inserting one more 1 in the form of

$$1 = \int\limits_0^\infty \mathrm{d}\sigma^2\, \delta(q^2 - \sigma^2) \tag{4.32}$$

yields

$$\begin{aligned} i\Delta'(x-y) &= \int\limits_0^\infty \mathrm{d}\sigma^2 \varrho(\sigma^2) \int \frac{\mathrm{d}^4q}{(2\pi)^3} \left[\Theta(q^0)\left(e^{-iq\cdot(x-y)} - e^{iq\cdot(x-y)}\right)\right] \\ &= \int\limits_0^\infty \mathrm{d}\sigma^2 \varrho(\sigma^2)\, \Delta(x-y)|_{\sigma^2} , \end{aligned} \tag{4.33}$$

where $\Delta(x-y)|_{\sigma^2}$ describes a commutator for the mass-shell given by σ^2. This implies that the function $\varrho(\sigma^2)$ describes a distribution in (mass)^2. By our assumptions about the states, the first state, which ϕ can create, has a minimum mass of m, the single-particle mass. If m_1 is the mass-threshold, where multi particle production starts, ϕ also cannot produce a state with mass between m and m_1. Therefore,

$$\varrho(\sigma^2) \sim \delta(\sigma^2 - m^2) \quad \text{for} \quad 0 \le \sigma^2 \le m_1^2 \,. \tag{4.35}$$

Since in the example under consideration there is only one particle species the multi-particle production threshold is at $m_1 = 2m$.
By taking the derivative of $\Delta'(x-y)$ w.r.t. the time and the limit $y^0 \to x^0$ the equal-time commutators of Eq.(4.10)

$$\lim_{y^0 \to x^0} \left[\frac{\partial}{\partial x^0} \Delta'(x-y) \right] = -i \left\langle 0 \right| \left[\dot{\phi}(x),\, \phi(y) \right] \left| 0 \right\rangle = \delta^3(\vec{x} - \vec{y}) \tag{4.36}$$

This in fact is identical to

$$\lim_{y^0 \to x^0} \left[\frac{\partial}{\partial x^0} \, \Delta(x-y)|_{\sigma^2} \right] = \delta^3(\vec{x} - \vec{y}) \,. \tag{4.37}$$

Applying the same operator, $\lim_{y^0 \to x^0} \partial/\partial x^0$ to Eq.(4.33) and comparing with the two previous equations, Eqs.(4.36) and (4.37), this ultimately implies that

$$\int\limits_0^\infty \mathrm{d}\sigma^2 \, \varrho(\sigma^2) = 1 \,. \tag{4.38}$$

For the interacting field $\phi(x)$ the spectral density $\varrho(\sigma^2) \neq 0$ for $\sigma^2 \geq 4m^2$. hence, a normalization Z has to be introduced acting on the one-particle state

[2] The same reasoning can also be applied to propagators instead of commutators. Starting from

$$\Delta_F'(x-y) = \langle 0 | T \phi(x) \phi(y) | 0 \rangle$$

one immediately finds

$$\Delta_F'(x-y) = \int\limits_0^\infty \mathrm{d}\sigma^2 \varrho(\sigma^2) \, \Delta_F(x-y)|_{\sigma^2} \;, \tag{4.34}$$

where now Δ_F is the free scalar propagator.

contribution to the spectral density. Therefore, in the region $0 \leq \sigma^2 < 4m^2$,

$$\varrho(\sigma^2,\, 0 \leq \sigma^2 < 4m^2) = Z\delta(\sigma^2 - m^2)\,, \tag{4.39}$$

where

$$Z = 1 - \int\limits_{4m^2}^{\infty} \mathrm{d}\sigma^2 \varrho(\sigma^2)\,. \tag{4.40}$$

Using that $\varrho(\sigma^2)$ is positive, it follows that

$$0 \leq Z \leq 1\,. \tag{4.41}$$

If $Z = 1$ then $\Delta'(x-y) = \Delta(x-y)$, $\phi(x) = \phi_{rmin}(x)$ and the theory essentially is a free theory. Returning now to Eq.(4.16),

$$\lim_{t\to-\infty} \phi(x) = \sqrt{Z}\,\phi_{\mathrm{in}}(x)\,, \tag{4.42}$$

it can be shown that this cannot consistently be implemented as an operator equation. At equal times, the commutators imply

$$\Big\langle 0 \Big| \Big[\dot{\phi}(t,\,\vec{x}),\, \phi(t,\,\vec{y})\Big] \Big| 0 \Big\rangle = -i\delta^3(\vec{x}-\vec{y})\,\langle 0|0\rangle = -i\delta^3(\vec{x}-\vec{y})\,, \tag{4.43}$$

independently of the actual value of t. Therefore, also

$$\lim_{t\to-\infty} \Big\langle 0 \Big| \Big[\dot{\phi}(t,\,\vec{x}),\, \phi(t,\,\vec{y})\Big] \Big| 0 \Big\rangle = -i\delta^3(\vec{x}-\vec{y})\,. \tag{4.44}$$

Due to the fact that the equal-time commutators coincide for the fields ϕ and ϕ_{in}, one is lead to the conclusion that $Z = 1$. This however, implies that $\varrho(\sigma^2) \propto \delta(\sigma^2 - m^2)$, and, hence, that $\phi(x) = \phi_{\mathrm{in}}(x)$, a free field theory. The way out is to implement Eq.(4.16) in a weaker form. In analogy to what has already been experienced in the case of the free photon field, one could implement Eq.(4.16) to hold true when sandwiched between normalizable states.

S-matrix

Coming back to the primary goal, i.e. calculating scattering cross sections, it is clear one has to calculate transition amplitudes T for the scattering of an in-state into an out-state,

$$T = \langle \beta_{\mathrm{out}} | \alpha_{\mathrm{in}} \rangle\,. \tag{4.45}$$

The probability for this process to happen is proportional to $|T|^2$, and, hence, the corresponding scattering cross section also is proportional to $|T|^2$. It is worth remembering that there is in principle no real difference between the in- and the out-states, since both field theories based on them essentially are the same - a free real scalar field theory of one particle type of physical mass m. In both cases the Fock states are built by the application of the field operators $a^{(\dagger)}_{\text{in, out}}$ on the same vacuum $|0\rangle$. Due to the fact that the theories are identical, the two fock spaces are isomorphic to each other. Then, there is a mapping S relating the in- and the out-states,

$$S|\alpha_{\text{out}}\rangle = |\alpha_{\text{in}}\rangle \,, \tag{4.46}$$

or, in terms of the actual fields,

$$\phi_{\text{out}}(x) = S^{-1}\,\phi_{\text{in}}(x)\,S\,. \tag{4.47}$$

This translation, of course, will depend on the reactions inside the interaction region and, consequently, on the interacting field theory. The transition amplitude can be linked to the individual S-matrix elements by

$$T = \langle\beta_{\text{out}}|\alpha_{\text{in}}\rangle = \langle\beta_{\text{in}}|S|\alpha_{\text{in}}\rangle = \langle\beta_{\text{out}}|S|\alpha_{\text{out}}\rangle = S_{\beta\alpha}\,. \tag{4.48}$$

The definition of S, Eq.(4.46), implies that it is unitary, because

$$\delta_{\alpha\beta} = \langle\beta_{\text{in}}|\alpha_{\text{in}}\rangle = \langle\beta_{\text{out}}|S^\dagger\,S|\alpha_{\text{out}}\rangle\,. \tag{4.49}$$

This has some further consequences: since the vacuum cannot scatter off itself, S translates the vacuum into itself, up to an unmeasurable phase that can be fixed to zero,

$$1 = \langle 0|0\rangle = \langle 0|S|0\rangle\,. \tag{4.50}$$

Also, a single stable particle has no other particle to initiate a process with, hence

$$\delta_{pp'} = \langle 1_{\text{out}}(p')|1_{\text{in}}(p)\rangle = \langle 1_{\text{in}}(p')|S|1_{\text{in}}(p)\rangle = \langle 1(p')|1(p)\rangle\,. \tag{4.51}$$

Finally, since the field operators $\phi_{\text{in, out}}$ are Lorentz-scalars, it follows that the S-matrix is a Lorentz-scalar, too.

4.2.2 LSZ formula

To calculate cross sections, one has to know S matrix elements. In other words, one has to relate the S matrix to the interacting fields ϕ. this happens through the reduction formula presented in this section, which was first derived by Lehmann, Symanzik and Zimmermann. More specifically, this reduction formula relates the S-matrix to time-ordered products of the fields in the vacuum, the Green's functions of the theory. To do so, first of all the creation and annihilation operators $a_{\rm in}^{(\dagger)}(k)$ have to be re-expressed in terms of the in-fields and their conjugate momenta, $\phi_{\rm in}(x)$ and $\pi_{\rm in}(x)$,respectively, cf. Eq.(3.17),

$$\begin{aligned} a_{\rm in}(\vec{k}) &= \int {\rm d}^3\vec{x}\left[\frac{e^{-ik\cdot x}}{\sqrt{(2\pi)^3 2k_0}}\left(k_0\phi_{\rm in}(x)+i\pi_{\rm in}(x)\right)\right] \\ &= \int {\rm d}^3\vec{x}\left[\frac{e^{-ik\cdot x}}{\sqrt{(2\pi)^3 2k_0}}\overset{\leftrightarrow}{\partial_0}\,\phi_{\rm in}(x)\right]. \end{aligned} \tag{4.52}$$

Here, the relations

$$k_0 e^{ik\cdot x} = -i\frac{\partial}{\partial t}e^{ik\cdot x} \quad \text{and} \quad \pi_{\rm in}(x) = \frac{\partial}{\partial t}\phi_{\rm in}(x) \tag{4.53}$$

have been used together with the definition

$$a\overset{\leftrightarrow}{\partial_0}\,b = -(\partial_0 a)b + a(\partial_0 b)\,. \tag{4.54}$$

In the following an S-matrix element for the transition of an n-particle in-state $|\alpha_{\rm in}\rangle$ into an m-particle out-state $|\beta_{\rm in}\rangle$ are considered. The momenta for the particles contained in these states are denoted as k_1^μ, k_2^μ, $\dots k_n^\mu$ and p_1^μ, p_2^μ, $\dots p_m^\mu$, respectively. To account for proper normalization, a smearing test function has to be added explicitly, but this will be left out in the following since it is not crucial for the following.

The idea behind the reduction formula is to pull out particle one-by-one from the in- and the out-states by writing out their creation operators explicitly. These operators are expressed in terms of the corresponding in- or out-operators, obviously so, since they create particles in the asymptotic states. The in- or out-operators are independent of time, hence, putting $t\to\pm\infty$ allows to replace them by the interacting field operators. Once, all particles in the in- and out-states are pulled to field operators, one is left with the vacuum expectation value of a product of interacting fields.

Pulling out one particle

Now, the first particle will be removed from the in-state:

$$\begin{aligned}
&\langle p_1 p_2 \dots p_m;\,\mathrm{out}|k_1 k_2 \dots k_n;\,\mathrm{in}\rangle \\
&\quad = \quad \left\langle p_1 p_2 \dots p_m;\,\mathrm{out}|a^\dagger_{\mathrm{in}}(k_1)|k_2 \dots k_n;\,\mathrm{in}\right\rangle \\
&\quad = \quad \left\langle p_1 p_2 \dots p_m;\,\mathrm{out}|a^\dagger_{\mathrm{in}}(k_1) - a^\dagger_{\mathrm{out}}(k_1)|k_2 \dots k_n;\,\mathrm{in}\right\rangle \\
&\qquad\quad + \left\langle p_1 p_2 \dots p_m;\,\mathrm{out}|a^\dagger_{\mathrm{out}}(k_1)|k_2 \dots k_n;\,\mathrm{in}\right\rangle\,. \qquad (4.55)
\end{aligned}$$

The second term of this expression will be zero, unless one of the p_i equals k_1. If one of the p_i matches, this term represents a disconnected process, i.e. the overall amplitude factorizes into two disconnected pieces. Assume now, just for the moment, that $p_1 = k_1$. Then, for the second term,

$$\begin{aligned}
&\left\langle p_1 p_2 \dots p_m;\,\mathrm{out}|a^\dagger_{\mathrm{out}}(k_1)|k_2 \dots k_n;\,\mathrm{in}\right\rangle \\
&\quad = \quad (2\pi)^3 2E_1 \delta^3(p_1 - k_1)\,\langle p_2 \dots p_m;\,\mathrm{out}|k_2 \dots k_n;\,\mathrm{in}\rangle\,, \qquad (4.56)
\end{aligned}$$

where E_1 is the energy related to k_1. This shows that it is sufficient to calculate the S matrix of connected processes only. Disconnected terms can just be written as connected terms multiplied with a momentum conservation factor or another connected term. In other words: only the connected terms are non-trivial. In the following connected terms only will be considered by implicitly assuming that the process under consideration is connected.
Now the first term of Eq.(4.55), $S^c_{\alpha\beta}$ is replaced by the field operators through the identity for the creation operator, Eq.(4.52). Omitting the normalization factor $\sqrt{(2\pi)^3 2E_1}$ in the following this yields

$$\begin{aligned}
S^c_{\beta\alpha} \;&=\; -i\left\langle p_1 p_2 \dots p_m;\,\mathrm{out}\left|\left\{\int \mathrm{d}^3\vec{x}_1\left[e^{-ik_1\cdot x_1}\overleftrightarrow{\partial_0}\,(\phi_{\mathrm{in}}(x_1) - \phi_{\mathrm{out}}(x_1))\right]\right\}\right.\right. \\
&\qquad\qquad\qquad\qquad\qquad\quad \left.\left|\, k_2 \dots k_n;\,\mathrm{in}\right.\right\rangle \\
&=\; \frac{i}{\sqrt{Z}}\left(\lim_{t\to\infty} - \lim_{t\to-\infty}\right)\left\{\int \mathrm{d}^3\vec{x}_1\left[e^{-ik_1\cdot x_1}\overleftrightarrow{\partial_0}\right.\right. \\
&\qquad\qquad\qquad\qquad \left.\left.\langle p_1 p_2 \dots p_m;\,\mathrm{out}\,|\phi(x_1)|\,k_2 \dots k_n;\,\mathrm{in}\rangle\right]\right\} \qquad (4.57)
\end{aligned}$$

The last line can be cast into a more convenient form by

$$\left(\lim_{t\to\infty} - \lim_{t\to-\infty}\right)\int \mathrm{d}^3\vec{x} f(x) = \lim_{\substack{t_f\,\to\,\infty \\ t_i\,\to\,-\infty}} \int_{t_i}^{t_f} \mathrm{d}t \frac{\partial}{\partial t}\int \mathrm{d}^3\vec{x} f(x)\,. \qquad (4.58)$$

Hence, employing the fact that plane waves are a solution to the Klein-Gordon equation, one finds

$$\begin{aligned}
S^c_{\beta\alpha} & \\
= & \frac{i}{\sqrt{Z}}\int\limits_{\infty}^{\infty} \mathrm{d}^4x_1 \partial_0 \left[e^{-ik_1\cdot x_1} \overleftrightarrow{\partial_0} \langle p_1 p_2 \ldots p_m ;\, \mathrm{out} \left|\phi(x_1)\right| k_2 \ldots k_n ;\, \mathrm{in}\rangle\right] \\
= & \frac{i}{\sqrt{Z}}\int\limits_{\infty}^{\infty} \mathrm{d}^4x_1 \left(\vec{\nabla}^2 + m^2\right) e^{-ik_1\cdot x_1} \langle p_1 p_2 \ldots p_m ;\, \mathrm{out} \left|\phi(x_1)\right| k_2 \ldots k_n ;\, \mathrm{in}\rangle \\
& + \frac{i}{\sqrt{Z}}\int\limits_{\infty}^{\infty} \mathrm{d}^4x_1 \partial_0^2 e^{-ik_1\cdot x_1} \langle p_1 p_2 \ldots p_m ;\, \mathrm{out} \left|\phi(x_1)\right| k_2 \ldots k_n ;\, \mathrm{in}\rangle \ . \qquad (4.59)
\end{aligned}$$

Integrating by parts to bring ∇^2 to ϕ can be achieved by assuming local interactions and throwing away any surface terms. Thus, finally

$$S^c_{\beta\alpha} = \frac{i}{\sqrt{Z}}\int\limits_{\infty}^{\infty} \mathrm{d}^4x_1 e^{-ik_1\cdot x_1} \left(\partial^2_{x_1} + m^2\right) \langle p_1 p_2 \ldots p_m ;\, \mathrm{out} \left|\phi(x_1)\right| k_2 \ldots k_n ;\, \mathrm{in}\rangle \ . \qquad (4.60)$$

Pulling out a second particle

To pull out another particle, this time from the out-state, one has to be careful with the operator ordering,

$$\begin{aligned}
& \langle p_1 p_2 \ldots p_m ;\, \mathrm{out} \left|\phi(x_1)\right| k_2 \ldots k_n ;\, \mathrm{in}\rangle \\
& \quad = \langle p_2 \ldots p_m ;\, \mathrm{out} \left|a_{\mathrm{out}}(p_1)\phi(x_1)\right| k_2 \ldots k_n ;\, \mathrm{in}\rangle \ . \qquad (4.61)
\end{aligned}$$

Applying the same tricks as before, one ends up with

$$\begin{aligned}
& \langle p_1 p_2 \ldots p_m ;\, \mathrm{out} \left|\phi(x_1)\right| k_2 \ldots k_n ;\, \mathrm{in}\rangle \\
= & \int \mathrm{d}^3\vec{y}_1 e^{p_1\cdot y_1} \overleftrightarrow{\partial_{0,y}} \\
& \qquad \langle p_2 \ldots p_m ;\, \mathrm{out} \left|\left(\phi_{\mathrm{out}}(y_1)\phi(x_1) - \phi(x_1)\phi_{\mathrm{in}}(y_1)\right)\right| k_2 \ldots k_n ;\, \mathrm{in}\rangle \ . \qquad (4.62)
\end{aligned}$$

To make use of the asymptotic relation between the interacting and non-interacting field operators, one has to put $y_0 \to \infty$ in the first term and

$y_0 \rightarrow infty$ in the second term. Then one has $\phi_{\text{out}}(y_1)$ to the left of $\phi(x_1)$ in the first term and $\phi_{\text{in}}(y_1)$ to the right of $\phi(x_1)$ in the second term, which is nothing else but a time-ordering. Then,

$$\begin{aligned}
&\langle p_1 p_2 \dots p_m;\, \text{out} \left|\phi(x_1)\right| k_2 \dots k_n;\, \text{in}\rangle \\
&\quad = \frac{i}{\sqrt{Z}}\left(\lim_{y_1^0\to\infty} - \lim_{y_1^0\to-\infty}\right) \\
&\qquad\qquad \int \mathrm{d}^3\vec{y}_1 e^{p_1\cdot y_1}\, \overleftrightarrow{\partial}_{0,y} \langle p_2 \dots p_m;\, \text{out} \left|T\phi(x_1)\phi(y_1)\right| k_2 \dots k_n;\, \text{in}\rangle\,.
\end{aligned} \tag{4.63}$$

Repeating the procedure above allows to re-write the S-matrix element as

$$\begin{aligned}
S^c_{\beta\alpha} &= \left(\frac{i}{Z}\right)^2 \int \mathrm{d}^4x_1 \mathrm{d}^4y_1 \left[e^{-ik_1\cdot x_1 + ip_1\cdot y_1}\left(\partial^2_{y_1} + m^2\right)\left(\partial^2_{x_1} + m^2\right)\right. \\
&\qquad\qquad \left.\langle p_2 \dots p_m;\, \text{out} \left|T\phi(x_1)\phi(y_1)\right| k_2 \dots k_n;\, \text{in}\rangle\right]\,.
\end{aligned} \tag{4.64}$$

Repeating this over and over again allows to pull out all fields and to fully reduce the S-matrix element to the vacuum expectation value of a time-ordered product of the fields times some pole terms.

4.2.3 Perturbation theory

Time-evolution

To evaluate these time-ordered products of the fields they have to be re-expressed in terms of the in- or out-fields. This can be accomplished using the fact that by assumption there are no bound states in the theory, allowing to relate them with, e.g., the in-fields through an unitary operator U. This operator can be defined by

$$\phi(t,\, \vec{x}) = U^\dagger(t)\phi_{\text{in}}(t,\, \vec{x})U(t)\,. \tag{4.65}$$

Again, due to the fact that Eq.(4.16) could not be implemented as an operator equation, also this relation does not hold true on the operator level. Instead, Eq.(4.65) only makes sense when sandwiched between properly normalized states. Plugging in this equation allows to re-express the vacuum expectation values of the time-ordered products of the fields ϕ_{in} as vacuum expectation

values of time-ordered products of in-fields. In deriving this connection, any normalization factor $\sqrt{Z}$ will be suppressed; in fact, leaving it out or not is connected to the choice of whether this relation is derived for the bare or renormalized fields, see Chapter 5. Leaving these factors $\sqrt{Z}$ out to some extent hides a basic difficulty of perturbation theory: They are divergent, and so are some terms of the perturbative expansion. This finally results in the fact that the final result of the following derivation needs to be modified once more: it needs to be renormalized.
In a first step, however, $U(t)$ has to be found. This can be achieved by constructing its equation of motion, which is possible by differentiating Eq.(4.65),

$$\begin{aligned}\frac{\partial}{\partial t}\phi_{\rm in}(t,\,\vec{x}) &= \frac{\partial}{\partial t}\left[U^\dagger(t)\phi(t,\,\vec{x})U(t)\right] \\ &= \left[\frac{\partial}{\partial t}U^\dagger(t)\right]\phi(t,\,\vec{x})U(t) + U^\dagger(t)\left[\frac{\partial}{\partial t}\phi(t,\,\vec{x})\right]U(t) + U^\dagger(t)\phi(t,\,\vec{x})\left[\frac{\partial}{\partial t}U(t)\right].\end{aligned} \tag{4.66}$$

Unitarity of the U-matrix implies that

$$\frac{\partial}{\partial t}\left(UU^\dagger\right) = 0 = \frac{\partial U}{\partial t}U^\dagger + U\frac{\partial U^\dagger}{\partial t}. \tag{4.67}$$

Therefore,

$$\left[\frac{\partial}{\partial t}U^\dagger\right]U = \left[\frac{\partial}{\partial t}U^\dagger\right]UU^\dagger\phi U = \left[\frac{\partial}{\partial t}U^\dagger\right]U\phi_{\rm in}. \tag{4.68}$$

Repeating the same exercise for the last term,

$$U^\dagger\phi\left[\frac{\partial}{\partial t}U\right] = U^\dagger\phi UU^\dagger\left[\frac{\partial}{\partial t}U\right] = -\phi_{\rm in}\left[\frac{\partial}{\partial t}U\right]U^\dagger. \tag{4.69}$$

Taken everything together yields

$$\frac{\partial}{\partial t}\phi_{\rm in} = \left[\dot{U}U^\dagger,\,\phi_{\rm in}\right] + U\dot{\phi}U^\dagger. \tag{4.70}$$

Using the equation of motion for ϕ,

$$\dot{\phi}(t,\,\vec{x}) = i\left[H(\phi,\,\pi),\,\phi\right] \tag{4.71}$$

on this allows to re-express the last term of the equation above as commutator,

$$U\dot{\phi}U^\dagger = iU\,[H,\,\phi]\,U^\dagger \tag{4.72}$$

Assuming that the Hamiltonian is normal-ordered (i.e. "subtracted") and inserting $UU^\dagger$ between H and ϕ takes the equation above to

$$U\dot{\phi}U^\dagger = i\left[UHU^\dagger,\,\phi\right]\,. \tag{4.73}$$

The trick now is to realize that the interacting Hamiltonian does not depend explicitly on time. It depends on time only implicitly through the time-evolution of the fields and their momenta. Therefore

$$U(t)H(\phi,\,\pi)U^\dagger(t) = H(\phi_{\rm in},\,\pi_{\rm in})\,. \tag{4.74}$$

Inserting this back into the equation for the time-derivative of the in-field,

$$\frac{\partial}{\partial t}\phi_{\rm in} = \left[\dot{U}U^\dagger,\,\phi_{\rm in}\right] + i\left[H(\phi_{\rm in},\,\pi_{\rm in}),\,\phi_{\rm in}\right]\,, \tag{4.75}$$

leading to

$$0 = \left[\dot{U}U^\dagger,\,\phi_{\rm in}\right] + i\left[H(\phi_{\rm in},\,\pi_{\rm in}) - H_{\rm in}(\phi_{\rm in},\,\pi_{\rm in}),\,\phi_{\rm in}\right]\,, \tag{4.76}$$

where the equation of motion for the in-field,

$$\frac{\partial}{\partial t}\phi_{\rm in} = i\left[H_{\rm in}(\phi_{\rm in},\,\pi_{\rm in}),\,\phi_{\rm in}\right]\,, \tag{4.77}$$

has been used. For the real scalar field theory under consideration this Hamiltonian is the free Hamiltonian,

$$H_{\rm in}(\phi_{\rm in},\,\pi_{\rm in}) = \frac{1}{2}\int {\rm d}^3\vec{x}\left[\pi_{\rm in}^2 + |\nabla\phi_{\rm in}|^2 + m^2\phi_{\rm in}^2\right]\,. \tag{4.78}$$

Identifying

$$H(\phi_{\rm in},\,\pi_{\rm in}) - H_{\rm in}(\phi_{\rm in},\,\pi_{\rm in}) = H_{\rm I}(\phi_{\rm in},\,\pi_{\rm in})\,, \tag{4.79}$$

just the interacting part of the Hamiltonian, tells that U satisfies

$$\left[\dot{U}U^\dagger + iH_{\rm I}(\phi_{\rm in},\,\pi_{\rm in}),\,\phi_{\rm in}\right] = 0\,. \tag{4.80}$$

Actually, a similar equation emerges when starting with the momentum conjugate to ϕ. Therefore, since $\dot{U}U^\dagger + iH_\mathrm{I}$ commutes with both the field and its conjugate momentum, this expression is not an operator but just a number,

$$\dot{U}U^\dagger + iH_\mathrm{I} = f(t)\,. \tag{4.81}$$

Defining

$$H_\mathrm{I}^f = H_\mathrm{I} - f \tag{4.82}$$

allows to multiply Eq.(4.81) from the right with U and to obtain a short form,

$$\dot{U}(t) = -iH_\mathrm{I}^f(t)U(t)\,. \tag{4.83}$$

The formal solution to this equation is

$$U(t) = U(t_0) - i\int\limits_{t_0}^{t} \mathrm{d}t' H_\mathrm{i}^f(t')U(t') \tag{4.84}$$

This integral equation can be solved in a Neumann series by iteration. Let $U^{(k+1)}$ be the $k+1$st approximation. Then

$$U^{(k+1)}(t) = U(t_0) - i\int\limits_{t_0}^{t} \mathrm{d}t' H_\mathrm{i}^f(t')U^{(k)}(t') \tag{4.85}$$

Fixing $U^{(0)}(t) = U(t_0)$, which is a constant, yields

$$U^{(1)}(t) = \left(1 - i\int\limits_{t_0}^{t} \mathrm{d}t' H_\mathrm{i}^f(t')\right) U(t_0) \tag{4.86}$$

and, for the next iteration step,

$$U^{(2)}(t) = \left(1 - i\int\limits_{t_0}^{t} \mathrm{d}t_1 H_\mathrm{i}^f(t_1) + (-i)^2 i\int\limits_{t_0}^{t} \mathrm{d}t_1 \int\limits_{t_0}^{t_1} \mathrm{d}t_2 H_\mathrm{i}^f(t_1)H_\mathrm{i}^f(t_2)\right) U(t_0) \tag{4.87}$$

Assuming that the series converges leads to

$$U(t) = \sum_{n=0}^{\infty} \left[(-i)^n \int\limits_{t_0}^{t} dt_1 \int\limits_{t_0}^{t_1} dt_2 \ldots \int\limits_{t_0}^{t_{n-1}} dt_n H_{\mathrm{i}}^f(t_1) H_{\mathrm{i}}^f(t_2) \ldots H_{\mathrm{i}}^f(t_n) \right] U(t_0) \tag{4.88}$$

involving an explicit time-ordering $t > t_1 > t_2 > \ldots > t_n$ as exhibited by the integration boundaries. Reordering of the times implies reordering of the operators. Since there are $n!$ permutations to do so,

$$\begin{aligned} U(t) &= \sum_{n=0}^{\infty} \left\{ \frac{(-i)^n}{n!} \int\limits_{t_0}^{t} dt_1 \int\limits_{t_0}^{t} dt_2 \ldots \int\limits_{t_0}^{t} dt_n T \left[H_{\mathrm{i}}^f(t_1) H_{\mathrm{i}}^f(t_2) \ldots H_{\mathrm{i}}^f(t_n) \right] \right\} U(t_0) \\ &= T \left[\exp \left(-i \int\limits_{t_0}^{t} \mathrm{d}t' : H_{\mathrm{I}}^f(t') : \right) \right] U(t_0) \, . \end{aligned} \tag{4.89}$$

Multiplying on the right with $U^{-1}(\tilde{t})$ and defining

$$U(t,\, \tilde{t}) = U(t) U^{-1}(\tilde{t}) \tag{4.90}$$

results in

$$U(t,\, t) = 1 \quad \text{and} \quad U(t,\, \tilde{t}) = T \left[\exp \left(-i \int\limits_{\tilde{t}}^{t} \mathrm{d}t' : H_{\mathrm{I}}^f(t') : \right) \right] . \tag{4.91}$$

Taking appropriate limits one recovers the S-matrix,

$$\lim_{\substack{\tilde{t} \to -\infty \\ t \to \infty}} U(t,\, \tilde{t}) = S \, , \tag{4.92}$$

leading to

$$S = T \left[\exp \left(-i \int\limits_{-\infty}^{\infty} \mathrm{d}t' : H_{\mathrm{I}}^f(t') : \right) \right] . \tag{4.93}$$

Time-ordered products

In order to calculate S-matrix elements one has to calculate vacuum expectation values of time-ordered products. Since the vacuum expectation values of the interacting fields are unknown – this would involve knowledge of the spectral density, i.e., solving the interacting field theory – the vacuum expectation values of the time-ordered products cannot be computed exactly. But both these objects are known for the in- and out-fields. Hence, they can be calculated perturbatively for the interacting fields. To map the vacuum expectation values of the interacting theory onto vacuum expectation values of the free theory, the U operators will be employed. For an n-point function $G^{(n)}$ this amounts to

$$
\begin{aligned}
&G^{(n)}(x_1,\,x_2,\,\ldots,\,x_n) := \langle 0\,|T\,[\phi(x_1)\phi(x_1)\ldots\phi(x_n)]|\,0\rangle \\
&= \left\langle 0\left|T\left[U^{-1}(t_1)\phi_{\rm in}(x_1)U(t_1)\ldots U^{-1}(t_n)\phi_{\rm in}(x_n)U(t_n)\right]\right|0\right\rangle \\
&= \left\langle 0\left|T\left[U^{-1}(t_1)\phi_{\rm in}(x_1)U(t_1,\,t_2)\ldots U(t_{n-1},\,t_n)\phi_{\rm in}(x_n)U(t_n)\right]\right|0\right\rangle\,.
\end{aligned}
\tag{4.94}
$$

In a next step, some factors $1 = U^{-1}(t)U(t)$ and $1 = U^{-1}(-t)U(-t)$ are inserted with t such that $t > t_i$ and $-t < t_i$. This allows to pull out the factors $U^{-1}(t)$ and $U(t)$ out of the time-ordering,

$$
\begin{aligned}
&G^{(n)}(x_1,\,x_2,\,\ldots,\,x_n) = \\
&\quad\left\langle 0\left|U^{-1}(t)\,T\left[\phi_{\rm in}(x_1)\ldots\phi_{\rm in}(x_n)\exp\left(\int\limits_{-t}^{t}{\rm d}t' H_{\rm I}^f(t')\right)\right]\,U(t)\right|0\right\rangle\,.
\end{aligned}
\tag{4.95}
$$

What remains to be done is to evaluate the effect of the operators U at times $\pm t \to \pm\infty$ acting on the vacuum bra or ket. Using the representation of the U-matrix, Eq.(4.89), for $t \to -\infty$ one realizes that

$$
U(-\infty)|0\rangle = \alpha|0\rangle\,. \tag{4.96}
$$

This is consistent with Eq(4.16) since the U matrix is always sandwiched between physical states. Also, since the vacuum is stable, the S matrix maps the vacuum back to itself up to a phase, and one has

$$
S|0\rangle = U(\infty)U^{-1}(-\infty)|0\rangle = \beta 0\rangle\,. \tag{4.97}
$$

This choice of initial condition and phase finally yields

$$G^{(n)}(x_1,\, x_2,\, \ldots,\, x_n) = \beta^*\alpha^*\alpha \left\langle 0 \middle| T\left[\phi_{\rm in}(x_1)\ldots\phi_{\rm in}(x_n)\exp\left(\int\limits_{-\infty}^{\infty} {\rm d}t' H_{\rm I}^f(t')\right)\right]\middle| 0\right\rangle\,. \tag{4.98}$$

The phases now can be fixed by setting an unmeasurable phase to zero. Consider a 0-point function, i.e. a vacuum-to-vacuum amplitude. Requiring that the S-matrix does not result in a phase when sandwiched between the vacuum translates into

$$1 = \beta^*\alpha^*\alpha \left\langle 0 \middle| T\left[\exp\left(\int\limits_{-\infty}^{\infty} {\rm d}t' H_{\rm I}^f(t')\right)\right]\middle| 0\right\rangle\,. \tag{4.99}$$

Hence, the normalization is just $\langle 0|S|0\rangle^{-1}$. Therefore,

$$G^{(n)}(x_1,\, x_2,\, \ldots,\, x_n) = \frac{\left\langle 0 \middle| T\left[\phi_{\rm in}(x_1)\ldots\phi_{\rm in}(x_n)\exp\left(\int\limits_{-\infty}^{\infty} {\rm d}t' H_{\rm I}^f(t')\right)\right]\middle| 0\right\rangle}{\left\langle 0 \middle| T\left[\exp\left(\int\limits_{-\infty}^{\infty} {\rm d}t' H_{\rm I}^f(t')\right)\right]\middle| 0\right\rangle}\,. \tag{4.100}$$

Finally, since it shows up both in the numerator and the denominator, the arbitrary function $f(t)$ in $H_{\rm I}^f(t)$ cancels out. This leads to the final result

$$G^{(n)}(x_1,\, x_2,\, \ldots,\, x_n) = \frac{\sum\limits_{m=0}^{\infty}\frac{(-i)^m}{m!}\left(\int\limits_{-\infty}^{\infty}\prod\limits_{i=1}^{m}{\rm d}^4y_i\right)\langle 0\left|T\left[\phi_{\rm in}(x_1)\ldots\phi_{\rm in}(x_n)\mathcal{H}_{\rm I}(y_1)\ldots\mathcal{H}_{\rm I}(y_m)\right]\right|0\rangle}{\sum\limits_{m=0}^{\infty}\frac{(-i)^m}{m!}\left(\int\limits_{-\infty}^{\infty}\prod\limits_{i=1}^{m}{\rm d}^4y_i\right)\langle 0\left|T\left[\mathcal{H}_{\rm I}(y_1)\ldots\mathcal{H}_{\rm I}(y_m)\right]\right|0\rangle}\,, \tag{4.101}$$

where $\mathcal{H}_{\rm I}(y) = \mathcal{H}_{\rm I}(\phi_{\rm in}(y))$ is the normal ordered Hamiltonian density, such that

$$H_{\rm I}(t) = \int {\rm d}^3\vec{x}\mathcal{H}(x)\,. \tag{4.102}$$

Even though it is called vacuum, the vacuum in a quantum field theory is a busy and complicated state; it just seems to be the vacuum because the fluctuations populating it by themselves cannot be measured. To manifest themselves, the fluctuations have to interact with something else, like, for instance, a particle crossing the vacuum. As such a particle propagates, it interacts with vacuum fluctuations giving rise to something coined (maybe falsely) self-interaction. This nomenclature is a reminder of the lacking capability to analyse an interacting field theory without by-passing the difficulties of the interactions through a perturbative expansion. Fluctuations, however, that do not interact should not affect any result, this is why they are cancelled out through the normalization.

Wick's theorem

Each term in the expansion of Eq.(4.101) can be represented by a graph or diagram. In fact,simple rules allow to reconstruct the perturbative term from the diagram alone, and this feature makes the Feynman diagrams a useful tool for perturbation theory. In other words, drawing the diagrams and evaluating the corresponding amplitudes is as good as formally manipulating the equation.
To do the calculation one has to analyze the time-ordered products of in-fields. A simple way to reduce such a calculation is to take into account that

$$a_{\rm in}(k)|0\rangle = 0\,. \tag{4.103}$$

In other words, having all annihilation operators to the right of all creation operators reduces a product to zero. This is exactly what normal-ordering does, it orders all positive frequency parts of an operator to the right of its negative frequency parts. To put into normal ordering a product of two fields, consider

$$\begin{aligned} :\phi_{\rm in}(x_1)\phi_{\rm in}(x_2): \; = \; & \phi^{(+)}_{\rm in}(x_1)\phi^{(+)}_{\rm in}(x_2) + \phi^{(-)}_{\rm in}(x_1)\phi^{(-)}_{\rm in}(x_2) + \\ & \phi^{(-)}_{\rm in}(x_1)\phi^{(+)}_{\rm in}(x_2) + \phi^{(-)}_{\rm in}(x_2)\phi^{(+)}_{\rm in}(x_1)\,, \end{aligned} \tag{4.104}$$

and add and subtract $\phi^{(+)}_{\rm in}(x_1)\phi^{(-)}_{\rm in}(x_2)$. Then,

$$\phi_{\rm in}(x_1)\phi_{\rm in}(x_2) =: \phi_{\rm in}(x_1)\phi_{\rm in}(x_2) : + \left[\phi^{(+)}_{\rm in}(x_1),\, \phi^{(-)}_{\rm in}(x_2)\right]\,. \tag{4.105}$$

Time-ordered and normal-ordered products are therefore connected through

$$
\begin{aligned}
T\left[\phi_{\rm in}(x_1)\phi_{\rm in}(x_2)\right] &= \left\{\begin{array}{ccc} \phi_{\rm in}(x_1)\phi_{\rm in}(x_2) & \text{for} & t_2<t_1 \\ \phi_{\rm in}(x_2)\phi_{\rm in}(x_1) & & t_1<t_2 \end{array}\right. \\
&= \left\{\begin{array}{ccc} :\phi_{\rm in}(x_1)\phi_{\rm in}(x_2): + \left[\phi^{(+)}_{\rm in}(x_1),\, \phi^{(-)}_{\rm in}(x_2)\right] & \text{for} & t_2<t_1 \\ :\phi_{\rm in}(x_1)\phi_{\rm in}(x_2): + \left[\phi^{(+)}_{\rm in}(x_2),\, \phi^{(-)}_{\rm in}(x_1)\right] & & t_1<t_2 \end{array}\right.
\end{aligned}
\tag{4.106}
$$

The commutators above can explicitly be calcualted by using the plane-wave expansion of the in-fields. Using

$$
\phi^{(\pm)}_{\rm in}(x) = \int \frac{{\rm d}^3\vec{k}}{(2\pi)^3 2\omega_k} a^{(,\dagger)}_{\rm in}(k) e^{\mp ik\cdot x}
\tag{4.107}
$$

one ends up with

$$
\begin{aligned}
\left[\phi^{(+)}_{\rm in}(x_1),\, \phi^{(-)}_{\rm in}(x_2)\right] &= \int \frac{{\rm d}^3\vec{k}}{(2\pi)^3 2\omega_k} \int \frac{{\rm d}^3\vec{k}'}{(2\pi)^3 2\omega_{k'}} \left[a_{\rm in}(k),\, a^\dagger_{\rm in}(k')\right] e^{-ik\cdot x_1 + ik'\cdot x_2} \\
&= \int \frac{{\rm d}^3\vec{k}}{(2\pi)^3 2\omega_k} e^{-ik\cdot(x_1-x_2)}\,.
\end{aligned}
\tag{4.108}
$$

Therefore,

$$
\begin{aligned}
&T\left[\phi_{\rm in}(x_1)\phi_{\rm in}(x_2)\right] = \\
&\quad = \; :\phi_{\rm in}(x_1)\phi_{\rm in}(x_2): + \\
&\qquad\quad \int \frac{{\rm d}^3\vec{k}}{(2\pi)^3 2\omega_k} \left[e^{-ik\cdot(x_1-x_2)}\Theta(t_1-t_2) + e^{-ik\cdot(x_2-x_1)}\Theta(t_2-t_1)\right] \\
&\quad = \; :\phi_{\rm in}(x_1)\phi_{\rm in}(x_2): + \langle 0\left|T\left[\phi_{\rm in}(x_1)\phi_{\rm in}(x_2)\right]\right|0\rangle \\
&\quad = \; :\phi_{\rm in}(x_1)\phi_{\rm in}(x_2): + i\Delta_F(x_1-x_2)
\end{aligned}
\tag{4.109}
$$

Alternatively, one could have noticed that the only difference of time- and normal-ordering of two scalar fields is the commuator of these two scalar fields, i.e. an ordinary function.

In a next step, consider the product of three fields. An expression for it through normal ordered products can be obtained simply by multiplying the product of two fields from the right with a third one,

$$
\begin{aligned}
&\phi_{\rm in}(x_1)\phi_{\rm in}(x_2)\phi_{\rm in}(x_3) = \\
&\qquad :\phi_{\rm in}(x_1)\phi_{\rm in}(x_2):\phi_{\rm in}(x_3) + \left[\phi^{(+)}_{\rm in}(x_1),\, \phi^{(-)}_{\rm in}(x_2)\right]\,\phi_{\rm in}(x_3) \\
&\qquad :\phi_{\rm in}(x_1)\phi_{\rm in}(x_2):\left(\phi^{(+)}_{\rm in}(x_3) + \phi^{(-)}_{\rm in}(x_3)\right) + \left[\phi^{(+)}_{\rm in}(x_1),\, \phi^{(-)}_{\rm in}(x_2)\right]\,\phi_{\rm in}(x_3)
\end{aligned}
\tag{4.110}
$$

Since the commutator is just a function, the second term implicitly is already normal ordered. For obvious reasons, the same holds true also for the $\phi^{(+)}_{\rm in}$-piece of the first term. What remains to be done is to cast the other, the $\phi^{(-)}_{\rm in}$-piece into normal-ordered form. The only part that needs some manipulations is of the form

$$
\left(\phi^{(+)}_{\rm in}(x_1)\phi^{(-)}_{\rm in}(x_2)\phi^{(-)}_{\rm in}(x_1)\phi^{(+)}_{\rm in}(x_2)\phi^{(-)}_{\rm in}(x_2)\phi^{(+)}_{\rm in}(x_1)\right)\,\phi^{(-)}_{\rm in}(x_3)\,. \tag{4.111}
$$

Again, adding and subtracting of tricky pieces is the cure to this. Such a procedure leads to

$$
\begin{aligned}
&\phi_{\rm in}(x_1)\phi_{\rm in}(x_2)\phi_{\rm in}(x_3) = \\
&\qquad :\phi_{\rm in}(x_1)\phi_{\rm in}(x_2)\phi_{\rm in}(x_3): + \left[\phi^{(+)}_{\rm in}(x_1),\, \phi^{(-)}_{\rm in}(x_2)\right]\,\phi_{\rm in}(x_3) \\
&\qquad + \left[\phi^{(+)}_{\rm in}(x_2),\, \phi^{(-)}_{\rm in}(x_3)\right]\,\phi_{\rm in}(x_1) + \left[\phi^{(+)}_{\rm in}(x_1),\, \phi^{(-)}_{\rm in}(x_3)\right]\,\phi_{\rm in}(x_2)
\end{aligned}
\tag{4.112}
$$

Therefore,

$$
\begin{aligned}
&T\left[\phi_{\rm in}(x_1)\phi_{\rm in}(x_2)\phi_{\rm in}(x_3)\right] = \\
&\qquad :\phi_{\rm in}(x_1)\phi_{\rm in}(x_2)\phi_{\rm in}(x_3): + \\
&\qquad i\Delta(x_1-x_2)\phi_{\rm in}(x_3) + i\Delta(x_2-x_3)\phi_{\rm in}(x_3) + i\Delta(x_1-x_3)\phi_{\rm in}(x_2)
\end{aligned}
\tag{4.113}
$$

Its vacuum expectation value vanishes.

Moving on to products of four fields is straightforward; just multiply the last equation from the right with $\phi_{\rm in}(x_4)$. Then, the second term of the result reads

$$
i\Delta(x_1-x_2)\phi_{\rm in}(x_3)\phi_{\rm in}(x_4) = i\Delta(x_1-x_2)\left[:\phi_{\rm in}(x_3)\phi_{\rm in}(x_4): + i\Delta(x_3-x_4)\right]\,, \tag{4.114}
$$

and, similarly, the other two terms involving the propagators can be cast into a form involving two propagators or a normal-oprdered product of two fields and a propagator of the other two. The only problem left to be solved is the first term. Multiplying from the right with $\phi_{\rm in}(x_4)$ and repeating previous steps results in

$$\begin{aligned}
&:\phi_{\rm in}(x_1)\phi_{\rm in}(x_2)\phi_{\rm in}(x_3):\phi_{\rm in}(x_4) = \\
&\quad :\phi_{\rm in}(x_1)\phi_{\rm in}(x_2)\phi_{\rm in}(x_3)\phi_{\rm in}(x_4): \\
&\quad + :\phi_{\rm in}(x_1)\phi_{\rm in}(x_2):\, T\left[\phi^{(+)}_{\rm in}(x_3),\, \phi^{(-)}_{\rm in}(x_4)\right] \\
&\quad + :\phi_{\rm in}(x_1)\phi_{\rm in}(x_3):\, T\left[\phi^{(+)}_{\rm in}(x_2),\, \phi^{(-)}_{\rm in}(x_4)\right] \\
&\quad + :\phi_{\rm in}(x_2)\phi_{\rm in}(x_3):\, T\left[\phi^{(+)}_{\rm in}(x_1),\, \phi^{(-)}_{\rm in}(x_4)\right]
\end{aligned} \tag{4.115}$$

Sticking everything together allows to calculate a time-ordered product of four fields. It reads

$$\begin{aligned}
&T\left[\phi_{\rm in}(x_1)\phi_{\rm in}(x_2)\phi_{\rm in}(x_3)\phi_{\rm in}(x_4)\right] = \\
&\quad :\phi_{\rm in}(x_1)\phi_{\rm in}(x_2)\phi_{\rm in}(x_3)\phi_{\rm in}(x_4): \\
&\quad +\sum_\sigma \left[:\phi_{\rm in}(x_{\sigma(1)})\phi_{\rm in}(x_{\sigma(2)}):\, i\Delta_F(x_{\sigma(3)}-x_{\sigma(4)})\right] \\
&\quad +(i)^2\left[\Delta_F(x_1-x_2)\Delta_F(x_3-x_4)+\Delta_F(x_1-x_3)\Delta_F(x_2-x_4)+\right. \\
&\qquad \left.\Delta_F(x_1-x_4)\Delta_F(x_2-x_3)\right]\,,
\end{aligned} \tag{4.116}$$

where the sum is over all permutations with $\sigma(1) < \sigma(2)$ and $\sigma(3) < \sigma(4)$. Taking the vacuum expectation value then leaves only the last term involving the sum over products of two propagators.

Going to more and more fields and relating time- and normal-ordered products proceeds through induction. The result is known as Wick's theorem,

$$\begin{aligned}
&T\left[\phi_{\rm in}(x_1)\dots\phi_{\rm in}(x_n)\right] \\
&\quad = \;:\phi_{\rm in}(x_1)\dots\phi_{\rm in}(x_n): \\
&\qquad +\sum_\sigma\left[i\Delta_F(x_{\sigma(1)}-x_{\sigma(2)})\;:\phi_{\rm in}(x_{\sigma(3)})\dots\phi_{\rm in}(x_{\sigma(n)}):\right] \\
&\qquad +\sum_\sigma\left[i\Delta_F(x_{\sigma(1)}-x_{\sigma(2)})\Delta_F(x_{\sigma(3)}-x_{\sigma(4)})\;:\phi_{\rm in}(x_{\sigma(5)})\dots\phi_{\rm in}(x_{\sigma(n)}):\right] \\
&\qquad +\dots \\
&\qquad +\left\{\begin{array}{ll} \sum\limits_\sigma\left[i\Delta_F(x_{\sigma(1)}-x_{\sigma(2)})\dots i\Delta_F(x_{\sigma(n-1)}-x_{\sigma(n)})\right] & \text{for}\quad \sigma=\text{even} \\ 0 & \text{for}\quad \sigma=\text{odd}\end{array}\right.
\end{aligned} \tag{4.117}$$

For vacuum expectation values of time-ordered products, Wick's theorem reads

$$
\begin{aligned}
&\langle 0 \,| T\left[\phi_{\rm in}(x_1)\ldots\phi_{\rm in}(x_n)\right] |\, 0\rangle \\
&= \left\{ \begin{array}{ll} \sum\limits_\sigma \left[i\Delta_F(x_{\sigma(1)}-x_{\sigma(2)})\ldots i\Delta_F(x_{\sigma(n-1)}-x_{\sigma(n)})\right] & \text{for} \quad \sigma = \text{even} \\ 0 & \text{for} \quad \sigma = \text{odd} \end{array} \right.
\end{aligned}
\tag{4.118}
$$

Again the permutations are such that $\sigma(1) < \sigma(2)$, $\sigma(3) < \sigma(4)$, ...

4.3 Feynman diagrams

4.3.1 Feynman rules

Adding in the interactions

Coming back to Wick's theorem, Eqs.(4.117) and (4.118) it can be seen that the time-ordered products of a number of fields can be re-expressed through the product of propagators, summed over all ordered permutations. The appearance of such a propagator is called a contraction of the two corresponding fields; in the vacuum expectation value of the n fields above, each of the fields is contracted $n-1$ times. Now, the terms to be evaluated are time-ordered products of in-fields stemming from pulling out the particles of the external states, and of in-fields stemming from the normal-ordered interaction Hamiltonian. Time-ordering an entire product of fields, no contractions will emerge of those fields that already appear inside a :: symbol. Specifically, apart from the examples already encountered,

$$
\begin{aligned}
&\langle 0 \,| T\left[:\phi_{\rm in}(x_1)\phi_{\rm in}(x_2) :: \phi_{\rm in}(x_3)\phi_{\rm in}(x_4):\right] |\, 0\rangle \\
&= \; i\Delta_F(x_1-x_3) i\Delta_F(x_2-x_4) + i\Delta_F(x_1-x_4) i\Delta_F(x_2-x_3)\,.
\end{aligned}
\tag{4.119}
$$

Neither the contraction of fields at x_1 and x_2 nor the contraction at x_3 and x_4 shows up. For the case of time-ordered products of fields and factors of the interaction Hamiltonian, the contractions that show up are limited by the normal ordering in the Hamiltonian. For a diagrammtic representation of the according expansion, each contraction or propagator is shown as a straight line, common endpoints of lines initiated by an interaction term are represented as a blob. For the time-ordered product of the four fields

$$
\begin{array}{c} 3 \text{———} 4 \\ 1 \text{———} 2 \end{array} + \begin{array}{c} 2 \text{———} 4 \\ 1 \text{———} 3 \end{array} + \begin{array}{c} 2 \text{———} 3 \\ 1 \text{———} 4 \end{array}
$$

Figure 4.1: The three surviving terms of Eq.(4.116) in diagrammatic representation. They constitue the first term of the series expansion of the four point function, see below.

of Eq.(4.116), there are therefore three diagrams, exhibited in Fig.4.1. The interaction Hamiltonian was not present so far, so this is just a free theory result. This is shown graphically by the fact that no two lines cross each other. The next term in the numerator of the four point function is

$$
-i \int_{-\infty}^{\infty} \mathrm{d}y \, \langle 0 \left| T \left[\phi_{\text{in}}(x_1) \ldots \phi_{\text{in}}(x_4) \mathcal{H}_{\text{I}}(y) \right] \right| 0 \rangle \, , \tag{4.120}
$$

where

$$
\mathcal{H}_{\text{I}}(y) = \frac{1}{2} \delta m^2 : \phi_{\text{in}}^2(y) : + \frac{\lambda_0}{4!} : \phi_{\text{in}}^4(y) : \, . \tag{4.121}
$$

Forgetting for a moment about the mass-like term, one ends up with

$$
\begin{aligned}
& - \frac{i\lambda_0}{4!} \int_{-\infty}^{\infty} \mathrm{d}y \left\langle 0 \left| T \left[\phi_{\text{in}}(x_1) \phi_{\text{in}}(x_2) \phi_{\text{in}}(x_3) \phi_{\text{in}}(x_4) : \phi_{\text{in}}^4(y) : \right] \right| 0 \right\rangle \\
= & \; -i\lambda_0 \int_{-\infty}^{\infty} \mathrm{d}y \Delta_F(x_1 - y) \Delta_F(x_2 - y) \Delta_F(x_3 - y) \Delta_F(x_4 - y) \, ,
\end{aligned} \tag{4.122}
$$

where the denominator 4! has been compensated by the possible permutations. This term is represented by Fig.4.2. The point y, where the four lines meet, is called an elementary vertex. Since it comes from the interaction Hamiltonian it will always have four lines attached to it. Of course, in other theories, other numbers and types of lines attached to a vertex may emerge. Consider now the other term in the interacton Hamiltonian, the one propertional to δm^2. Attaching it to the four-point function translates into

$$
- \frac{i\delta m^2}{2} \int_{-\infty}^{\infty} \mathrm{d}y \left\langle 0 \left| T \left[\phi_{\text{in}}(x_1) \phi_{\text{in}}(x_2) \phi_{\text{in}}(x_3) \phi_{\text{in}}(x_4) : \phi_{\text{in}}^2(y) : \right] \right| 0 \right\rangle
$$

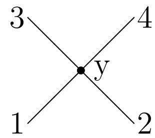


Figure 4.2: The first interaction term of the four-point function, Eq.(4.122) in diagrammatic representation.

$$= \quad -i\delta m^2 \int\limits_{-\infty}^{\infty} \mathrm{d}y \, [\Delta_F(x_1 - x_2)\Delta_F(x_3 - y)\Delta_F(x_4 - y) + \text{perm.}] \, , \tag{4.123}$$

Graphically, this translates to corrections to the three graphs of Fig.4.1, signified by a “crossed” line, cf. Fig.4.3.

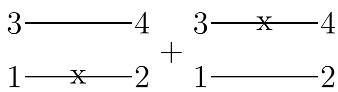


Figure 4.3: The correction terms of Eq.(4.123) in diagrammatic representation. Only the diagrams corresponding to the first graphs of Fig.4.1 are shown. They - and the other ones - constitute a correction term of the series expansion of the four point function, see below.

Taken together: the result of adding in the interaction Hamiltonian into the time-ordered products is the creation of vertices. Graphs therefore consist of lines, connecting vertices. For each factor of $\mathcal{H}_\mathrm{I}$ there will be exactly one vertex. Each diagram of contributing in the perturbative expansion to an n-point function will have n propagators with one end not connected to a vertex. These lines are called external legs.

Stating the Feynman rules

To compute an n-point function at a given perturbative order m therefore all distinct diagrams with n external legs and m vertices have to be drawn; the corresponding amplitudes have to be summed. The value of a diagram is calculated by associating a propagator $\Delta_F(x_i - x_j)$ to each line connecting the

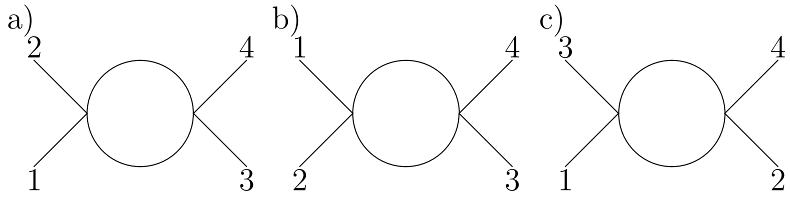


Figure 4.4: Examples for topologically identical and different diagrams. Diagrams a) and b) are topologically identical, hence they are not distinct, and it suffices to take into account only one of them. In contrast, diagram c) is distinct from a) and b), so it has to be taken into account.

points x_i and x_j. To each vertex a factor $-i\lambda_0 \int \mathrm{d}^4 y_i$ is associated, to each δm^2 corresponds a factor $-i\delta m^2 \int \mathrm{d}^4 y_i$. The factors 1/4! or 1/2! are cancelled by the number of possible permutations in applying Wick's theorem. Potentially symmetry factors hav e to be multiplied. In summary, the Feynman rules for $\lambda\phi^4$ theory in coordinate space are:

1. Draw all distinct diagrams with n legs and m vertices.

2. For each line connecting x_i and x_j multiply with a propagator $\Delta_F(x_i - x_j)$.

3. For each vertex, multiply with $-i\lambda_0 \int \mathrm{d}^4 y_i$, for each mass correction associate a factor $-i\delta m^2 \int \mathrm{d}^4 y_i$.

4. Divide the diagrams by symmetry factors.

"Distinct diagrams" in this context denote diagrams that are topologically different after relabeling the external legs. This is exemplified in Fig.4.4. There the diagrams a) and b) are not distinct, because, although the labels for the legs 1 and 2 have been permuted, they are still connected to the same vertex. In contrast a) and c) or b) and c) are distinct, because the legs 2 and 3 are attached to different vertices in the respective two diagrams. In fact, the labeling of the vertices is unimportant for this. Since one integrates over the positions of the vertices, each permutation of the corresponding y_i is just as good and contributes an identical result. This, however, is cancelled by the factor $1/m!$ stemming from the expansion of the n-point function, cf. Eq.(4.101).

In most cases, it is much easier to compute the Fourier transform of the n-point function. This is mainly because the free field propagators in momentum space take on a much nicer form,

$$i\Delta_F(x'-x) := \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{ie^{-ip\cdot(x-x')}}{p^2-m^2+i\epsilon} \longrightarrow \frac{i}{p^2-m^2+i\epsilon} =: i\Delta_F(p)\,. \tag{4.124}$$

To calculate an n-point function in momentum space, the Fourier-transfrom of the Feynman rules above have to be applied:

1. Draw all distinct diagrams with n legs and m vertices. By convention, all incoming momenta point into the diagram, all outgoing momenta point out of the diagram – other conventions are as good, one just has to use them throughout and consistently.

2. Label direction and momentum of the propagators. Assoiciate a factor $i\Delta_F(p_j)$ to the j-th external leg with momentum p_j. For internal lines, the momentum k_l is to be integrated over. Hence, attach a factor

$$i\int\frac{\mathrm{d}^4k_l}{(2\pi)^4}\Delta_F(k_l)\,.$$

3. Associate a factor

$$-i\lambda_0\,(2\pi)^4\delta^4\left(\sum_{j=1}^{4}q_j\right)$$

   for each vertex. The δ-function ensures momentum conservation and is over all momenta entering the vertex. Here, of course, the directions of the momenta are important.

   Similarly, associate a factor

$$-i\delta m^2\,(2\pi)^4\delta^4\left(\sum_{j=1}^{2}q_j\right)$$

   to each mass-correction term.

4. Divide the diagrams by symmetry factors.

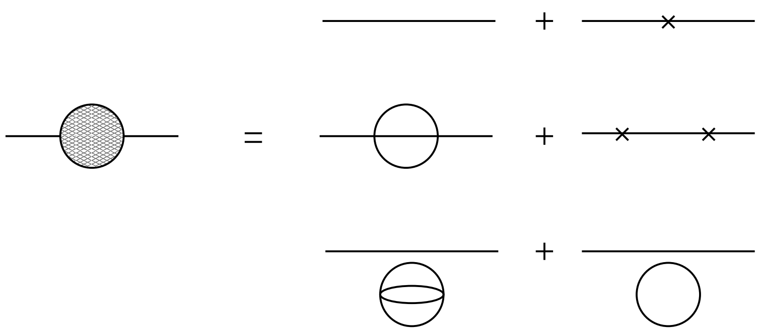


Figure 4.5: Diagrams for the expansion of the two-point function up to second order.

5. Multiply the overall result by

$$(2\pi)^4\delta^4\left(\sum_{j=1}^{n} p_j\right) , \tag{4.125}$$

the overall momentum conservation. Make sure to use the correct directions and signs.

The origin of the last rule is the overall translation invariance of n-point functions, leading to four-momentum conservation.

4.3.2 2- and 4-point functions

As first examples of the perturbative expansion of n-point functions through Feynman diagram techniques the 2- and 4-point functions will be considered.

2-point function

Examining the 2-point function, the diagrams corresponding to the numerator of Eq.(4.101) have two external legs, and, for expansion up to second order, at most two vertices. The diagrams are exhibited in Fig.4.5. The denominator of Eq.(4.101) is the same for all n-point functions. It does not depend on any external fields, consequently it does not have any external legs. Hence it can be represented as a vacuum amplitude, see Fig.4.6. Since in coordinate space an integation over all positions of the vertices is

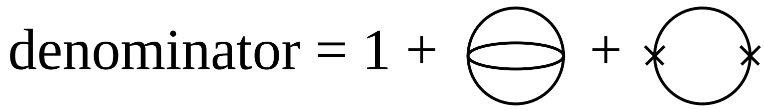


Figure 4.6: Diagrammatic expansion of the denominator of Eq.(4.101) up to second order.

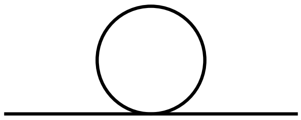


Figure 4.7: A tadpole diagram, contributing to the 2-point function. Such diagrams are not present when the interaction Hamiltonian is in normal-ordering. If this is not the case, they must be included.

performed, the diagrams in the denoimator are just numbers, representing disconnected vacuum processes, dubbed "vacuum bubbles". Turning back to the numerator, Fig.4.5 one then immediately realizes that the two diagrams in the lower line containing vacuum bubbles are just numbers times the free field propagator. These numbers, however, are cancelled against the denominator. This cancellation takes place to all orders, resulting in the fact that disconnected graphs in the numerator can be written as the 2-point function to some order times a vacuum bubble. Factoring out all vacuum bubbles and cancelling them against the denominator then implies that one really has to calculate only connected graphs. It is worth stressing that this makes sense in physical terms: the vacuum bubbles are just self-interactions of the vacuum which are disconnected to the particles propagating through it. So there's really no point why the particles should be affected.
Before the Feynman rules will be applied in order to evaluate the connected diagrams of Fig.4.5, consider the diagram of Fig.4.7. Such "tadpole" diagrams are not present when the interaction Hamiltonian is in normal-ordering. If this is not the case, they must be included and contribute to the mass renormalization.
However, to start with the contributing diagrams, consider the first diagram, the free field propagator. Applying the rules for momentum space the result

is

$$G^{(2,0)}(p,\,p) = \frac{i}{p^2 - m^2 + i\epsilon}(2\pi)^4\delta^4(p-p)\,, \tag{4.126}$$

where the first superscript is for the $n = 2$-point function, and the second one denotes the perturbative order. In the following the final δ-function will be suppressed. The next diagram involves one δm^2 vertex; it is similar to the free field graph and reads

$$G^{(2,1)}(p_1,\,p_2) = \frac{i}{p_1^2 - m^2 + i\epsilon}i\delta m^2\frac{i}{p_2^2 - m^2 + i\epsilon}(2\pi)^4\delta^4(p_1-p_2)\,, \tag{4.127}$$

or, using the δ-function stemming from the vertex,

$$G^{(2,1)}(p,\,p) = \frac{i}{p^2 - m^2 + i\epsilon}i\delta m^2\frac{i}{p^2 - m^2 + i\epsilon} = i\delta m^2\left(\frac{i}{p^2 - m^2 + i\epsilon}\right)^2\,. \tag{4.128}$$

The result for the diagram with two δm^2 vertices is similar, simple consideration yields

$$G^{(2,2_\mathrm{a})}(p,\,p) = \left(i\delta m^2\right)^2\left(\frac{i}{p^2 - m^2 + i\epsilon}\right)^3\,. \tag{4.129}$$

These diagrams just guarantee through renormalization that the physical mass remains at m, they stem from terms dubbed "counter-terms"m, see below. There, δm^2 is to bne adjusted such that it reproduces the physical mass. Finally, turn to the "setting sun" diagram. It is the first non-trivial, interesting graph, for the momentum flow, cf. Fig4.8. It is equal to

$$\begin{aligned}
&G^{(2,2_\mathrm{b})}(p,\,p)\\
&= \frac{i\lambda_0^2}{3!}\left(\frac{i}{p^2 - m^2 + i\epsilon}\right)^2\int\frac{\mathrm{d}^4k_1}{(2\pi)^4}\frac{i}{k_1^2 - m^2 + i\epsilon}\int\frac{\mathrm{d}^4k_2}{(2\pi)^4}\frac{i}{k_2^2 - m^2 + i\epsilon}\\
&\quad\cdot\int\frac{\mathrm{d}^4k_3}{(2\pi)^4}\frac{i}{k_3^2 - m^2 + i\epsilon}\left[(2\pi)^4\delta^4(p-k_1-k_2-k_3)\right]^2\\
&= \frac{i\lambda_0^2}{3!}\left(\frac{i}{p^2 - m^2 + i\epsilon}\right)^2\int\frac{\mathrm{d}^4k_1}{(2\pi)^4}\frac{i}{k_1^2 - m^2 + i\epsilon}\\
&\quad\cdot\int\frac{\mathrm{d}^4k_2}{(2\pi)^4}\frac{i}{k_2^2 - m^2 + i\epsilon}\frac{i}{(p-k_1-k_2)^2 - m^2 + i\epsilon}\,.
\end{aligned} \tag{4.130}$$

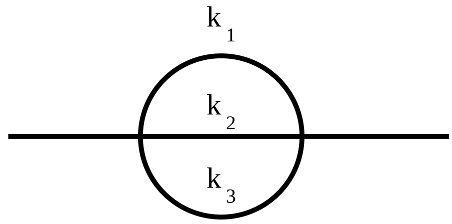


Figure 4.8: A "settingsun" diagram, contributing to the 2-point function. It is the first non-trivial, interesting graph in the perturbative expansino of the 2-point function.

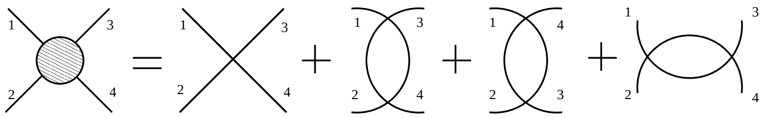


Figure 4.9: Diagrams for the expansion of the four-point function up to second order.

The symmetry factor of this diagram is 1/3! since there are as many different ways to permute the internal lines. The integrals occuring are not so easy to compute. In fact, for large loop-momenta k_1 or k_2 they naively diverge – in fact it is only one of them that gives rise to such an ultraviolet divergence. To cure these divergencies and to keep the quantum field theory physically sensible the procedure of renormalizing the theory has to be performed, see below.

4-point function

Making use of the results of the previous section, it is clear that for the four point function only connected diagrams have to be taken into account. Up to second order they are exhibited in Fig.4.9. Labeling the incoming and outgoing momenta as $p_{1,\,2}$ and $p_{3,\,4}$, respectively, the leading order connected diagram yields

$$G^{(4,\,1)}(p_1,\,p_2,\,p_3,\,p_4) = -i\lambda_0(2\pi)^4\delta^4(p_1+p_2-p_3-p_4)\prod_{i=1}^{4}\left(\frac{i}{p_i^2-m^2+i\epsilon}\right) \tag{4.131}$$

The diagram containing the δm^2 vertex is pretty trivial, consider therefore only the non-trivial one involving a closed loop. It reads

$$
\begin{aligned}
G^{(4,\,2)}(p_1,\, p_2,\, p_3,\, p_4) &= G^{(4,\,1)}(p_1,\, p_2,\, p_3,\, p_4) \\
&\quad \cdot \left(-\frac{i\lambda_0}{2}\right) \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{i^2}{[k^2 - m^2 + i\epsilon][(p_1 + p_2 - k)^2 - m^2 + i\epsilon]}\,.
\end{aligned}
\tag{4.132}
$$

Of course, there are several permutations for it, i.e. several distinct topologies. They all have to do with the question, which momenta are combined to enter the loop.

4.3.3 Cross sections

The S-matrix element

In this section the first cross section, the one for elastic scattering of two real scalar particles, is computed to first non-zero order in λ_0. To this end, to compute the S-matrix, one has to caluclte the four-point function to order λ_0. This has already been done in the previous section, cf. Eq.(4.131),

$$
G^{(4,\,1)}(p_1,\, p_2,\, p_3,\, p_4) = -i\lambda_0 (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \prod_{i=1}^{4} \left(\frac{i}{p_i^2 - m^2 + i\epsilon}\right)
\tag{4.133}
$$

Applying the reduction formula, Eq.(4.64), for an $n \to m$ process in momentum space, one has

$$
\begin{aligned}
S^c_{\beta\alpha} &= \left(\frac{-i}{\sqrt{Z}}\right)^{n+m=4} \left[\prod_{i=1}^{n=2} (p_i^2 - m^2) \prod_{i=n+1}^{n+m=4} (p_i^2 - m^2)\right] G^{(4,\,1)}(p_1,\, p_2,\, p_3,\, p_4) \\
&= -\frac{i\lambda_0}{Z^2} (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4)\,.
\end{aligned}
\tag{4.134}
$$

Here, Z the one-particle amplitude of ϕ is determined through Eqs.(4.34) and (4.35)

$$
\lim_{p^2 \to m^2} \Delta'_F(p) = Z \delta_F(p)\,,
\tag{4.135}
$$

where Δ'_F is the interacting two-point function. However, at leading order in λ_0 this is identical to the free two-point function, and, hence, $Z = 1$ in this case.

Definition of cross section

The cross section, usually denoted by σ is the transition rate per scatterer in the target, per unit incident flux. The reaction rate is the transition probability per unit time, hence

$$\begin{aligned}\sigma &= \text{transition rate} \cdot \frac{1}{\#\text{ of scartterers in target}} \cdot \frac{1}{\text{unit incident flux}} \\ &= \frac{\text{transition prob.}}{\text{unit time}} \cdot \frac{1}{\#\text{ of scartterers in target}} \cdot \frac{1}{\text{unit incident flux}}\end{aligned} \tag{4.136}$$

The incident flux is defined as the number of particles, A, crossing an area, A, in a certain amount of time, T. In typical scattering experiments the incoming particles are usually very carefully prepared, such that they all have nearly identical velocity $\vec{v}$. Considering the target particle to be at rest, then the incoming flux is

$$\text{flux} = \frac{\#\text{ of particles}|\vec{v}|}{V}\,, \tag{4.137}$$

where V is the volume containing the number of particles equal in value to the flux. In the present situation with just one incoming and one target particle, the flux becomes

$$\text{flux} = \frac{|\vec{v}|}{V}\,. \tag{4.138}$$

Therefore the cross section is

$$\sigma = \frac{\text{transition prob.}}{\text{unit time} \times \text{unit volume}} \frac{1}{|\vec{v}|}\,. \tag{4.139}$$

The transition probability is related to the S-matrix element as follows

$$\text{transition prob.} = \int \mathrm{d}^3 p_3 \mathrm{d}^3 p_4 \, \frac{\left|S^c_{fi}\right|^2}{\langle f|f\rangle\langle i|i\rangle}\,, \tag{4.140}$$

where the normalization factors of the momentum eigenstates have explicitly been added, since they are not unity. In fact,

$$\langle p_3 p_4 | p_3 p_4\rangle = 4 p_3^0 p_4^0 (2\pi)^6\,, \tag{4.141}$$

and similar for the incoming particles. When squaring the S-matrix element, also the δ function for momentum conservation is squared. Conventionally this boils down to

$$\left|(2\pi)^4\delta^4\left(\sum_i p_i\right)\right| = (2\pi)^4\delta^4\left(\sum_i p_i\right)(2\pi)^4\delta^4\left(0\right) . \tag{4.142}$$

By examining the integral definition of the δ function, it becomes obvious that $(2\pi)^4\delta^4\left(0\right)$ is the volume of space-time. Taking everything together, the cross sectio for elastoc scattering of two real scalar particles is

$$\begin{aligned} \mathrm{d}\sigma &= \frac{1}{|\vec{v}|}\frac{1}{4p_1^0p_2^0}|-i\lambda_0|^2\int\frac{\mathrm{d}^3p_3}{(2\pi)^3 2p_3^0}\frac{\mathrm{d}^3p_4}{(2\pi)^3 2p_4^0}(2\pi)^4\delta^4(p_1+p_2-p_3-p_4) \\ &= \frac{1}{2|\vec{v}|}\frac{1}{4p_1^0 m}|-i\lambda_0|^2\int\frac{\mathrm{d}^3p_3}{(2\pi)^3 2p_3^0}\frac{\mathrm{d}^3p_4}{(2\pi)^3 2p_4^0}(2\pi)^4\delta^4(p_1+p_2-p_3-p_4)\,, \end{aligned} \tag{4.143}$$

where the factor of $1/2$ emerges due to the symmetry of the final state. If the target particles were not at rest, $|\vec{v}|$ represents the relative velocity. Expressed in invariant form one then has

$$\begin{aligned} \mathrm{d}\sigma &= \frac{1}{2}\frac{1}{4\sqrt{(p_1\cdot p_2)^2-m^2m^2}}|-i\lambda_0|^2 \\ &\quad\cdot\int\frac{\mathrm{d}^3p_3}{(2\pi)^3 2p_3^0}\frac{\mathrm{d}^3p_4}{(2\pi)^3 2p_4^0}\,(2\pi)^4\delta^4(p_1+p_2-p_3-p_4) \end{aligned} \tag{4.144}$$

Chapter 5

Hiding the infinities: $\lambda\phi^4$

In this chapter, light will be shed on the question of how to deal with ultraviolet divergences that occur at higher orders of the perturbative expansion of field theories. As an example, the simplest case, namely self-interacting scalar fields are considerred, more specifically, $\lambda\phi^4$ in four dimensions is used to exemplify the concept of dimensional regularisation and renormalisation. Having understood the origin of ultraviolet divergences in four dimensions, i.e. the eventual divergence of diagrams containing loops, in a first step these divergences are regularised. To do so, the number of dimensions is set to d, for four-dimensional field theory, the divergences then show up as poles in the form $1/(4-d)$. In a next step, the Lagrangian of the theory is modified such that these poles vanish. This is achieved by adding terms, aptly called counter-terms, to the original Lagrangian. The special feature of renormalisable field theories such as $\lambda\phi^4$ theory is that the structure of these counter-terms is identical to other terms already present in the Lagrangian. Renormalisation can thus be achieved multiplicatively, resulting in a re-definition of fields and parameters (coupling constants and masses) of the theory. This re-definition finally leads to the effect that the parameters are not fixed any longer but depend on a scale. This phenomenon is called running coupling constant etc..

5.1 Regularisation

5.1.1 Divergences in Feynman diagrams

Examples

In the last chapter, Feynman rules for $\lambda\phi^4$ have been derived, which allowed a perturbative expansion of n-point functions. Such n-point functions can be represented as Feynman diagrams, and they ultimately allow to analyse interacting field theories, i.e. to calculate cross sections etc.. However, already the first non-trivial contributions lead to divergent contributions, related to integrals over loops internal to diagrams. For instance, the "tadpole"-graph, although not existing when the interacting Hamiltonian is taken to be normal-ordered, leads to an integral of the form

$$\int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{k^2 - m^2 + i\epsilon}\,, \tag{5.1}$$

which, for $|k| \to \infty$, diverges. The integral is said to diverge quadratically in the ultraviolet, due to the fact that for large k it behaves like d^4k/k^2. This region of large momenta is equivalent to small distances in coordinate space, in other words, these divergencies stem from $\Delta(\delta x)$, where $\delta x \to 0$. Another example occurred when considering the "fish"-diagram, leading to

$$\begin{aligned}
&\frac{\lambda_0^2}{2} \int \frac{\mathrm{d}^4 k_1}{(2\pi)^4} \frac{\mathrm{d}^4 k_2}{(2\pi)^4} \frac{(2\pi)^4 \delta^4(p_1 + p_2 - k_1 - k_2)}{(k_1^2 - m^2 + i\epsilon)(k_2^2 - m^2 + i\epsilon)} \\
&= \frac{\lambda_0^2}{2} \int \frac{\mathrm{d}^4 k_1}{(2\pi)^4} \frac{1}{(k_1^2 - m^2 + i\epsilon)[(p_1 + p_2 - k_1)^2 - m^2 + i\epsilon]}\,,
\end{aligned} \tag{5.2}$$

which, again, diverges for $|k_1| \to \infty$, since in this case the integral behaves like d^4k_1/k_1^4. In contrast to these ultraviolet divergences also infrared ones might occur, related to the limit $|k_1| \to 0$. In the example above, the fish diagram, this could happen in the massless case. However, in what follows infrared divergences are neglected.

Classification of divergences

A mixture of power-counting of the integration measures and the propagators and of topology will come to help in the question where to find these divergences. Consider a Feynman diagram of real scalar theory with V vertices,

E external lines and I internal ones. The number of independent internal momenta, which finally have to be integrated over, is identical to the number of independent loops, L. Since the internal lines satisfy $V-1$ relations among themselves (the -1 is due to overall momentum conservation), there are

$$L = I - V + 1 \tag{5.3}$$

loop momenta. It is easy to check that this holds true for the simple diagrams encountered so far, tadpole (I=1, V=1, L=1), fish (I=2, V=2, L=2), and setting sun (I=3, V=2, L=2). This relation however, allows to define a "naive" degree of divergence, also dubbed the superficial or apparent degree of divergence, D. To compute it, two ingredients are needed: first of all, L independent integrations, each of which provides d powers of momenta, where d is the number of dimensions – in most cases $d=4$. Furthermore, there are I internal propagators, each of which provides two inverse powers of momenta. Taken together this leads to

$$D = d \cdot L - 2 \cdot I\,. \tag{5.4}$$

In fact, there are more relations between L, I and E, allowing to be more specific about D. Suppose, V_N denotes the number of vertices with N legs. In a diagram with such vertices only, there are V_N lines, which are either internal or external. Since internal lines have two vertices as endpoints they count twice. Therefore

$$\sum_N N \cdot V_N = E + 2 \cdot I\,. \tag{5.5}$$

Plugging this into the expression for D yields

$$\begin{aligned} D &= d - \frac{d-2}{2} \cdot E + \sum_N \left[V_N \left(\frac{N-2}{2} d - N \right) \right] \\ &\stackrel{d \to 4}{\longrightarrow} 4 - E + (N-4) V_N \end{aligned} \tag{5.6}$$

In $\lambda\phi^4$ theory, when the δm^2 insertions are neglected, there's only one type of vertex with $N=4$. This immediately implies that

$$D_{\lambda\phi^4}^{(d=4)} = 4 - E\,, \tag{5.7}$$

in other words, for $D \geq 0$ there are only two candidates in this theory:

1. $G^{(2)}$ with $D = 2$ superficial degree of divergence (i.e. quadratic), and

2. $G^{(4)}$ with $D = 0$ superficial degree of divergence (i.e logarithmic).

This analysis, however, does not prove that $G^{(n>4)}$ are convergent, that's why D is called superficial or naive.

n-particle (ir)reducible diagrams

There is a reason, however, why D is called the naive or superficial degree of divergence and not the "meaningless" degree of divergence. For this consider a diagram with E external lines which cannot be disconnected by cutting n lines. Such a diagram is called n-particle irreducible. Examples are: tadpole and fish are one-particle irreducible, the setting sun is two-particle irreducible. In general, reducing a diagram by cutting n lines results in two blobs connected by the n lines. Of course, by construction, each of the blobs is at least n-particle irreducible. Then, for the total diagram

$$D = D_1 + D_2 + 4(n-1) - 2n\,, \tag{5.8}$$

where D_1 and D_2 are the superficial degrees of divergence of the two blobs. The other two terms are due to the fact that there are $(n-1)$ independent loops cut open and $2n$ propagators.
With $n = 1$ the diagrams could be described through $D_1 = D_2 = 0$ leading to $D = -2$, leading to the idea that the diagram is apparently convergent. An example of this is the left diagram in Fig.5.1, cut along the dashed lines. This cut leaves to vertex corrections as sub-diagrams, both of which are naively divergent, whereas the full diagram naively converges. Another example of a one-particle irreducible (1PI) graph is the right diagram in the same figure, Fig.5.1. Again cutting along the dashed line leads to $D_1 = 2$ and $D_2 = -2$, i.e. the tadpole is quadratically divergent, whereas the tree-level (no loops) six-point sub-diagram is naively convergent, as is then the full diagram.
When $n = 2$, one can similarly have either of the two sub-diagrams D_1 or D_2 with a sufficiently negative naive degree of divergence, i.e. sufficiently convergent to live up for the divergences of the other substructure. An example for this is the graph in Fig.5.2.
In $\lambda\phi^4$ theory, diagrams can be at most three-particle irreducible. This is because by cutting three - eventually internal - lines one can always isolate a vertex connected to an external leg. The procedure for hunting down

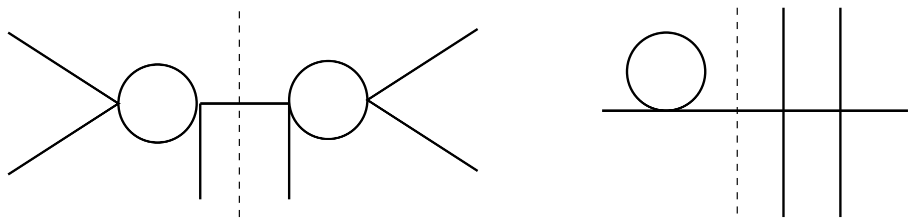


Figure 5.1: Examples for one-particle reducible graphs.

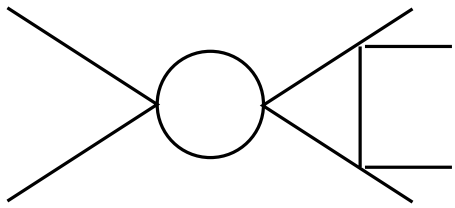


Figure 5.2: A two-particle reducible graph.

divergencies is now clear. Take any diagram and catalogue it in terms of its reducibility, i.e according to whether it is one-, two-, or three-particle reducible. If it is three-particle reducible, decompose it into units of the form exhibited in Fig.5.3, where $N \geq 5$. In this type of diagram, there is a naively divergent four-point function, the left blob; the right blob can be decomposed further.
Similarly, two-particle reducible graphs exhibit their divergences when they are decomposed according to the two examples of Fig.5.4. In both cases the left blob carries naively divergent structures, the right blob again has to be

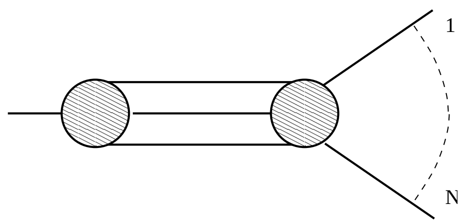


Figure 5.3: The prototypical three-particle reducible graph.

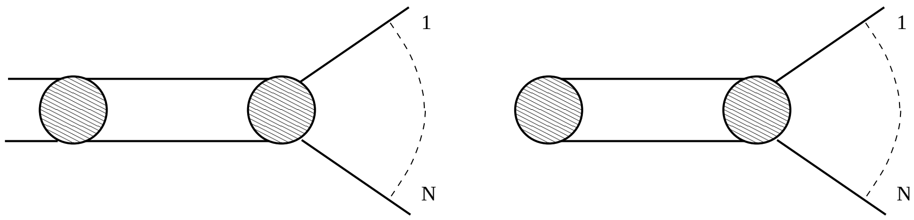


Figure 5.4: The prototypical two-particle reducible graphs.

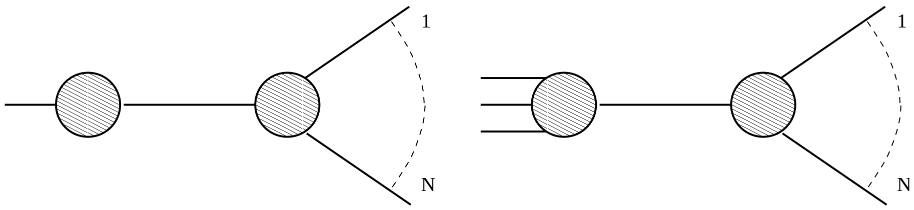


Figure 5.5: The prototypical one-particle reducible graphs.

decomposed further. In these examples, $N > 4$ for the left diagram and $N > 2$ for the right diagram, respectively.
Finally, one-particle reducible graphs can be decomposed according to the two patterns depicted in Fig.5.5. There, for the left diagram, $N > 5$, for the right graph $N > 3$. Again, the left blobs will then carry naive divergences, and again the right blobs have to be decomposed further.
This exhaustive catalogue shows that truly convergent diagrams have no divergent two- or four-point sub-diagram. The origin of the ultraviolet divergences hidden in two- and four-point functions only in the framework of $\lambda\phi^4$-theory can be understood also in a more pedestrian fashion. Consider any loop residing somewhere inside a diagram: the momentum integration will yield four powers of momenta. This leads to divergences only, if the loop is bound by at most two propagators (internal lines). Any more lines will result in ultraviolet convergence. In this way it is clear that, due to the vertices in this theory, the only divergent structures are two- and four-point functions, at one-loop these are the tadpole-, fish- and setting sun-graphs. UV divergence inside a diagram then originates from such graphs inside two- or four-point functions nested inside the diagram.
Thus, for $\lambda\phi^4$-theory a diagram is truly convergent, if its superficial degree

of divergence is negative (i.e. if it is superficially convergent) and if it cannot be decomposed into three-, two- or one-particle reducible parts like the one depicted in the figures above which might contain isolated two- or four-point functions. In other words: a diagram is truly convergent if it is superficial convergent and if all its subgraphs are as well.
This immediately translates into the following: controlling the divergent structures, i.e. the two- and four-point functions means to control all divergences of the theory. Those graphs that contain the generic divergences are called primitively divergent. The fact that in $\lambda\phi^4$-theory the primitively divergent interactions are finite in number and of the types already appearing in the Lagrangian allows for a successful removal of ultraviolet divergences through suitable redefinitions of the entities entering the Lagrangian. A theory which contains only a limited number of primitively divergent interactions which can be absorbed by suitable redefinitions of interactions in the Lagrangian is called renormalisable. From the consideration above, it is clear that only a few theories satisfy this criterion. To highlight this, consider the following examples:

- When $d = 4$, it is clear that D as given by Eq.(5.6) quickly grows with the number of vertices for which $N > 4$, i.e. for those vertices that have N lines attached to them. Hence, in $d = 4$ dimensions theories with interactions of the type $\phi^{N>4}$, although perfectly reasonable on the classical level, lead to an infinite number of primitively divergent diagrams. This is because for such theories, diagrams become the more divergent the more vertices they contain. In such cases, one can abandon all hope to keep track of the situation and to renormalise the theory. In fact, as an aside, it should be noted that in such cases due to the mass-dimension of the Lagrangian and of the fields, the coupling constant must have negative dimensions of mass. Often this is a sign for "bad" ultraviolet behaviour.

- In contrast, when $d = 2$, Eq.(5.6) becomes

$$D = 2 - 2 \cdot V_N \,, \tag{5.9}$$

a totally reversed situation, since the more vertices a diagram contains, the more divergent it is. There, the convergence behaviour is also independent of the type of interaction and of the number of external legs. Instead, the only primitively divergent diagrams are those with zero or one vertex. Loop diagrams with one vertex, however, disappear when the interaction Hamiltonian is normal-ordered.

- Finally, it is worth noting that for $d \geq 7$ there is no quantum field theory of point particles with a finite number of primitively divergent graphs. The last renormalisable theory in higher dimensions is $\lambda\phi^3$ in six dimensions; there λ is dimension-less, since in six dimensions the scalar fields have mass dimension of two.

5.1.2 Dimensional regularisation

Basic idea

In the following, the ultra-violet divergent (one-) loop integrals will be evaluated. These loop integrals look like

$$I_{d=4}(k) = \int \mathrm{d}^{d=4} l F(l,\, k)\,, \tag{5.10}$$

where F behaves like $1/l^2$ or $1/l^4$ for $l \to \infty$. The basic idea of dimensional regularisation is to lower the number of dimensions, d, such that the integral becomes convergent. Divergences are then expressed as poles of the form $1/(d-4)$. For instance, if $F \to 1/l^4$, the integral converges at the ultraviolet end if $d < 4$. Mathematically, one can then write

$$I_d(k) = I(d,\, k) = \int \mathrm{d}^d l F(l,\, k)\,, \tag{5.11}$$

as a function of the (even complex) variable d. One can then invent a function $I'(d,\, k)$ with well-defined singularities outside the d-region of convergence, which reduces to I inside this region. Then, by analytic continuation, both functions are the same. This is nicely exemplified by the Euler- and the Weierstrass-representation of the Γ-function. For $\mathcal{R}e(z) > 0$, the Euler-representation reads

$$\Gamma(z) = \int\limits_0^\infty \mathrm{d}t e^{-t} t^{z-1}\,, \tag{5.12}$$

which diverges for $\mathcal{R}e(z) \leq 0$ as $t \to \infty$. In this limit, the integral behaves like $\mathrm{d}t/t^{1+\mathcal{R}e(z)}$, leading to an infinite result. But starting from the definition above, one can split up the region of integration and write

$$\begin{aligned}\Gamma(z) &= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_0^{t_0}\mathrm{d}t\, t^{n+z-1} + \int_{t_0}^{\infty}\mathrm{d}t\, e^{-t}t^{z-1} \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{t_0^{n+z}}{z+n} + \int_{t_0}^{\infty}\mathrm{d}t\, e^{-t}t^{z-1}\,. \end{aligned} \tag{5.13}$$

with some arbitrary t_0. This form is valid everywhere, for all values of z. Furthermore, it should not (and it does not) depend on the cut-off t_0. This can be checked by verifying that $\mathrm{d}\Gamma/\mathrm{d}t_0 = 0$. With $t_0 = 1$ this is the Weierstrass representation of the Γ function. To isolate the singularities one had to introduce an arbitrary scale t_0, the result, however, does not depend on its choice.

The problem with the integral expressions like the one of Eq.(5.11) is very similar to the one with the Euler representation of the Γ-function. In the actual case, one wants to find an equivalent of Weierstrass' representation. The procedure for this is as follows:

1. Establish a finite domain of convergence for the loop integral, typically to the left of $d = 4$.

2. Construct a new function which overlaps with the loop integral in its domain of convergence, but is defined for more values of d, including $d = 4$.

3. Take the limit $d \to 4$.

Integration in d dimensions

Now one has to perform a loop integration in d dimensions. This looks like a complicated task; but in fact, it is rather straight forward after some manipulations with the function to be integrated over. Consider for instance the function $1/(k^2 - m^2 + i\epsilon)^n$. It can be cast into a much nicer form by identifying

$$\frac{1}{(k^2 - m^2 + i\epsilon)^n} = \frac{(-i)^n}{\Gamma(n)}\int_0^{\infty}\mathrm{d}z\, z^{n-1}e^{iz(k^2-m^2+i\epsilon)} \tag{5.14}$$

Then, a Wick-rotation can be performed, i.e. $k \to K = (ik_0, \vec{k})$. This manipulation basically allows to use Euclidean metric for the K, such that $K^2 = -(k_0^2 + \vec{k}^2) < 0$. Using the trick above, one can reduce the task to integrate an algebraic function in d dimensions to a Gaussian integral in d dimensions,

$$\begin{aligned}
\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2 - m^2 + i\epsilon)^n} &= \int \frac{\mathrm{d}^d K}{(2\pi)^d} \frac{i}{(K^2 - m^2 + i\epsilon)^n} = \\
&= \frac{i(-i)^n}{\Gamma(n)} \int\limits_0^\infty \mathrm{d}z\, z^{n-1} \int \frac{\mathrm{d}^d K}{(2\pi)^d}\, e^{iz(K^2 - m^2 + i\epsilon)} \\
&= \frac{i(-i)^n}{\Gamma(n)} \frac{\pi^{d/2}}{i^{d/2}(2\pi)^d} \int\limits_0^\infty \mathrm{d}z\, z^{n-1-d/2} e^{-iz(m^2 + i\epsilon)} \\
&= \frac{i(-i)^n}{\Gamma(n)} \frac{\pi^{d/2}}{i^{d/2}(2\pi)^d} \frac{\Gamma\left(n - \frac{d}{2}\right)}{(im^2)^{n-d/2}} \\
&= \frac{i(-1)^n}{(4\pi)^{d/2}} \frac{\Gamma\left(n - \frac{d}{2}\right)}{\Gamma(n)} \frac{1}{(m^2)^{n-d/2}}
\end{aligned} \tag{5.15}$$

The divergences of the integral now show up, when taking the limit $d \to 4$. To do so, replace $d \to 4 - 2\eta$ and expand around small η. For the last line, i.e. the result of the integral above, one finds

$$\begin{aligned}
&\frac{i(-1)^n}{(4\pi)^{d/2}} \frac{1}{(m^2)^{n-d/2}} \frac{\Gamma\left(n - \frac{d}{2}\right)}{\Gamma(n)} \\
&= \frac{i(-1)^n}{(4\pi)^2} \frac{1}{(m^2)^{n-2}} \frac{1}{\Gamma(n)} \left[(4\pi)^\eta m^{2\eta} \Gamma\left(n - 2 + \eta\right)\right] ,
\end{aligned} \tag{5.16}$$

hence, the divergences show up encoded as poles of the Γ-function for integer negative arguments. For the following considerations it will be useful to remember some of its properties such as its definition

$$\Gamma(x) = \int\limits_0^\infty \mathrm{d}y\, y^{x-1} e^{-y} \tag{5.17}$$

and

$$\Gamma(n+1) \;=\; n! \quad \text{for} \quad n \in N ,$$

$$
\begin{aligned}
\Gamma(x+1) &= x\Gamma(x) \quad \text{for} \quad x > 0\,, \\
\Gamma(x)\Gamma(1-x) &= \frac{\pi}{\sin(\pi x)}\,, \\
\Gamma(1) &= 1\,.
\end{aligned} \tag{5.18}
$$

Using its definition, the first derivatives of $\Gamma(x)$ can be calculated for $x = 1$, namely

$$
\Gamma'(1) = -\gamma_E \quad \text{and} \quad \Gamma''(1) = \gamma_E^2 + \frac{1}{6}\pi^2\,, \tag{5.19}
$$

where γ_E is Eulers constant satisfying

$$
\gamma_E = \lim_{n\to infty}\left[-\log(n) + 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right] = 0.5772157\,, \tag{5.20}
$$

or

$$
\gamma_E = -\int\limits_0^\infty \mathrm{d}x e^{-x}\log(x)\,. \tag{5.21}
$$

Then, the Γ function can be expanded around $x = 1$ as

$$
\begin{aligned}
\Gamma(1+\eta) &= 1 - \gamma_E\,\eta + \frac{1}{2}\left(\gamma_E^2 + \frac{\pi^2}{6}\right)\eta^2 + \ldots \\
\log\left[\Gamma(1+\eta)\right] &= -\gamma_E\,\eta + \frac{\zeta(2)}{2}\,\eta^2 - \frac{\zeta(3)}{3}\,\eta^3 \ldots\,,
\end{aligned} \tag{5.22}
$$

where

$$
\zeta(p) = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \ldots = \sum_{i=1}^\infty \frac{1}{i^p} \tag{5.23}
$$

is Riemanns ζ-function of order p. It fulfils

$$
\zeta(2) = \frac{1}{6}\,\pi^2\,. \tag{5.24}
$$

In general, expanding $\Gamma(-n+\eta)$ around $\eta = 0$ for $n \in N$ yields

$$
\begin{aligned}
&\Gamma(-n+\eta) \\
&= \frac{(-1)^n}{n!}\left\{\frac{1}{\eta} + \psi(n+1) + \frac{1}{2}\eta\left[\frac{\pi^2}{3} + \psi^2(n+1) - \psi'(n+1)\right] + \mathcal{O}(\eta^2)\right\}\,,
\end{aligned} \tag{5.25}
$$

where

$$\begin{aligned}\psi(x) &= \frac{\mathrm{d}\log\Gamma(x)}{\mathrm{d}x} \Longrightarrow \psi(n+1) = 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \gamma_E \\ \psi'((n+1) &= \frac{\pi^2}{6} - \sum_{k=1}^{n}\frac{1}{k^2}\,.\end{aligned} \tag{5.26}$$

Feynman vs. Schwinger

Before turning to the evaluation of the first few real diagrams, another little nuisance will be discussed. So far, the discussion focused around the question how to integrate functions of the form $1/(k^2 - m^2 + i\epsilon)^n$ in d dimensions. In real life, however, the propagator structure of diagrams might and will not look so pleasant. Instead, at one-loop level on has to be prepared to tackle structures of the form

$$\frac{k^\mu k^\nu \ldots}{[(k+p_1)^2 - m_1^2 + i\epsilon][(k+p_2)^2 - m_2^2 + i\epsilon]\ldots}\,. \tag{5.27}$$

The natural question to be answered is: how can one bring such expressions to the form already successfully tackled ?

1. In fact, there are two answers to this. One dates back to Feynman and works as follows: consider the expression $1/(a \cdot b)$. This can be rewritten as

$$\frac{1}{a \cdot b} = \int_0^1 \frac{\mathrm{d}x}{[ax + (1-x)b]^2} = \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \frac{\delta(1-x-y)}{[ax+yb]^2}\,. \tag{5.28}$$

   Starting from there, one can generalise to arbitrary powers of the denominator terms or to arbitrary numbers of them. This results in

$$\begin{aligned}\frac{1}{a^N b^M} &= \frac{\Gamma(N+M)}{\Gamma(N)\Gamma(M)} \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y\, \frac{x^{N-1} y^{M-1} \delta(1-x-y)}{[ax+yb]^{N+M}} \\ &= \frac{\Gamma(N+M)}{\Gamma(N)\Gamma(M)} \int_0^1 \mathrm{d}x\, \frac{x^{N-1}(1-x)^{M-1}}{[ax+(1-x)b]^{N+M}}\,, \\ \frac{1}{abc} &= \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z\, \frac{2\delta(1-x-y-z)}{[ax+by+cz]^3}\end{aligned}$$

$$\frac{1}{a_1 a_2 \ldots a_n} = \Gamma(n) \left[\prod_{i=1}^{n} \int_0^1 \mathrm{d}z_i\right] \left[\frac{\delta\left(1 - \sum\limits_{i=1}^{n} z_i\right)}{\left(\sum\limits_{i=1}^{n} z_i a_i\right)^n}\right] \tag{5.29}$$

Applying these rules any denominator of the form in Eq.(5.27) can be cast into

$$\begin{aligned}
&\frac{k^\mu k^\nu \ldots}{[(k+p_1)^2 - m_1^2 + i\epsilon][(k+p_2)^2 - m_2^2 + i\epsilon]\ldots} \\
&= \left[\prod_{i=1}^{n} \int_0^1 \mathrm{d}z_i\right] \left\{\frac{(k^\mu k^\nu \ldots)\,\Gamma(n)\delta\left(1 - \sum\limits_{i=1}^{n} z_i\right)}{\left[\sum\limits_{i=1}^{n} z_i\left(k^2 + 2kp_i + p_i^2 - m_i^2\right)\right]^n}\right\} \\
&= \left[\prod_{i=1}^{n} \int_0^1 \mathrm{d}z_i\right] \left\{\frac{(k^\mu k^\nu \ldots)\,\Gamma(n)\delta\left(1 - \sum\limits_{i=1}^{n} z_i\right)}{\left[k^2 + 2k\left(\sum\limits_{i=1}^{n} z_i p_i\right) + \left(\sum\limits_{i=1}^{n} z_i(p_i^2 - m_i^2)\right)\right]^n}\right\}
\end{aligned} \tag{5.30}$$

When integrating over d^4k one can now substitute

$$k^\mu \to K^\mu = k^\mu + \sum_{i=1}^{n} z_i p_i^\mu \tag{5.31}$$

and cast the denominator into a form quadratic in K,

$$\begin{aligned}
&\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{k^\mu k^\nu \ldots}{[(k+p_1)^2 - m_1^2 + i\epsilon][(k+p_2)^2 - m_2^2 + i\epsilon]\ldots} \\
&= \int_0^1 \mathrm{d}\mathcal{Z} \int \frac{\mathrm{d}^d K}{(2\pi)^d} \frac{(K^\mu - \mathcal{P}^\mu)(K^\nu - \mathcal{P}^\nu)\ldots}{[K^2 - \mathcal{P}^2 + \mathcal{M}^2]^n}
\end{aligned} \tag{5.32}$$

where the abbreviations

$$\begin{aligned}
\int_0^1 \mathrm{d}\mathcal{Z} &= \Gamma(n) \left[\prod_{i=1}^{n} \int_0^1 \mathrm{d}z_i\right] \delta\left(1 - \sum_{i=1}^{n} z_i\right) \\
\mathcal{P}^\mu &= \sum_{i=1}^{n} z_i p_i^\mu \\
\mathcal{M}^2 &= \sum_{i=1}^{n} z_i (p_i^2 - m_i^2)
\end{aligned} \tag{5.33}$$

have been used. From the form of Eq.(5.33) it is clear, that all odd powers of K^μ and similar in the numerator vanish due to the symmetry of the integration region. Therefore, one can average pairwise over the directions of the K^μ, which results in the replacement

$$K^\mu K^\nu \longrightarrow K^2 \frac{g^{\mu\nu}}{d} \tag{5.34}$$

for each possible pair of K's in the integral. This reduces the K-integral of Eq.(5.33) to

$$\int \frac{\mathrm{d}^d K}{(2\pi)^d} \frac{(K^2)^N}{\left[K^2 - C^2\right]^M} \, . \tag{5.35}$$

To evaluate it, consider the case $N = 1$. It can be reduced to the known result for $N = 0$ by

$$\begin{aligned}
\int \frac{\mathrm{d}^d K}{(2\pi)^d} \frac{(K^2)^{N=1}}{\left[K^2 - C^2\right]^M} &= \int \frac{\mathrm{d}^d K}{(2\pi)^d} \frac{(K^2 - C^2 + C^2)}{\left[K^2 - C^2\right]^M} \\
&= \int \frac{\mathrm{d}^d K}{(2\pi)^d} \left\{ \frac{1}{\left[K^2 - C^2\right]^{M-1}} + \frac{C^2}{\left[K^2 - C^2\right]^M} \right\} \\
&= \frac{i(-1)^{M-1}}{(4\pi)^{d/2}} \left\{ \frac{1}{(C^2)^{M-1-d/2}} \cdot \frac{\Gamma\left(M - 1 - \frac{d}{2}\right)}{\Gamma\left(M-1\right)} - \right. \\
&\qquad \left. \frac{C^2}{(C^2)^{M-d/2}} \cdot \frac{\Gamma\left(M - \frac{d}{2}\right)}{\Gamma\left(M\right)} \right\} \\
&= \frac{i(-1)^{M-1}}{(4\pi)^{d/2}} \frac{1}{(C^2)^{M-1-d/2}} \frac{\Gamma\left(M - 1 - \frac{d}{2}\right)}{\Gamma\left(M\right)} \left[M - 1 - \left(M - 1 - \frac{d}{2} \right) \right] \\
&= \frac{i(-1)^{M-1}}{(4\pi)^{d/2}} \frac{1}{(C^2)^{M-1-d/2}} \frac{\Gamma\left(M - 1 - \frac{d}{2}\right)}{\Gamma\left(M\right)} \frac{d}{2} \\
&= \frac{i(-1)^{M-1}}{(4\pi)^{d/2}} \frac{1}{(C^2)^{M-1-d/2}} \frac{\Gamma\left(M - 1 - \frac{d}{2}\right)}{\Gamma\left(M\right)} \frac{\Gamma\left(1 + \frac{d}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} \, .
\end{aligned} \tag{5.36}$$

Repeating this procedure leads to the result

$$\int \frac{\mathrm{d}^d K}{(2\pi)^d} \frac{(K^2)^N}{\left[K^2 - C^2\right]^M}$$

$$= \quad \frac{i(-1)^{M-N}}{(4\pi)^{d/2}} \frac{1}{(C^2)^{M-N-d/2}} \frac{\Gamma\left(M-N-\frac{d}{2}\right)}{\Gamma\left(M\right)} \frac{\Gamma\left(N+\frac{d}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} \, . \tag{5.37}$$

Taken together, this allows to evaluate all d-dimensional one-loop integrals; in fact the major difficulty usually arises from the finite integrations over the z_i.

2. Another trick to deal with integrals of the function in Eq.(5.27),

$$\frac{k^\mu k^\nu \dots}{[(k+p_1)^2 - m_1^2 + i\epsilon][(k+p_2)^2 - m_2^2 + i\epsilon] \dots}\,, \tag{5.38}$$

goes back to Schwinger. Using Eq.(5.14), for all denominators,

$$\frac{1}{(a+i\epsilon)^n} = \frac{(-i)^n}{\Gamma(n)} \int\limits_0^\infty \mathrm{d}z \, z^{n-1} e^{iz(a+i\epsilon)} \tag{5.39}$$

and representing the k^μ of the numerators through

$$k^\mu = -i \left.\frac{\partial}{\partial x_\mu} e^{ix\cdot k}\right|_{x=0} = i \left.\frac{\partial}{\partial x_\mu} e^{-ix\cdot k}\right|_{x=0} \tag{5.40}$$

allows to cast Eq.(5.27) into a Gaussian exponential,

$$\begin{aligned}
&\frac{k^\mu k^\nu \dots}{[(k+p_1)^2 - m_1^2 + i\epsilon][(k+p_2)^2 - m_2^2 + i\epsilon] \dots} \\
&= \left(i\frac{\partial}{\partial x_\mu}\right)\left(i\frac{\partial}{\partial x_\nu}\right) \dots \left[\prod_{i=1}^n \int\limits_0^\infty \mathrm{d}z_i\right] \\
&\qquad (-i)^n \exp\left\{ i \left[\sum_{i=1}^n z_i[(k+p_i)^2 - m_i^2]\right] - ix\cdot k \right\}_{x=0}
\end{aligned} \tag{5.41}$$

Then, by replacing

$$\left[\prod_{i=1}^n \int\limits_0^\infty \mathrm{d}z_i\right] = \int\limits_0^\infty dZ \left[\prod_{i=1}^n \int\limits_0^1 \mathrm{d}z_i\right] \delta\left(1 - \sum_{i=1}^n z_i\right) \tag{5.42}$$

one has

$$(-i)^n \int \frac{\mathrm{d}^d k}{(2\pi)^d} \int\limits_0^\infty dZ \left[\prod_{i=1}^n \int\limits_0^1 \mathrm{d}z_i\right] \delta\left(1 - \sum_{i=1}^n z_i\right) \left[i^l \prod_{i=1}^l \frac{\partial}{\partial x_{\mu_l}}\right]$$

$$
\begin{aligned}
& \exp\left\{iZk^2 + 2iZk\left[\sum_{i=1}^{n} z_i p_i\right]] + iZ\left[\sum_{i=1}^{n} z_i(p_i^2 - m_i^2)\right] - ix\cdot k\right\}_{x=0} \\
= \;& i(-i)^n \int \frac{\mathrm{d}^d K}{(2\pi)^d} \int_0^\infty dZ Z^n \int_0^1 \frac{\mathrm{d}\mathcal{Z}}{\Gamma(n)} \left[i^l \prod_{i=1}^{l} \frac{\partial}{\partial x_{\mu_l}}\right] \\
& \exp\left\{iZ\left[K^2 - \left(\mathcal{P} - \frac{x}{2Z}\right)^2 + \mathcal{M}^2\right]\right\}_{x=0} \\
= \;& \frac{i(-i)^n}{i^{d/2}(4\pi)^{d/2}} \int_0^\infty dZ Z^{n-d/2} \int_0^1 \frac{\mathrm{d}\mathcal{Z}}{\Gamma(n)} \left[i^l \prod_{i=1}^{l} \frac{\partial}{\partial x_{\mu_l}}\right] \\
& \exp\left\{-iZ\left[\left(\mathcal{P} - \frac{x}{2Z}\right)^2 + \mathcal{M}^2\right]\right\}_{x=0}
\end{aligned}
\tag{5.43}
$$

Then, after taking the derivatives w.r.t. the various x_{μ_l} and after integrating over Z, one ends up with the expression one would have obtained with the Feynman method. To exemplify this, consider the cases of no or of two derivatives:

$$
\begin{aligned}
I_0 \;=\;& \frac{i(-i)^n}{i^{d/2}(4\pi)^{d/2}} \int_0^1 \frac{\mathrm{d}\mathcal{Z}}{\Gamma(n)} \\
& \int_0^\infty dZ Z^{n-1-d/2} \exp\left\{-iZ\left[\left(\mathcal{P} - \frac{x}{2Z}\right)^2 + \mathcal{M}^2\right]\right\}_{x=0} \\
=\;& \frac{i(-i)^n}{i^{d/2}(4\pi)^{d/2}} \int_0^1 \frac{\mathrm{d}\mathcal{Z}}{\Gamma(n)} \int_0^\infty dZ Z^{n-d/2-1} \exp\left\{-iZ\left[\mathcal{P}^2 + \mathcal{M}^2\right]\right\} \\
=\;& \frac{i(-i)^n}{i^{d/2}(4\pi)^{d/2}} \int_0^1 \frac{\mathrm{d}\mathcal{Z}}{\Gamma(n)} \frac{(-i)^{n-d/2}\Gamma\left(n - \frac{d}{2}\right)}{\left(\mathcal{P}^2 + \mathcal{M}^2\right)^{n-d/2}} \\
=\;& \frac{i(-1)^n}{(4\pi)^{d/2}} \frac{\Gamma\left(n - \frac{d}{2}\right)}{\Gamma(n)} \int_0^1 \frac{\mathrm{d}\mathcal{Z}}{\left(\mathcal{P}^2 + \mathcal{M}^2\right)^{n-d/2}}
\end{aligned}
\tag{5.44}
$$

and

$$
I_2 \;=\; \frac{i(-i)^n}{i^{d/2}(4\pi)^{d/2}} \int_0^1 \frac{\mathrm{d}\mathcal{Z}}{\Gamma(n)}
$$

$$
\begin{aligned}
&\int_0^\infty dZ Z^{n-1-d/2}\left[i^2\frac{\partial^2}{\partial x_\mu \partial x_\nu}\right]\exp\left\{-iZ\left[\left(\mathcal{P}-\frac{x}{2Z}\right)^2+\mathcal{M}^2\right]\right\}_{x=0}\\
=\;& \frac{-i(-i)^n}{i^{d/2}(4\pi)^{d/2}}\int_0^1 \frac{\mathrm{d}\mathcal{Z}}{\Gamma(n)}\\
&\int_0^\infty dZ Z^{n-1-d/2}\left[-i\frac{g^{\mu\nu}}{2Z}-\mathcal{P}^\mu\mathcal{P}^\nu\right]\exp\left\{-iZ\left[\mathcal{P}^2+\mathcal{M}^2\right]\right\}\\
=\;& \frac{-i(-i)^n}{i^{d/2}(4\pi)^{d/2}}\int_0^1 \frac{\mathrm{d}\mathcal{Z}}{\Gamma(n)}\left\{-\frac{ig^{\mu\nu}}{2}\frac{(-i)^{n-1-d/2}\Gamma\left(n-1-\frac{d}{2}\right)}{(\mathcal{P}^2+\mathcal{M}^2)^{n-1-d/2}}\right.\\
&\left.\qquad -\mathcal{P}^\mu\mathcal{P}^\nu\frac{(-i)^{n-d/2}\Gamma\left(n-\frac{d}{2}\right)}{(\mathcal{P}^2+\mathcal{M}^2)^{n-d/2}}\right\}\\
=\;& \frac{-i(-1)^n}{(4\pi)^{d/2}}\left\{\frac{g^{\mu\nu}}{2}\frac{\Gamma\left(n-1-\frac{d}{2}\right)}{\Gamma(n)}\int_0^1 \mathrm{d}\mathcal{Z}\frac{1}{(\mathcal{P}^2+\mathcal{M}^2)^{n-1-d/2}}\right.\\
&\left.\qquad -\frac{\Gamma\left(n-\frac{d}{2}\right)}{\Gamma(n)}\int_0^1 \mathrm{d}\mathcal{Z}\frac{\mathcal{P}^\mu\mathcal{P}^\nu}{(\mathcal{P}^2+\mathcal{M}^2)^{n-1-d/2}}\right\}.
\end{aligned}
\tag{5.45}
$$

5.1.3 Real diagrams and their results

Having derived some master integrals, one can now start to regularise the divergences of $\lambda\phi^4$ theory at the one-loop level.

Tadpole

As a warm-up, consider the tadpole diagram, cf. Fig.5.6. In d dimensions, it contributes to the two-point function as

$$
\begin{aligned}
G^{(2)}_{\text{tadpole}}(p) &= \frac{-i\lambda_0}{2}\mu^{4-d}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{i}{k^2-m^2+i\epsilon}\\
&= \frac{i\lambda_0}{2}\cdot\frac{i(\mu^2)^{2-d/2}}{(4\pi)^{d/2}}\frac{1}{(m^2)^{1-d/2}}\frac{\Gamma\left(1-\frac{d}{2}\right)}{\Gamma(1)},
\end{aligned}
\tag{5.46}
$$

where a symmetry factor 1/2 has been added and μ^{4-d} has been introduced to ensure the correct dimensions of mass for this correction term. In fact,

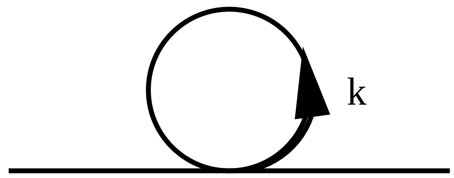


Figure 5.6: Tadpole, once again.

the renormalisation scale μ will play a crucial role in fixing the physics of the renormalised theory. Setting $d = 4 - 2\eta$ and expanding around $\eta = 0$ yields, up to finite terms,

$$\begin{aligned} G^{(2)}_{\rm tadpole}(p) &= -\frac{m^2}{2}\frac{\lambda_0}{(4\pi)^2}\left(-\frac{1}{\eta} - \psi(2)\right)\left(1 + \eta\log\frac{\mu^2}{m^2}\right)\left(1 + (4\pi)^\eta\right) \\ &= \frac{m^2}{2}\frac{\lambda_0}{(4\pi)^2}\left[\frac{1}{\eta}\left(1 + \eta\log\frac{4\pi e^{-\gamma_E}\mu^2}{m^2}\right) + 1\right] \\ &= \frac{m^2}{2}\frac{\lambda_0}{(4\pi)^2}\left[\frac{1}{\eta}\left(1 + \eta\log\frac{\hat{\mu}^2}{m^2}\right) + 1\right] \end{aligned} \tag{5.47}$$

Already in this simple first example the structure of UV-divergences in dimensional regularisation at the one-loop level becomes apparent: the divergence is encoded in a simple pole, proportional to $1/(4-d)$, the finite parts are, at this moment, completely arbitrary because of the arbitrary scale μ.

Fish

The next-simple example is the fish-diagram of Fig.5.7, which is the first higher-order correction to the four-point function. In $d = 4 - 2\eta$ dimensions, and with $p = p_1 + p_2$, it reads

$$\begin{aligned} G^{(4)}_{\rm fish}(p_1,\, p_2) &= \frac{(-i\lambda_0)^2}{2}\mu^{4-d}\int\frac{{\rm d}^dk}{(2\pi)^d}\frac{i}{k^2 - m^2 + i\epsilon}\,\frac{i}{(k+p)^2 - m^2 + i\epsilon} \\ &= \frac{\lambda_0^2\mu^{4-d}}{2}\int\frac{{\rm d}^dk}{(2\pi)^d}\int\limits_0^1{\rm d}x\frac{1}{\left[k^2 + 2xp\cdot k + xp^2 - m^2\right]^2} \end{aligned} \tag{5.48}$$

Substituting $K^\mu = k^\mu + xp^\mu$ and integrating over K leads to

$$G^{(4)}_{\rm fish}(p_1,\, p_2) = \frac{i\lambda_0^2\mu^{4-d}\Gamma\left(2 - \frac{d}{2}\right)}{2(4\pi)^{d/2}}\int\limits_0^1{\rm d}x\frac{1}{\left[m^2 + p^2x(x-1)\right]^{2-d/2}} \tag{5.49}$$

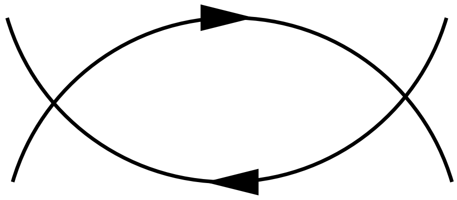


Figure 5.7: Fish diagram.

Expanding, as before, around $\eta = 0$ leads to

$$\begin{aligned} G^{(4)}_{\text{fish}}(p_1,\, p_2) &= \frac{i\lambda_0^2}{2(4\pi)^2}\left\{\frac{1}{\eta} + \psi(1) - \int\limits_0^1 \mathrm{d}x \log\left[\frac{m^2 + p^2 x(x-1)}{4\pi\mu^2}\right]\right\} \\ &= \frac{i\lambda_0^2}{2(4\pi)^2}\left\{\frac{1}{\eta} + \psi(1) - \left[-2 - \log\frac{\mu^2}{m^2}\right.\right. \\ &\qquad\qquad \left.\left. + \sqrt{1 - \frac{4m^2}{p^2}} \log\frac{\sqrt{1-\frac{4m^2}{p^2}}+1}{\sqrt{1-\frac{4m^2}{p^2}}-1}\right]\right\} \\ &= \frac{i\lambda_0^2}{2(4\pi)^2}\left\{\frac{1}{\eta} + 2 + \log\frac{\hat{\mu}^2}{m^2} + \sqrt{1-\frac{4m^2}{p^2}} \log\frac{\sqrt{1-\frac{4m^2}{p^2}}+1}{\sqrt{1-\frac{4m^2}{p^2}}-1}\right\}. \end{aligned} \tag{5.50}$$

Here, in the evaluation of the finite x integral, it has been observed that due to $0 \le x(1-x) \le 1/4$ the integrand is always positive in the range of integration. Then, one can use

$$\int\limits_0^1 \mathrm{d}x \log\left[1 + \frac{4x(x-1)}{a}\right] = -2 - \sqrt{1+a}\, \log\frac{\sqrt{1+a}-1}{\sqrt{1+a}+1}\,. \tag{5.51}$$

Double scoop

Going to one more loop, the simplest diagram is the "double scoop", cf. Fig.5.8. It contributes to the two-point function, employing the same tech-

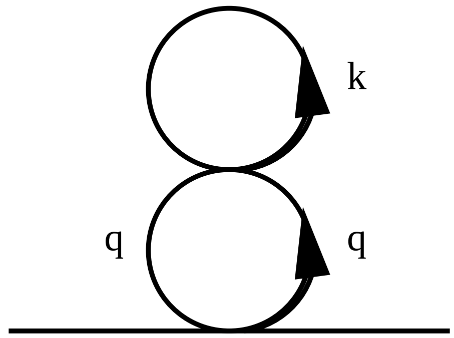


Figure 5.8: "Double scoop" diagram.

niques as before, it reads

$$
\begin{aligned}
&G^{(2)}_{\text{doublescoop}} \\
&= \frac{(-i\lambda_0)^2}{4}\mu^{8-2d}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{i}{k^2-m^2+i\epsilon}\int\frac{\mathrm{d}^dq}{(2\pi)^d}\frac{i}{[q^2-m^2+i\epsilon]^2} \\
&= \frac{i^2(-1)\lambda_0^2\mu^{8-2d}}{4(4\pi)^d}\frac{\Gamma\left(1-\frac{d}{2}\right)}{\Gamma(1)}\frac{1}{(m^2)^{1-d/2}}\frac{\Gamma\left(2-\frac{d}{2}\right)}{\Gamma(2)}\frac{1}{(m^2)^{2-d/2}} \\
&= \frac{m^2\lambda_0^2}{4(4\pi)^4}\left[-\frac{1}{\eta}-\left(\log\frac{\hat{\mu}^2}{m^2}+1\right)-\frac{\eta}{2}\left(\log\frac{\hat{\mu}^2}{m^2}+1\right)^2-\frac{\eta}{2}\left(1+\frac{\pi^2}{6}\right)\right] \\
&\qquad\left[\frac{1}{\eta}+\log\frac{\hat{\mu}^2}{m^2}+\frac{\eta}{2}\left(\log^2\frac{\hat{\mu}^2}{m^2}+\frac{\pi^2}{6}\right)\right]. \qquad (5.52)
\end{aligned}
$$

Even without multiplying, one realizes immediately that there is a double pole, since there are two individually divergent diagrams. The arbitrary scale then renders the single pole already arbitrary.

Setting sun

As the final enterprise in this section, the setting sun diagram will be calculated. It also contributes to the second order correction of the two point function. Since this is the first non-trivial (non-vanishing under normal-ordering) self-energy correction it will be dubbed $\Sigma(p)$, in accordance with the nomenclature of QED and other theories. Suppressing the terms $+i\epsilon$ in

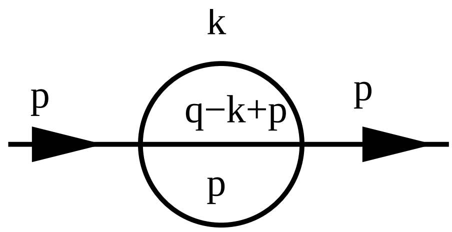


Figure 5.9: Setting sun, once again.

the propagators, it reads

$$\Sigma(p) \;=\; \frac{(i\lambda_0)^2}{6}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\int\frac{\mathrm{d}^dq}{(2\pi)^d}\frac{i^3(\mu^2)^{4-d}}{[k^2-m^2]\,[q^2-m^2]\,[(k-q+p)^2-m^2]}\,. \tag{5.53}$$

In diagrams with several, nested loops, the divergences usually will show up also in the parametric integrals at the end. to keep things as easy as possible, one would prefer to have as little nuisances as possible in these last integrals; most of the time, they're tricky enough. To alleviate this burden, introduce a 1, this time of the form

$$1 = \frac{1}{2d}\left[\frac{\partial k_\mu}{\partial k_\mu}+\frac{\partial q_\mu}{\partial q_\mu}\right]\,, \tag{5.54}$$

integrate by parts and forget about the (vanishing) surface terms. One ends up with

$$\Sigma(p) \;=\; -\frac{i\lambda_0^2(\mu^2)^{4-d}}{6\cdot(2d)}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\int\frac{\mathrm{d}^dq}{(2\pi)^d}\left\{\left[k_\mu\frac{\partial}{\partial k_\mu}+q_\mu\frac{\partial}{\partial q_\mu}\right]\frac{1}{[k^2-m^2]\,[q^2-m^2]\,[(k-q+p)^2-m^2]}\right\}\,. \tag{5.55}$$

Explicit differentiation and the freedom in applying linear changes in the integration variables yields

$$\Sigma(p) \;=\; -\frac{i\lambda_0^2(\mu^2)^{4-d}}{6\cdot(d-3)}\left[3m^2K(p^2)+p^\mu K_\mu(p^2)\right]\,, \tag{5.56}$$

where

$$\begin{aligned}
K(p^2) &= \int \frac{\mathrm{d}^d k}{(2\pi)^d} \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{1}{\left[k^2-m^2\right]^2 \left[q^2-m^2\right] \left[(k-q+p)^2-m^2\right]} \\
K_\mu(p^2) &= \int \frac{\mathrm{d}^d k}{(2\pi)^d} \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{(k-q+p)_\mu}{\left[k^2-m^2\right] \left[q^2-m^2\right] \left[(k-q+p)^2-m^2\right]^2}
\end{aligned} \tag{5.57}$$

Naively, $K(p^2)$ is logarithmically divergent, and $K_\mu(p^2)$ is linearly divergent. First of all, $K(p^2)$ is dealt with. Due to lorentz-invariance it depends on p^2 rather than on p^μ. Doing Feynman parameters and the integrals loop by loop one finds, with the eventual substitution $q^\mu \to Q^\mu = q^\mu - x(p+k)^\mu$ and by heavily relying on Eqs.(5.29) and (5.37)

$$\begin{aligned}
K(p^2) &= \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{\left[k^2-m^2\right]^2} \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{1}{\left[q^2-m^2\right] \left[(k-q+p)^2-m^2\right]} \\
&= \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{\left[k^2-m^2\right]^2} \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^d Q}{(2\pi)^d} \frac{1}{\left[Q^2-m^2+x(1-x)(k+p)^2\right]^2} \\
&= \frac{i\Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^{d/2}} \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{\left[k^2-m^2\right]^2 \left[m^2-x(1-x)(k+p)^2\right]^{2-d/2}} \\
&= \frac{i\Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^{d/2}} \frac{\Gamma\left(2+2-\frac{d}{2}\right)}{\Gamma(2)\,\Gamma\left(2-\frac{d}{2}\right)} \int_0^1 \mathrm{d}x \left[x(1-x)\right]^{d/2-2} \int_0^1 \mathrm{d}y y^{1-d/2}(1-y) \\
&\qquad \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{\left[k^2+p^2 y(1-y)-m^2\left(1-y+\frac{y}{x(1-x)}\right)\right]^{4-d/2}} \\
&= \frac{-\Gamma(4-d)}{(4\pi)^d} \int_0^1 \mathrm{d}x \left[x(1-x)\right]^{d/2-2} \int_0^1 \mathrm{d}y y^{1-d/2}(1-y) \\
&\qquad \left[p^2 y(1-y)-m^2\left(1-y+\frac{y}{x(1-x)}\right)\right]^{d-4}
\end{aligned} \tag{5.58}$$

This is a tricky integral, and apart from the obvious divergence contained in the Γ-function, there is yet another one in the parametric integral over y. This is clear, since for $y \to 0$ the integral behaves roughly like dy/ym^2.

Therefore, one has to set $d = 4 - 2\eta$. When employing

$$y^{-1+\eta} = \frac{1}{\eta}\frac{\mathrm{d}y^{\eta}}{\mathrm{d}y} \tag{5.59}$$

and integrating by parts, this yields

$$\begin{aligned} K(p^2) &= \frac{-\Gamma(2\eta)}{(4\pi)^{4-2\eta}} \int_0^1 \mathrm{d}x\,[x(1-x)]^{-\eta} \int_0^1 \mathrm{d}y y^{-1+\eta}(1-y) \\ &\qquad \left[p^2 y(1-y) - m^2\left(1 - y + \frac{y}{x(1-x)}\right)\right]^{-2\eta} \\ &= \frac{-\Gamma(2\eta)}{(4\pi)^{4-2\eta}} \int_0^1 \mathrm{d}x\,[x(1-x)]^{-\eta} \int_0^1 \mathrm{d}y y^{\eta} A^{-2\eta}(x,\,y,\,p^2) \\ &\qquad \left\{1 + 2\eta(1-y)\frac{\mathrm{d}y^{\eta}}{\mathrm{d}y}\log[A(x,\,y,\,p^2)]\right\}\,, \end{aligned} \tag{5.60}$$

where the abbreviation

$$A(x,\,p^2) = \left[p^2 y(1-y) - m^2\left(1 - y + \frac{y}{x(1-x)}\right)\right] \tag{5.61}$$

has been introduced. Expanding around $\eta = 0$, one ends up with

$$K(p^2) = \frac{-\Gamma(2\eta)}{(4\pi)^{4-2\eta}}\frac{1}{\eta}\left[1 + \eta - 2\eta\log m^2 + \mathcal{O}(\eta^2)\right]\,. \tag{5.62}$$

In a similar fashion $K_\mu(p^2)$ can be evaluated. Then,

$$\begin{aligned} \frac{p^\mu K_\mu(p^2)}{p^2} &= \frac{-\Gamma(2\eta)}{(4\pi)^{4-2\eta}} \int_0^1 \mathrm{d}x\,[x(1-x)]^{-\eta} \int_0^1 \mathrm{d}y y^{\eta}(1-y)\,A^{-2\eta}(x,\,y,\,p^2) \\ &= \frac{-\Gamma(2\eta)}{(4\pi)^{4-2\eta}}\left[\frac{1}{2} + \mathcal{O}(\eta)\right]\,. \end{aligned} \tag{5.63}$$

Putting it all together yields for the setting sun

$$\begin{aligned} \Sigma(p) &= -\frac{i\lambda_0^2(\mu^2)^{4-d}}{6\cdot(d-3)}\left[3m^2 K(p^2) + p^\mu K_\mu(p^2)\right] \\ &= \frac{i\lambda_0^2}{6(4\pi)^4}\left[\frac{3m^2}{2\eta^2} + \frac{3m^2}{\eta}\left(\frac{3}{2} + \log\frac{\hat{\mu}^2}{m^2}\right) + \frac{p^2}{4\eta} + \text{finite}\right]\,. \end{aligned} \tag{5.64}$$

The evaluation of the finite terms is a very difficult task. It involves the dilogarithm,

$$\mathrm{Li}_2(x) = -\int\limits_0^1 \frac{\mathrm{d}t}{t}\log(1-xt)\,, \tag{5.65}$$

a notorious result for two-loop calculations.

5.2 Renormalisation

5.2.1 Counter-terms and renormalisation conditions

In the previous section, it was shown, how to evaluate Feynman diagrams involving loops, which carry divergences. In $\lambda\phi^4$ theory, some diagrams – those related to two- and four-point functions – are UV divergent. In dimensional regularisation , these divergences appear as poles $1/(4-d)$ in the plane of dimension d. In addition, the non-leading terms (poles of lower order or finite terms) were arbitrary, stemming from the introduction of the arbitrary mass scale μ.
Having regularised the diagrams, in the next step the divergences are eliminated order by order in λ_0 by modifying the Feynman such that finite results are guaranteed. Basically, this amounts to modifying the Lagrangian by adding terms. These additional terms are called counter-terms, because they are used to cancel the divergences.

First example: Tadpole

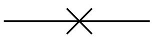
As a first example, consider the tadpole. Its contribution to the two point function, as evaluated in Eq.(5.47) reads

$$G^{(2)}_{\mathrm{tadpole}}(p) = \frac{m^2}{2}\frac{\lambda_0}{(4\pi)^2}\left[\frac{1}{\eta}\left(1+\eta\log\frac{\hat{\mu}^2}{m^2}\right)+1\right]\,. \tag{5.66}$$

The pole in $1/\eta$ can be hidden by adding to the Lagrangian the term

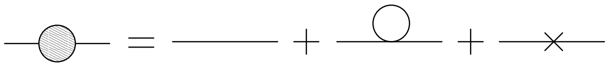
$$\mathcal{L}^{\mathrm{c.t.}}_{\mathrm{tadpole}} = \frac{m^2}{4}\frac{\lambda_0}{(4\pi)^2}\left[\frac{1}{\eta}+F_1(\eta,\,m^2)\right]\phi^2\,, \tag{5.67}$$

where $F_1(\eta, m^2)$ is an arbitrary, dimensionless function, which has no poles when $\eta \to 0$. It is present to reflect the arbitrariness of the full procedure at this point. Adding this term adds a new Feynman rule,



$$= -\frac{m^2}{2}\frac{\lambda_0}{(4\pi)^2}\left[\frac{1}{\eta} + F_1(\eta,\, m^2)\right] . \tag{5.68}$$

Then computing the propagator to first order in λ_0 diagrammatically yields



(5.69)

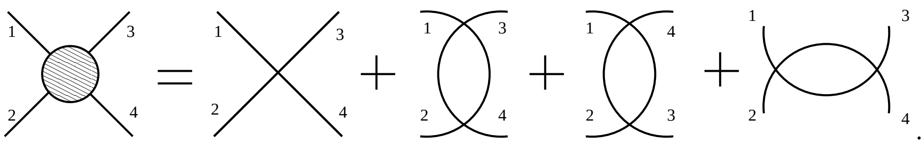
The inverse propagator then reads (the minus-sign stems from inversion)

$$\left[\tilde{\Gamma}^{(2)}_{\text{c.t.}}\right]^{-1} = p^2 + m^2\left[1 - \frac{\lambda_0}{2(4\pi)^2}\left(\log\frac{\hat{\mu}^2}{m^2} - F_1(\eta,\, m^2)\right)\right] + \mathcal{O}(\lambda_0^2)\,, \tag{5.70}$$

and, by construction, the poles have vanished. This very naive procedure obviously makes the result finite at the given order. It is crucial to note that the structure of the counter term is the same as of another term in the original Lagrangian, the mass term. Hence, no new structure had to be added.

Second example: Fish

Going to a higher order in λ_0, i.e. $\mathcal{O}(\lambda_0^2)$, there is a one-particle irreducible contribution to the four-point function. Using the result of Eq.(5.50) the four-point function at this order reads



(5.71)

and in d dimensions it is given by

$$-\lambda_0\left[1 - \frac{3}{2}\frac{\lambda_0}{(4\pi)^2}\left(\frac{1}{\eta} + \log\frac{\hat{\mu}^2}{m^2} + 2 - \frac{1}{3}A(s,\, t,\, u)\right)\right] , \tag{5.72}$$

where

$$A(s,\, t,\, u) = \sum_{\{ij\}=\{12\},\,\{13\},\,\{14\}} \sqrt{1+\frac{4m^2}{s_{ij}}}\,\log\frac{\sqrt{1+\frac{4m^2}{s_{ij}}}+1}{\sqrt{1+\frac{4m^2}{s_{ij}}}-1} \tag{5.73}$$

and

$$s_{ij} = (p_i + p_j)^2\,. \tag{5.74}$$

As was to be expected, the four-point function so far is divergent. To change this, and to ensure convergence of the four-point function at $\mathcal{O}(\lambda_0^2)$, one adds to the Lagrangian a term

$$\mathcal{L}_{\text{fish}}^{\text{c.t.}} = \frac{\mu^{2\eta}\lambda_0}{(4!}\frac{3\lambda_0}{2(4\pi)^2}\left[\frac{1}{\eta}+G_1(\eta,\, m^2)\right]\phi^4\,, \tag{5.75}$$

which results in a new Feynman rule, namely

$$= -\frac{3}{2}\frac{\mu^{2\eta}\lambda_0^2}{(4\pi)^2}\left[\frac{1}{\eta}+G_1(\eta,\, m^2)\right]\,. \tag{5.76}$$

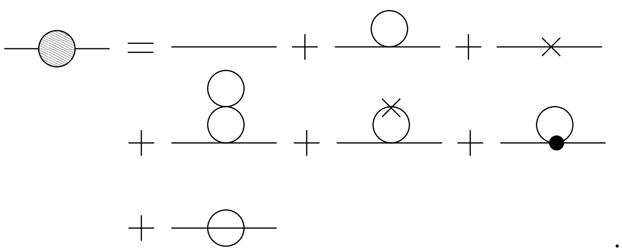
Again, adding such a counter-diagram to the previously divergent result renders it finite,

$$\tilde{\Gamma}^{(4)}_{\text{c.t.}} = -\mu^{2\eta}\lambda\left[1-\frac{3}{2}\frac{\lambda_0}{4\pi}\left(\log\frac{\hat{\mu}^2}{m^2}+2-G_1(\eta,\, m^2)-\frac{1}{3}A(s,\, t,\, u)\right)\right]\,. \tag{5.77}$$

Again, the counter term had the same structure as another, original term from the Lagrangian, namely the interaction term.

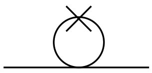
Self-energy at $\mathcal{O}(\lambda_0^2)$

The $\mathcal{O}(\lambda_0^2)$ contributions to the self energy, i.e. to the two-point function can be composed in a similar way. The only little caveat here is that the two counter terms induce corrections which look like “tadpole + counter-term”. The inverse two-point function can then be represented as

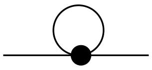


(5.78)

The effect of the two new Feynman rules clearly can be seen. The resulting corrections are simple to calculate. Exhibiting only the poles, one has



$$= \frac{m^2\lambda_0^2}{4(4\pi)^4}\left[\frac{1}{\eta^2} + \frac{1}{\eta}\left(\log\frac{\hat{\mu}^2}{m^2} + F_1(\eta,\, m^2)\right) + \ldots\right] \tag{5.79}$$

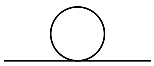
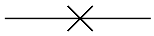


$$= \frac{3m^2\lambda_0^2}{4(4\pi)^4}\left[\frac{1}{\eta^2} + \frac{1}{\eta}\left(1 + \log\frac{\hat{\mu}^2}{m^2} + G_1(\eta,\, m^2)\right) + \ldots\right] \tag{5.80}$$

Comparing Eq.(5.79) with the result of the double scoop, Eq.(5.52),

$$G^{(2)}_{\text{doublescoop}} = -\frac{m^2\lambda_0^2}{4(4\pi)^4}\left[\frac{1}{\eta^2} + \frac{1}{\eta}\left(1 - \log\frac{\hat{\mu}^2}{m^2}\right) + \ldots\right] \tag{5.81}$$

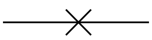
one realizes immediately that the double poles of both diagrams cancel, leaving only some simple poles. This, however, was to be expected, because the sum of the tadpole and its counter-term is, by construction, finite,

 $+$  $=$ finite. (5.82)

Adding all diagrams up to $\mathcal{O}(\lambda^2)$, one finds

$$G^{(2)}(p) = -\frac{\lambda^2}{(4\pi)^4}\frac{p^2}{24\eta} + \frac{\lambda^2}{(4\pi)^4}\frac{m^2}{2}\left[\frac{1}{\eta^2} + \frac{1}{2\eta}\left(F_1 + 3G_1 - 1\right) + \ldots\right], \tag{5.83}$$

where, again, finite parts have been omitted. The interesting thing about this expression is that the $\log\mu^2$ terms are not present in the simple poles. To cancel the poles in this expression, two counter-terms will be introduced. First of all, another mass counter-term is necessary, modifying the Feynman rule of Eq.(5.68) to read



$$\begin{aligned} = \;& -\frac{m^2}{2}\left\{\frac{1}{\eta^2}\frac{\lambda^2}{(4\pi)^4} + \frac{1}{\eta}\left[\frac{\lambda}{(4\pi)^2} + \frac{\lambda^2}{(4\pi)^4}\left(F_1 + 3G_1 - 1\right)\right]\right. \\ & \left. + \frac{\lambda^2}{(4\pi)^4}F_2 + \frac{\lambda}{(4\pi)^2}F_1\right\}, \end{aligned} \tag{5.84}$$

where all F and G are functions of η and m and finite as $\eta \to 0$. The corresponding counter-term reads

$$\mathcal{L}_m^{\text{c.t.}} = \frac{m^2}{4}\phi^2 \left\{ \frac{\hat{\lambda}^2}{\eta^2} + \frac{1}{\eta}\left[\hat{\lambda}+)\frac{\hat{\lambda}^2}{4}(F_1 + 3G_1 - 1)\right] + \hat{\lambda}^2 F_2 + \hat{\lambda} F_1 \right\}, \tag{5.85}$$

where $\hat{\lambda} = \lambda/(4\pi)^2$ has been introduced. To compensate for the other term of the two-point function above, yet another term is added, namely

$$\mathcal{L}_p^{\text{c.t.}} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi\left[-\frac{\hat{\lambda}^2}{24\eta} - \hat{\lambda}^2 H_2\right], \tag{5.86}$$

where the new function H_2 is also a function of η and m, again finite as $\eta \to 0$.

Renormalised theory

Of course this game of identifying the poles term by term and order by order and introducing counter-terms before going to the next order can be continued to arbitrary length. In each order, new counter-terms will emerge; the remarkable thing so far – which holds actually true for arbitrary orders in this theory – is that no new interactions have to be introduced, only the coefficients grow. If this feature continues to all orders in a theory the theory is called renormalisable. The outline of a proof of renormalisability clearly is beyond the scope of this lecture. Suffice to point out that terms of the form $\log p^2$ or similar, which may emerge from overlapping divergences, are a danger to renormalisability, since they cannot be absorbed in local structures. One can now add the original Lagrangian and the counter-terms. The resulting Lagrangian called the renormalised Lagrangian then yields finite results at this order,

$$\mathcal{L}^{\text{ren}} = \mathcal{L} + \mathcal{L}^{\text{c.t.}}, \tag{5.87}$$

where the original and the counter-term Lagrangian read

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\mu^{2\eta}\phi^4 \tag{5.88}$$

and

$$\mathcal{L}^{\text{c.t.}} = A\cdot\frac{1}{2}\partial^\mu\phi\partial_\mu\phi + B\cdot\frac{1}{2}m^2\phi^2 + C\cdot\frac{\lambda}{4!}\mu^{2\eta}\phi^4, \tag{5.89}$$

respectively. The prefactors A, B, and C are chosen such that the result is finite as $\eta \to 0$. By redefinition one can thus write

$$\mathcal{L}^{\rm ren} = \frac{1}{2}\partial^\mu \phi_0 \partial_\mu \phi_0 + \frac{1}{2} m_0^2 \phi_0^2 + \frac{\lambda_0}{4!}\mu^{2\eta}\phi_0^4 \,, \tag{5.90}$$

where

$$\begin{aligned} \phi_0 &= (1+A)^{1/2}\phi = Z_\phi^{1/2}\phi \\ m_0^2 &= \frac{1+B}{1+A} m^2 = \frac{1+B}{Z_\phi} m^2 \\ \lambda_0 &= \frac{1+C}{(1+A)^2}\lambda\mu^{2\eta} = \frac{1+C}{Z_\phi^2}\lambda\mu^{2\eta} \,. \end{aligned} \tag{5.91}$$

The quantities ϕ_0, m_0, and λ_0 are called the bare fields, mass, and coupling constant, respectively, and they are infinite as $\eta \to 0$. The renormalised quantities ϕ, m, and λ, in contrast, are finite. But so far, these quantities are also arbitrary. Later on, they will be identified with physical parameters and thus they will be linked to measurements.

5.2.2 The renormalisation group

n-point functions

Starting from the renormalised Lagrangian, one obtains the Green's functions (or n-point functions) of the previous sections by just replacing the renormalised with the bare quantities. These Green's functions $\Gamma_0^{(n)}$ are finite. The connection to the original ones is given through

$$\Gamma_0^{(n)}(p_1,\, \ldots,\, p_n;\, \lambda_0,\, m_0;\, \eta) = Z_\phi^{-n/2}\Gamma^{(n)}(p_1,\, \ldots,\, p_n;\, \lambda,\, m;\, \mu,\, \eta)\,, \tag{5.92}$$

where the effect of the external field renormalisation is embedded in the prefactor $Z_\phi^{-n/2}$. Both Green's functions are finite as $\eta \to 0$, and one can understand either the bare or the renormalised parameters as the independent ones. However, obviously the left hand side of the equation is independent of the arbitrary mass-scale μ, whereas the right hand side depends explicitly and implicitly (through λ and m) on μ. Therefore, differentiating the above equation with respect to μ leads to a differential equation that shows the full power of renormalisation,

$$\left[\mu\frac{\partial}{\partial\mu} + \mu\frac{\partial\lambda}{\partial\mu}\frac{\partial}{\partial\lambda} + \mu\frac{\partial m}{\partial\mu}\frac{\partial}{\partial m} - \frac{n}{2}\mu\frac{\partial \log Z_\phi}{\partial\mu}\right]\Gamma^{(n)} = 0\,. \tag{5.93}$$

The important thing to note on this equation is that it involves only the renormalised Green' function which is finite as $\eta \to 0$. Defining the also finite, dimensionless coefficient functions

$$\begin{aligned}
\beta\left(\lambda,\, \frac{\mu}{m},\, \eta\right) &= \mu \frac{\partial \lambda}{\partial \mu} \\
\gamma_m\left(\lambda,\, \frac{\mu}{m},\, \eta\right) &= \frac{\mu}{2} \frac{\partial m^2}{\partial \mu} \\
\gamma_d\left(\lambda,\, \frac{\mu}{m},\, \eta\right) &= \frac{\mu}{2} \frac{\partial \log Z_\phi}{\partial \mu}\,,
\end{aligned} \tag{5.94}$$

one can rewrite the renormalisation group equation in the following way:

$$\left[\mu \frac{\partial}{\partial \mu} + \beta\left(\lambda,\, \frac{\mu}{m},\, \eta\right) \frac{\partial}{\partial \lambda} + \gamma_m\left(\lambda,\, \frac{\mu}{m},\, \eta\right) \frac{\partial}{\partial m} - n\gamma_d\left(\lambda,\, \frac{\mu}{m},\, \eta\right)\right] \Gamma^{(n)} = 0\,. \tag{5.95}$$

On the other hand, $\Gamma^{(n)}$ has a dimension equal to $4 - n + \eta(n-2)$. which can be deduced as the sum of its degree of homogeneity in its dimensionful parameter(s),

$$\left[\mu \frac{\partial}{\partial \mu} + s\frac{\partial}{\partial s} + m\frac{\partial}{\partial m} - [4 - n + \eta(n-2)]\right] \Gamma^{(n)}(sp,\, m,\, \lambda;\, ,\, \mu,\, \eta) = 0\,. \tag{5.96}$$

Combining both equations and eliminating $\mu\partial/\partial\mu$ leads, for $\eta \to 0$, to

$$\begin{aligned}
\Bigg\{-s\frac{\partial}{\partial s} &+ \beta\left(\lambda,\, \frac{\mu}{m},\, \eta\right) \frac{\partial}{\partial \lambda} + \left[\gamma_m\left(\lambda,\, \frac{\mu}{m},\, \eta\right) - 1\right] m\frac{\partial}{\partial m} \\
&- n\gamma_d\left(\lambda,\, \frac{\mu}{m},\, \eta\right) + 4 - n\Bigg\} \Gamma^{(n)}(sp,\, m,\, \lambda;\, ,\, \mu) = 0\,.
\end{aligned} \tag{5.97}$$

This equation provides an information on what happens for the n-point functions when the momenta are scaled by an overall factor s. On of the main difficulty in solving it is that the coefficient functions in it depend on two parameters, λ and μ/m. In principle they can be computed order by order in perturbation theory, but so far they are quite arbitrary since the finite pieces of the renormalisation procedure have not been specified yet. And of course, these functions depend on the finite pieces.

The bare parameters can be expressed as a Laurent series in η through the renormalised parameters and thus exhibit their property of being infinite for $\eta \to 0$:

$$\begin{aligned}
\lambda_0 &= \mu^{2\eta} \left[\sum_{k=0}^{\infty} \frac{a_k\left(\lambda,\, \frac{\mu}{m},\, \eta\right)}{\eta^k} \right] \\
m_0^2 &= m^2 \left[\sum_{k=0}^{\infty} \frac{b_k\left(\lambda,\, \frac{\mu}{m},\, \eta\right)}{\eta^k} \right] \\
Z_\phi &= \left[\sum_{k=0}^{\infty} \frac{c_k\left(\lambda,\, \frac{\mu}{m},\, \eta\right)}{\eta^k} \right] ,
\end{aligned} \tag{5.98}$$

where the a_k, b_k, and c_k are finite as $\eta \to 0$. In fact, since this would allow to shift them for $k \geq 1$ they do not depend on η at all. The first coefficients are up to $\mathcal{O}(\lambda^2)$ given by

$$\begin{aligned}
a_0\left(\lambda,\, \frac{\mu}{m},\, \eta\right) &= \lambda \left(1 + \frac{3}{2} \frac{\lambda}{(4\pi)^2}\, G_1 \right) \\
a_1\left(\lambda,\, \frac{\mu}{m},\, \eta\right) &= \frac{3}{2} \frac{\lambda^2}{(4\pi)^2} \\
b_0\left(\lambda,\, \frac{\mu}{m},\, \eta\right) &= 1 + \frac{1}{2} \left(\frac{\lambda}{(4\pi)^2}\, F_1 + \frac{\lambda^2}{(4\pi)^4}\, F_2 \right) + \frac{\lambda^2}{(4\pi)^4}\, H_2 \\
b_1\left(\lambda,\, \frac{\mu}{m},\, \eta\right) &= \frac{1}{2} \left(\frac{\lambda}{(4\pi)^2} + \frac{\lambda^2}{4(4\pi)^4}\, (F_1 + 3G_1 - 1) \right) + \frac{\lambda^2}{24(4\pi)^4} \\
b_2\left(\lambda,\, \frac{\mu}{m},\, \eta\right) &= \frac{\lambda^2}{2(4\pi)^4} \\
c_0\left(\lambda,\, \frac{\mu}{m},\, \eta\right) &= 1 - \frac{\lambda^2}{(4\pi)^4}\, H_2 \\
c_1\left(\lambda,\, \frac{\mu}{m},\, \eta\right) &= -\frac{\lambda^2}{24(4\pi)^4} \,.
\end{aligned} \tag{5.99}$$

Comparing the single terms it becomes clear that the coefficients depend on μ/m only through the so far unspecified functions $F_{1,2}$, G_1 and H_1 introduced before. This is easy to understand: since the counter-terms are constructed such that they come into full effect in the ultraviolet, the masses should not matter - they are just not the relevant scales. Thus, as long as all finite parts are set to zero, the bare quantities are not expected to depend on the masses or the scale μ. Therefore, before discussing any feature of the solutions of the renormalisation group equation, renormalisation prescriptions and their effect on the finite pieces are briefly discussed.

Renormalisation prescriptions

Having renormalised a theory by introducing the concept of bare and renormalised quantities it remained to be seen how the contact of them and to physical quantities is established. This usually is achieved by specifying a scale at which the renormalised Green's function has a specific value. Such considerations therefore define the renormalised quantities and their interplay inside the Green's functions and thus the finite pieces of the subtraction terms. They are therefore called subtraction methods or renormalisation prescriptions. In the following two such prescriptions will be presented:

1. Define the parameters through

$$\begin{aligned} \Gamma^{(2)}(p,\,m) &= p^2 - m^2 \quad \text{at} \quad p^2 = 0\,, \\ \Gamma^{(4)}(p_1,\,p_2,\,p_3,\,p_4) &= \mu^{2\eta}\lambda \quad \text{at} \quad p_i = 0\,. \end{aligned} \tag{5.100}$$

   The first of these equations actually fixes the mass as well as the fields renormalisation. The corresponding subtraction terms read

$$F_1 = 1 + \log\frac{\hat{\mu}^2}{m^2}\,,\quad G_1 = \log\frac{\hat{\mu}^2}{m^2}, \tag{5.101}$$

   and all others being equal to zero.

2. Set all subtraction terms to zero, a prescription dubbed mass-independent scheme for obvious reasons. This allows for a very simple and beautiful solution of the renormalisation group equation.

The β-function and the running coupling

Using the mass-independent subtraction scheme one can express the bare coupling constant as

$$\lambda_0 = \mu^{2\eta}\left[\lambda + \sum_{k=1}^{\infty}\frac{a_k(\lambda)}{\eta^k}\right]\,. \tag{5.102}$$

Differentiating w.r.t. μ at fixed λ_0 yields

$$0 = 2\eta \left[\lambda + \sum_{k=1}^{\infty} \frac{a_k(\lambda)}{\eta^k}\right] + \mu \frac{\partial \lambda}{\partial \mu} \left[1 + \sum_{k=1}^{\infty} \frac{a'_k(\lambda)}{\eta^k}\right] , \tag{5.103}$$

where the prime here denotes differentiation w.r.t. λ. Here, both λ and $\partial\mu/\partial\lambda$ are analytic in η and especially for $\eta \to 0$. From the finite piece, one has, for the β-function,

$$\beta(\lambda) = \mu \frac{\partial \lambda}{\partial \mu} = -2 \left(1 - \lambda \frac{\mathrm{d}}{\mathrm{d}\lambda}\right) a_1(\lambda) . \tag{5.104}$$

Using Eq.(5.104) and the fact that the poles in Eq.(5.103) must mutually cancel results in the recursion relation for the a_i,

$$\left(1 - \lambda \frac{\mathrm{d}}{\mathrm{d}\lambda}\right) a_{k+1}(\lambda) = a'_k(\lambda) \left(1 - \lambda \frac{\mathrm{d}}{\mathrm{d}\lambda}\right) a_1(\lambda) . \tag{5.105}$$

The meaning of this is clear: any successful renormalisation means that the bare couplings do not depend on what μ is, instead, any change in μ is accompanied with a corresponding change in the coupling, λ. Plugging in the result of the calculations before, one finds

$$\beta(\lambda) = \mu \frac{\partial \lambda}{\partial \mu} = \frac{3\lambda}{(4\pi)^2} + \mathcal{O}(\lambda^3) . \tag{5.106}$$

Straightforward integration of this simple differential equation yields

$$\lambda = \frac{\lambda_s}{1 - \frac{3\lambda_s}{(4\pi)^2} \log \frac{\mu}{\mu_s}} . \tag{5.107}$$

Here, λ_s is the coupling fixed at a scale μ_s. From the above equation, it is clear that λb increases with the scale. Therefore, at ever larger scales one has to take into account more and more orders of perturbation theory since the coupling constant, the parameter of expansion, becomes larger. This is especially so for

$$\mu = \mu_s \exp \left[\frac{16\pi^2}{3\lambda_s}\right] , \tag{5.108}$$

where the coupling constant at this order of perturbation theory develops a pole called Landau pole. It should be stressed, however, that there is no

reason to assume first order (i.e. one-loop) results to hold true for large values of λ. In fact, in this region it is not known how to calculate the β-function. On the other hand, the running of the coupling constant with the scale leads to a small coupling constant at small scales, that is, at large distances. The perturbative approach of defining asymptotic states thus can be trusted for this theory. Should the sign of the r.h.s. of Eq.(5.106) be negative, then the opposite would be true: it would be impossible to construct asymptotic states, since the coupling constant would become larger and larger at increasing distances, in contrast, perturbation theory would be great at short distances. This, in fact, is realized in QCD.
To dwell on the meaning of the β-function, consider a few cases, all of which start out with $\beta = 0$ for $\lambda = 0$.

1. $\beta(\lambda)$ stays positive for large λ. The λ keeps increasing with scale. If $\beta(\lambda)$ explodes at some point, λ itself blows up and develops the Landau pole.

2. $\beta(\lambda)$ starts out positive for comparably low λ, then bends over to become negative. It will be crossing the axis at some λ_F. This is called a fix point, since, if for some accidental reason λ would have started out at this value, it would remain there. To analyse the behaviour of λ in the vicinity of λ_F, expand there. One has

$$\mu\frac{\partial\lambda}{\partial\mu} = (\lambda - \lambda_F)\beta'(\lambda_F)\,. \tag{5.109}$$

   One immediately realizes the importance of the sign of the first derivative of β w.r.t. λ at λ_F. If it is negative, $\mu\partial\lambda/\partial\mu$ is larger than zero to the left of the fix point and λ will be pushed into the fix point. Then, also $\mu\partial\lambda/\partial\mu$ is smaller than zero to the right of the fix point and λ will be pushed into the fix point from this side, too. Hence, in such a single case, λ will be driven to λ_F as μ increases. Such a fix point is called ultraviolet stable.

3. $\beta(\lambda)$ starts out negative for low λ, decreasing monotonically. Then λ also decreases, with $\log\mu$. In this case, perturbation theory becomes better and better at shorter and shorter distances. In this case, λ is driven to $\lambda_F = 0$, which in turn is an ultraviolet stable fix point. Such a behaviour is characteristic for non-Abelian gauge theories in four dimensions, this phenomenon is called asymptotic freedom.

The running mass

Similar to the coupling constant, also the masses change with scale. In the mass independent scheme, the differential equation describing this variation is

$$\mu \frac{\partial m^2}{\partial \mu} = 2\lambda m^2 \frac{\mathrm{d}b_1}{\mathrm{d}\lambda}\,. \tag{5.110}$$

Hence,

$$\gamma_m(\lambda) = 2\lambda \frac{\mathrm{d}b_1}{\mathrm{d}\lambda}\,, \tag{5.111}$$

leading to

$$\gamma_m(\lambda) = \frac{\lambda}{(4\pi)^2} + \frac{7}{12}\frac{\lambda^2}{(4\pi)^4} \tag{5.112}$$

in $\lambda\phi^4$. There is also a recursion relation for the higher poles, namely

$$\lambda \frac{\mathrm{d}b_{k+1}}{\mathrm{d}\lambda} = b_k \lambda \frac{\mathrm{d}b_1}{\mathrm{d}\lambda} - \frac{\mathrm{d}b_k}{\mathrm{d}\lambda}\left(1 - \lambda\frac{\mathrm{d}}{\mathrm{d}\lambda}\right) a_1(\lambda)\,. \tag{5.113}$$

Chapter 6

Path integral formulation

In this chapter an alternative representation of quantum field theory will be presented, the path integral formalism. The discussion will start by first reviewing the formalism for quantum mechanics. First, it will be defined through the construction of total transition amplitudes and its usage will be exemplified by the calculation of expectation values. Then, the path integral for a simple harmonic oscillator will be discussed. Then the focus shifts on field theories, and free quantum field theories for scalar, vector and Dirac spinor fields will be evaluated once more, i.e. the corresponding free field propagators will be constructed. Through the introduction of sources, interacting field theories will be made accesible for perturbation theory, and, accordingly, the first terms of the perturbative series for $\lambda\phi^4$ theory will be given for some n-point functions. By the inspection of QED in the Coulomb gauge the path integral formulation of gauge theories will be visited the first time. The chapter closes with a discussion of the treatment of general gauge theories in this formalism, and how local gauge invariance can be dealt with.

6.1 Path integral representation of quantum mechanics

6.1.1 Constructive definition

Total amplitude in quantum mechanics

Consider a system with one particle contained in a potential $V(x)$. The time-dependent state vector of the particle is $|\psi(t)\rangle$. The probability density for this particle to be at position x at time t is

$$\mathcal{P}(x,\, t) = |\psi(x,\, t)|^2 = |\langle x|\psi(x,\, t)\rangle|^2 \,. \tag{6.1}$$

Here, $\psi(x,\, t)$ is the wave function, representing $|\psi(t)\rangle$ in terms of position eigenvectors. $|\psi(t)\rangle$ satisfies the Schrödinger equation,

$$i\frac{\partial}{\partial t}|\psi(t)\rangle = \mathcal{H}|\psi(t)\rangle \,. \tag{6.2}$$

Its formal solution is

$$|\psi(t)\rangle = e^{-i\mathcal{H}(t-t_0)}|\psi(t_0)\rangle \,. \tag{6.3}$$

In position space this becomes

$$\begin{aligned} \psi(x,\, t) = \langle x|\psi(t)\rangle &= \langle x|e^{-i\mathcal{H}(t-t_0)}|\psi(t_0)\rangle \\ &= \int \mathrm{d}x_1 \langle x|e^{-i\mathcal{H}(t-t_0)}|x_1\rangle\langle x_1|\psi(t_0)\rangle \\ &= \int \mathrm{d}x_1 G(x,\, t;\, x_1,\, t_0)\psi(x_1,\, t_0) \,. \end{aligned} \tag{6.4}$$

Here $G(x,\, t;\, x_0,\, t_0)$ denotes the total amplitude in position space. Again, the term $\exp\left[-i\mathcal{H}(t-t_0)\right]$ is the time-evolution operator, which propagates the state $|\psi(t_0)\rangle$ forward in time. The basic observation, that leads to the path integral formalism is that the total amplitude can be represented as a path integral. To see this, suppose the particle is localised at x_0 at time t_0, i.e. $|\psi(t_0)\rangle = |x_0\rangle$. Then,

$$\psi(x_1,\, t_0) = \langle x_1|\psi(t_0)\rangle = \langle x_1|x_0\rangle = \delta(x_1 - x_0) \,. \tag{6.5}$$

Consequently,

$$\psi(x,\, t) = \int \mathrm{d}x_1 G(x,\, t;\, x_1,\, t_0)\delta(x_1 - x_0) = G(x,\, t;\, x_0,\, t_0) \,, \tag{6.6}$$

and, hence, $G(x,\, t;\, x_0,\, t_0)$ is the quantum mechanical amplitude for the particle to go from $(x_0,\, t_0)$ to $(x,\, t)$. However, in this definition it is not clear, how the particle reached its destination. In fact, without any further measurement, quantum mechanics postulates that the particle took all paths (weighted by their amplitudes) and that the amplitude is a superposition of the individual amplitudes of all paths. The total amplitude is nothing but the sum over all amplitudes of all paths.

One can now break up the propagation into smaller steps, i.e. one can decompose $G(x,\, t;\, x_0,\, t_0)$ such that it is a product of amplitudes for the particle to go from $(x_0,\, t_0)$ to $(x_1,\, t_1)$ and then to $(x,\, t)$. With similar arguments as above one can write

$$G(x,\, t;\, x_0,\, t_0) = \int \mathrm{d}x_1 G(x,\, t;\, x_1,\, t_1) G(x_1,\, t_1;\, x_0,\, t_0)\,, \tag{6.7}$$

where $t_0 < t_1 < t$ is understood.

Construction of a total amplitude

This procedure can be repeated over and over again and one has

$$\begin{aligned} &G(x,\, t;\, x_0,\, t_0) \\ &= \int \mathrm{d}x_N \ldots \mathrm{d}x_1 G(x,\, t;\, x_N,\, t_N) G(x_N,\, t_N;\, x_{N-1},\, t_{N-1}) \ldots G(x_1,\, t_1;\, x_0,\, t_0)\,. \end{aligned} \tag{6.8}$$

Taking the limit $N \to \infty$ one can identify the individual integrals over the x_i as the functional integral with respect to the paths $x(t)$,

$$\lim_{N\to\infty} \int \prod_{i=1}^{N} \mathrm{d}x_1 \to \int \mathcal{D}x(t)\,. \tag{6.9}$$

The task left is to re-write the integrand, i.e. the product of single amplitudes in infinitesimal distances in a better, more compact form. To do so, assume that $t_{i+1} - t_i = \epsilon$ is small. Then, one can expand

$$\begin{aligned} G(x_{i+1},\, t_{i+1};\, x_i,\, t_i) &= \left\langle x_{i+1} \left| e^{-i\mathcal{H}(t_{i+1}-t_i)} \right| x_i \right\rangle \\ &= \left\langle x_{i+1} \left| [1 - i\epsilon\mathcal{H} + \ldots] \right| x_i \right\rangle\,. \end{aligned} \tag{6.10}$$

Now, analyse the matrix elements to find

$$\begin{aligned}\langle x_{i+1}|x_i\rangle &= \delta(x_{i+1}-x_i) = \int\limits_{-\infty}^{\infty}\frac{\mathrm{d}p}{2\pi}e^{ip(x_{i+1}-x_i)}\\ \langle x_{i+1}\,|\mathcal{H}|\,x_i\rangle &= \int\limits_{-\infty}^{\infty}\frac{\mathrm{d}p}{2\pi}\mathcal{H}(p_i,\,\bar{x}_i)e^{ip(x_{i+1}-x_i)}\,, \end{aligned} \tag{6.11}$$

where $\bar{x}_i$ is the average x, in this case

$$\bar{x}_i = \frac{x_{i+1}+x_i}{2}\,. \tag{6.12}$$

Therefore,

$$\begin{aligned}G(x_{i+1},\,t_{i+1};\,x_i,\,t_i) &= \int\limits_{-\infty}^{\infty}\frac{\mathrm{d}p}{2\pi}\left(1-i\epsilon\mathcal{H}(p_i,\,\bar{x}_i)+\ldots\right)e^{ip(x_{i+1}-x_i)}\\ &\approx \int\limits_{-\infty}^{\infty}\frac{\mathrm{d}p}{2\pi}e^{-i\epsilon\mathcal{H}(p_i,\,\bar{x}_i)}e^{ip(x_{i+1}-x_i)}\,, \end{aligned} \tag{6.13}$$

and the product of amplitudes can be cast into the following form, representing one of the paths,

$$\begin{aligned}&\lim_{N\to\infty}\prod_{i=0}^{N}G(x_{i+1},\,t_{i+1},\,x_i,\,t_i)\\ &= \lim_{N\to\infty}\int\limits_{-\infty}^{\infty}\prod_{j=0}^{N}\frac{\mathrm{d}p_j}{2\pi}\exp\left\{\left[i\sum_{j=0}^{N}p_j(x_{j+1}-x_j)\right]-\left(\frac{t-t_0}{N+1}\right)\mathcal{H}(p_j,\,\bar{x}_j)\right\}\\ &= \int\mathcal{D}p\exp\left[i\int\limits_{t_0}^{t}\mathrm{d}t'\,(p\dot{x}-H(p,\,x))\right]\,. \end{aligned} \tag{6.14}$$

The entire amplitude therefore is

$$G(x,\,t;\,x_0,\,t_0) = \int\mathcal{D}\bar{x}\mathcal{D}p\exp\left[i\int\limits_{t_0}^{t}\mathrm{d}t'\,(p\dot{\bar{x}}-H(p,\,\bar{x}))\right] \tag{6.15}$$

Assume now the standard form of a Hamiltonian with no unorthodox powers of the momentum, $\mathcal{H}=p^2/2+V(x)$. Then, completing the squares in one of the mini-amplitudes yields

$$
\begin{aligned}
G(x_{i+1},\, t_{i+1};\, x_i,\, t_i) &= \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}p_i}{2\pi} e^{ip_i(x_{i+1}-x_i)} e^{-i\epsilon(p_i^2/2+V(\bar{x}_i))} \\
&= \sqrt{\frac{1}{2\pi i\epsilon}} \exp\left[i\epsilon\left(\frac{\dot{\bar{x}}_i^2}{2} - V(\bar{x}_i)\right)\right] , \qquad (6.16)
\end{aligned}
$$

where $\dot{\bar{x}}_i = (x_{i+1} - x_i)/\epsilon$. Then the total amplitude becomes

$$
\begin{aligned}
G(x,\, t;\, x_0,\, t_0) &= \int \mathcal{D}\bar{x} \exp\left[i \int\limits_{t_0}^{t} \mathrm{d}t' \mathcal{L}(\dot{\bar{x}},\, \bar{x},\, t)\right] \\
&= \int \mathcal{D}\bar{x} \exp\left[i\mathcal{S}[\bar{x}]\right] , \qquad (6.17)
\end{aligned}
$$

where $\mathcal{S}$ is the classical action functional of the theory. The interpretation of this is that the amplitude of a single path in quantum mechanics is the exponential of i times the classical action related to it. A comment concerning the average $\bar{x}$ and operator ordering is in order here. Straightforward calculation shows that taking the average amounts to

$$
\left\langle x_{i+1} \left| \frac{1}{2}(xp + px) \right| x_i \right\rangle = \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}p_i}{2\pi} e^{ip_i(x_{i+1}-x_i)} p_i \bar{x}_i \,. \qquad (6.18)
$$

Transition amplitudes and expectation values

The very nature of the path integral renders it a useful representation to calculate transition matrix elements, vacuum expectation values and coordinate or momentum space representations of matrix elements. As one easily anticipates, it is also extremely useful in quantum field theory. However, the calculation of transition matrix elements is a straightforward generalisation of what has been achieved so far. If the system is in state $|\psi\rangle$ at some time t_0 then it will evolve to the state $|\phi\rangle$ at some later time t, where

$$
|\phi\rangle = e^{-i\mathcal{H}(t-t_0)}|\psi\rangle \,. \qquad (6.19)
$$

The amplitude to be in the state $|\chi\rangle$ instead at time t is $\langle\chi|\phi\rangle$,

$$
\langle\chi|\phi\rangle = \left\langle \chi \left| e^{-i\mathcal{H}(t-t_0)} \right| \psi \right\rangle
$$

$$
\begin{aligned}
&= \int \mathrm{d}x_0 \mathrm{d}x \left\langle \chi \middle| x \right\rangle \left\langle x \middle| e^{-i\mathcal{H}(t-t_0)} \middle| x_0 \right\rangle \left\langle x_0 \middle| \psi \right\rangle \\
&= \int \mathrm{d}x_0 \mathrm{d}x \chi^*(x) G(x,\, t;\, x_0,\, t_0) \phi(x_0) \\
&= \int \mathrm{d}x_0 \mathrm{d}x \int \mathcal{D}\bar{x}\, \chi^*(x) \exp\left(i\mathcal{S}[\bar{x}]\right) \phi(x_0)\,, \qquad (6.20)
\end{aligned}
$$

where all paths satisfy $\bar{x}(t_0) = x_0$ and $\bar{x}(t) = x$.
To arrive at the path integral representation of a vacuum expectation value of an operator, one has to rewrite the total amplitude in terms of energy eigenstates, $|\phi_n\rangle$, of the Hamiltonian. Then

$$
\mathcal{H}|\phi_n\rangle = E_n|\phi_n\rangle \qquad (6.21)
$$

and it is assumed that the ϕ_n form a complete set, i.e.

$$
\sum_n |\phi_n\rangle\langle\phi_n| = 1\,. \qquad (6.22)
$$

Using this completeness relation to insert to suitable 1s, one can write

$$
\begin{aligned}
G(x,\, t;\, x_0,\, t_0) &= \left\langle x \middle| e^{-i\mathcal{H}(t-t_0)} \middle| x_0 \right\rangle \\
&= \sum_{n,m} \phi_n(x) e^{-iE_n t} \langle\phi_n|\phi_m\rangle e^{iE_m t_0} \phi_m^*(x_0) \\
&= \sum_n e^{-iE_n(t-t_0)} \phi_n^*(x_0)\phi_n(x)\,. \qquad (6.23)
\end{aligned}
$$

Setting $t_0 = 0$ and letting t go to infinity, all contributions vanish apart from the one related to the lowest energy state, $n = 0$, provided that there is a gap between E_1 and E_0. In contrast, any other term with $E_{n\geq 1}$ in the sum will oscillate as t becomes large, and each term above the ground state will decay exponentially. Therefore

$$
\begin{aligned}
\lim_{t\to\infty} G(x,\, t;\, x,\, 0) &= \phi_0^*(x)\phi_0(x)\,, \\
\lim_{t\to\infty} G(x,\, t;\, 0,\, 0) &\approx \phi_0(x)\,. \qquad (6.24)
\end{aligned}
$$

This can be used to evaluate the vacuum expectation value of an operator $\mathcal{O}(x)$ as

$$
\begin{aligned}
\langle\mathcal{O}\rangle &= \lim_{t\to\infty} \int \mathrm{d}x\, G(x\, t;\, x\,, 0)\mathcal{O}(x) \\
&= \lim_{t\to\infty} \int \mathrm{d}x\, G(x\, t;\, x\,, -t)\mathcal{O}(x) \\
&= \int \mathrm{d}x \int \mathcal{D}\bar{x}\, \mathcal{O}(\bar{x}) \exp\left[i \int_{-\infty}^{\infty} \mathrm{d}t' \mathcal{L}(\dot{\bar{x}},\, \bar{x},\, t') \right]\,, \qquad (6.25)
\end{aligned}
$$

where all paths satisfy $\bar{x}(-\infty) = \bar{x}(\infty) = x$.

Simple example: Free particle

In the following the simple example of a free particle will be calculated explicitly. For a free particle of mass m, the Hamiltonian is simply

$$\mathcal{H} = \frac{p^2}{2m}\,. \tag{6.26}$$

Hence,

$$G(x,\,t;\,x_0,\,t_0) \;=\; \int \mathcal{D}p\mathcal{D}\bar{x}\,\exp\left[i\int\limits_{t_0}^{t}\mathrm{d}t'\left(p\dot{\bar{x}} - \frac{p^2}{2m}\right)\right]\,, \tag{6.27}$$

where all paths start at $(x_0,\,t_0)$ and end at $(x,\,t)$. Using the definition of Eq.(6.15) for the total amplitude plugging in the expression of Eq.(6.14) for a single path, and completing the squares in p_j one ends up with

$$\begin{aligned} G(x,\,t;\,x_0,\,t_0) &= \lim_{N\to\infty}\int\prod_{j=0}^{N}\frac{\mathrm{d}p_j}{2\pi}\prod_{i=1}^{N}\mathrm{d}\bar{x}_i\,\exp\left\{i\sum_{j=0}^{N}\epsilon p_j\left(\frac{\bar{x}_{j+1}-\bar{x}_j}{\epsilon}\right) - \epsilon\frac{p_j^2}{2m}\right\} \\ &= \lim_{N\to\infty}\int\prod_{i=1}^{N}\mathrm{d}\bar{x}_i\left(\frac{m}{2\pi i\epsilon}\right)^{\frac{N+1}{2}}\exp\left\{i\frac{m\epsilon}{2}\sum_{j=0}^{N}\left(\frac{\bar{x}_{j+1}-\bar{x}_j}{\epsilon}\right)^2\right\}\,. \end{aligned} \tag{6.28}$$

This is the discretised form of the free particle action,

$$S = \int\limits_{t_0}^{t}\mathrm{d}t'\,\frac{m\dot{x}^2}{2}\,. \tag{6.29}$$

Now, the remaining N Gaussian integrals over the $\bar{x}_i$ have to be done before the limit $N \to \infty$ is to be taken. This can be done by successively completing the squares. Starting with $i = 1$ one finds

$$\begin{aligned} &G(x,\,t;\,x_0,\,t_0) \\ &= \lim_{N\to\infty}\left(\frac{m}{2\pi i\epsilon}\right)^{\frac{N+1}{2}}\int\prod_{i=2}^{N}\mathrm{d}\bar{x}_i\,\exp\left\{i\frac{m\epsilon}{2}\sum_{j=2}^{N}\left(\frac{\bar{x}_{j+1}-\bar{x}_j}{\epsilon}\right)^2\right\} \end{aligned}$$

$$\int_{-\infty}^{\infty} \mathrm{d}\bar{x}_1 \exp\left[\frac{im\epsilon}{2}\left((\bar{x}_2-\bar{x}_1)^2-(\bar{x}_1-\bar{x}_0)^2\right)\right]$$

$$= \lim_{N\to\infty}\left(\frac{m}{2\pi i\epsilon}\right)^{\frac{N+1}{2}}\left(\frac{\pi\epsilon}{im}\right)^{\frac{1}{2}}$$
$$\int\prod_{i=2}^{N}\mathrm{d}\bar{x}_i \exp\left\{i\frac{m\epsilon}{2}\sum_{j=2}^{N}\left(\frac{\bar{x}_{j+1}-\bar{x}_j}{\epsilon}\right)^2\right\}\exp\left\{\frac{im}{4\epsilon}\left(\bar{x}_2-\bar{x}_0\right)^2\right\} \tag{6.30}$$

Taking out the $\bar{x}_2$-integral in a similar fashion one finds

$$G(x,\,t;\,x_0,\,t_0) = \lim_{N\to\infty}\left(\frac{m}{2\pi i\epsilon}\right)^{\frac{N+1}{2}}\left(\frac{1}{2}\cdot\frac{2\pi\epsilon}{im}\right)^{\frac{1}{2}}\left(\frac{2}{3}\cdot\frac{2\pi\epsilon}{im}\right)^{\frac{1}{2}}$$
$$\int\prod_{i=3}^{N}\mathrm{d}\bar{x}_i \exp\left\{i\frac{m\epsilon}{2}\sum_{j=3}^{N}\left(\frac{\bar{x}_{j+1}-\bar{x}_j}{\epsilon}\right)^2\right\}\exp\left\{\frac{im}{2(3\epsilon)}\left(\bar{x}_3-\bar{x}_0\right)^2\right\}. \tag{6.31}$$

Here, the funny factor 2/3 stems from completing the squares of the $\bar{x}_2$-integral in Eq.(6.30),

$$\frac{im}{4\epsilon}\left(\bar{x}_2-\bar{x}_0\right)^2+\frac{im}{2\epsilon}\left(\bar{x}_3-\bar{x}_2\right)^2$$
$$= \frac{im}{2\epsilon}\left\{\frac{3}{2}\left[\bar{x}_2-\frac{2\bar{x}_3+\bar{x}_0}{3}\right]^2+\frac{1}{3}\left[\bar{x}_3-\bar{x}_0\right]^2\right\} \tag{6.32}$$

This also gives some hindsight on the emerging pattern. After all N integrals have been done, the final result reads

$$G(x,\,t;\,x_0,\,t_0)$$
$$= \lim_{N\to\infty}\left(\frac{m}{2\pi i\epsilon}\right)^{\frac{N+1}{2}}\prod_{j=1}^{N}\left[\frac{j}{j+1}\cdot\frac{2\pi\epsilon}{im}\right]^{\frac{1}{2}}\exp\left\{\frac{im}{2(N+1)\epsilon}\left(\bar{x}_{N+1}-\bar{x}_0\right)^2\right\}. \tag{6.33}$$

Identifying $\bar{x}_{N+1}=\bar{x}$, $(N+1)\epsilon = t-t_0$ and

$$\left(\frac{m}{2\pi i\epsilon}\right)^{\frac{N+1}{2}}\prod_{j=1}^{N}\left[\frac{j}{j+1}\cdot\frac{2\pi\epsilon}{im}\right]^{\frac{1}{2}} = \left(\frac{m}{2\pi i\epsilon}\right)^{\frac{N+1}{2}}\left(\frac{-2\pi i\epsilon}{m}\right)^{\frac{N}{2}}\left(\frac{1}{N+1}\right)^{\frac{N}{2}} \tag{6.34}$$

one finally has

$$G(x,\,t;\,x_0,\,t_0) = \left(\frac{m}{2i\pi(t-t_0)}\right)^{\frac{1}{2}} \exp\left(\frac{im(\bar{x}-\bar{x}_0)^2}{2(t-t_0)}\right) . \tag{6.35}$$

Next simple example: Harmonic oscillator

Turning to the next simple example, the harmonic oscillator, its Hamiltonian reads

$$\mathcal{H} = \frac{p^2}{2m} + m\omega^2 x^2 \, . \tag{6.36}$$

The total amplitude is

$$G(x,\,t;\,x_0,\,t_0) = \int \mathcal{D}\bar{x} \exp\left\{i\mathcal{S}[\bar{x}]\right\} = \int \mathcal{D}\bar{x} \exp\left\{i \int\limits_{t_0}^{t} \mathrm{d}t' \frac{m}{2}\left[\dot{\bar{x}}^2 - \omega^2\bar{x}^2\right]\right\} , \tag{6.37}$$

where, as usual, $\bar{x}(t_0) = x_0$ and $\bar{x}(t) = x$. Consider now a Taylor expansion of the action functional around the classical path, $\bar{x}_{\mathrm{cl.}}$.

$$\mathcal{S}[\bar{x}] = \mathcal{S}[\bar{x}_{\mathrm{cl.}}] + \left.\frac{\delta\mathcal{S}}{\delta\bar{x}}\right|_{\bar{x}=\bar{x}_{\mathrm{cl.}}} (\bar{x}-\bar{x}_{\mathrm{cl.}}) + \left.\frac{\delta^2\mathcal{S}}{\delta\bar{x}^2}\right|_{\bar{x}=\bar{x}_{\mathrm{cl.}}} \frac{(\bar{x}-\bar{x}_{\mathrm{cl.}})^2}{2} . \tag{6.38}$$

Terms higher than second order vanish, due to the orders present in the action. To do a stationary phase approximation one chooses the classical path such that

$$\left.\frac{\delta\mathcal{S}}{\delta\bar{x}}\right|_{\bar{x}=\bar{x}_{\mathrm{cl.}}} = 0 \, . \tag{6.39}$$

This is a consistent choice, since evaluation of the variation $\delta\mathcal{S}/\delta\bar{x} = 0$ just yields the classical equations of motion. For an harmonic oscillator starting at $(x_0,\,t_0)$ and going through $(x_1,\,t_1)$ this path is just given by

$$\bar{x}_{\mathrm{cl.}}(t) = x_0 \frac{\sin\left(\omega(t_1-t)\right)}{\sin\left(\omega(t_1-t_0)\right)} + x_1 \frac{\sin\left(\omega(t-t_0)\right)}{\sin\left(\omega(t_1-t_0)\right)} \, . \tag{6.40}$$

Hence, the first term in the Taylor expansion reads

$$\mathcal{S}[\bar{x}_{\mathrm{cl.}}] = \frac{m}{2}\left[\frac{\omega}{\sin\left(\omega(t_1-t_0)\right)}\right]\left[(x_1+x_0)^2\cos\left(\omega(t_1-t_0)\right) - 2x_1x_0\right] . \tag{6.41}$$

Plugging the expansion of the action into the expression of the total amplitude, the latter becomes

$$\begin{aligned} & G(x,\, t;\, x_0,\, t_0) \\ &= \exp\left\{ i\mathcal{S}[\bar{x}_{\text{cl.}}]\right\} \int \mathcal{D}\bar{x}\, \exp\left\{ \frac{1}{2}\int\limits_{t_0}^{t} \mathrm{d}t' \, \left[(\bar{x}-\bar{x}_{\text{cl.}}) \left(\partial_t^2+\omega^2\right)(\bar{x}-\bar{x}_{\text{cl.}})\right]\right\} \end{aligned} \tag{6.42}$$

To do this remaining integral one switches from paths $\bar{x}$ to fluctuations $y = \bar{x} - \bar{x}_{\text{cl.}}$ around the classical path. In this notation, $y(t) = y(t_0) = 0$ and one can think of using periodic boundary conditions, and, hence, paths with period $t - t_0$. Using some little functional calculus, one then has

$$\begin{aligned} G(x,\, t;\, x_0,\, t_0) &= \det{}^{-\frac{1}{2}}\left[\frac{im(\partial_t^2+\omega^2)}{2\pi}\right] \\ \cdot \exp&\left\{\frac{im}{2}\left[\frac{\omega}{\sin\left(\omega(t_1-t_0)\right)}\right]\left[(x_1+x_0)^2\cos\left(\omega(t_1-t_0)\right)-2x_1x_0\right]\,.\right\} \end{aligned} \tag{6.43}$$

The determinant of an operator is the product of its eigenvalues. Since periodic paths with period $T = t - t_0$ are integrated over, the eigenfunctions of $\partial_t^2 + \omega^2$ are

$$\sin\frac{n\pi t}{T} \quad \text{with} \quad n = 1,\, 2,\, \dots \tag{6.44}$$

leading to eigenvalues

$$\lambda_n = -\left(\frac{n\pi}{T}\right)^2 + \omega^2\,. \tag{6.45}$$

Hence,

$$\begin{aligned} \det\left[\frac{im(\partial_t^2+\omega^2)}{2\pi}\right] &= \prod_n \frac{m}{2\pi i}\left[\left(\frac{n\pi}{T}\right)^2-\omega^2\right] \\ &= \prod_n\left[\frac{m}{2\pi i}\left(\frac{n\pi}{T}\right)^2\right]\prod_n\left[1-\frac{\omega^2T^2}{(n\pi)^2}\right]\,. \end{aligned} \tag{6.46}$$

The surprising fact of this equation is that, if a free particle is considered, $\omega = 0$, and plugging in the result for this case, one finds

$$\prod_n\left[\frac{m}{2\pi i}\left(\frac{n\pi}{T}\right)^2\right] = \frac{2\pi i T}{m}\,. \tag{6.47}$$

According to Euler, the remaining piece is nothing but

$$\prod_n \left[1 - \frac{\omega^2 T^2}{(n\pi)^2}\right] = \frac{\sin(\omega T)}{\omega T}\,. \tag{6.48}$$

Taken everything together the total amplitude for a harmonic oscillator is given by

$$\begin{aligned} G(x,\, t;\, x_0,\, t_0) = & \left(\frac{m\omega}{2\pi i \sin[\omega(t - t_0)]}\right)^{\frac{1}{2}} \\ & \cdot \exp\left\{\frac{im}{2}\left[\frac{\omega}{\sin\left(\omega(t_1 - t_0)\right)}\right]\left[(x_1 + x_0)^2 \cos\left(\omega(t_1 - t_0)\right) - 2x_1 x_0\right]\,.\right\} \end{aligned} \tag{6.49}$$

As a first application one can calculate the energy eigenfunctions from this total amplitude. First of all, to alleviate the task to take the limit $iT \to \infty$ rewrite the total amplitude such that an exponential of iT times something appears in front.

$$\begin{aligned} & G(x,\, T;\, x_0,\, 0) \\ = \quad & \left(\frac{m\omega}{\pi}\right)^{\frac{1}{2}} e^{-\frac{i\omega T}{2}} \left(1 - e^{-2i\omega T}\right)^{-\frac{1}{2}} \\ & \exp\left\{-\frac{m\omega}{2}\left[(x^2 + x_0^2)\left(\frac{1 + e^{-2i\omega T}}{1 - e^{-2i\omega T}} - \frac{4xx_0 e^{-i\omega T}}{1 - e^{-2i\omega T}}\right)\right]\right\} \\ \xrightarrow{iT\to\infty} \quad & \left(\frac{m\omega}{\pi}\right)^{\frac{1}{2}} e^{-\frac{i\omega T}{2}} \exp\left(-\frac{m\omega(x^2 + x_0^2)}{2}\right) \\ = \quad & e^{-iE_0 T}\phi_0^*(x)\phi_0(x_0)\,. \end{aligned} \tag{6.50}$$

Therefore,

$$E_0 = \frac{\omega}{2} \quad \text{and} \quad \phi_0(x) = \left(\frac{m\omega}{\pi}\right)^{\frac{1}{4}} \exp\left(-\frac{m\omega x^2}{2}\right) \tag{6.51}$$

Adding external forces

When an external force $F(t)$ is added to the harmonic oscillator, its action becomes

$$\mathcal{S}[x(\bar{t})] = \int_{t_0}^{t} \mathrm{d}\bar{t}\left[\frac{1}{2}m^2\left(\dot{x}^2 - \omega^2 x^2\right) + xF(\bar{t})\right] \tag{6.52}$$

leading to the equation of motion

$$\frac{\delta \mathcal{S}}{\delta x} = m\left(\partial_t^2 + \omega^2\right) x(t) - F(t) = 0\,, \tag{6.53}$$

as expected. Using the classical boundary conditions $x_{\rm cl}^F(t_0) = x_0$ and $x_{\rm cl}^F(t) = x$ one finds the general solution

$$x_{\rm cl}^F(t') = x_{\rm cl}(t') - \int\limits_{t_0}^{t'} {\rm d}\bar{t} g(t',\, \bar{t}) F(\bar{t})\,, \tag{6.54}$$

where, again, $x_{\rm cl}$ is the classical path in the absence of the force. It has been calculated already in Eq.(6.40). In the equation, above, $g(t',\, \bar{t})$ is the Green's function of the free harmonic oscillator and satisfies

$$(\partial_{t'}^2 + \omega^2) g(t',\, \bar{t}) = -\delta(t' - \bar{t})\,. \tag{6.55}$$

An explicit expression for the Green's function is

$$g(t',\, \bar{t}) = \begin{cases} \dfrac{\sin[\omega(t-t')]\sin[\omega(\bar{t}-t_0)]}{\omega \sin[\omega(t-t_0)]} & \text{for} \quad t' \le \bar{t} \\ \dfrac{\sin[\omega(t-\bar{t})]\sin[\omega(t'-t_0)]}{\omega \sin[\omega(t-t_0)]} & \text{for} \quad \bar{t} \le t'\,. \end{cases} \tag{6.56}$$

As one can see $g(t',\, \bar{t})$ is an admixture of advanced and retarded signals, a precursor of the Feynman propagator. The total amplitude then becomes

$$\begin{aligned} G_F(x,\, t;\, x_0,\, t_0) &= \int \mathcal{D}x \exp\left\{i\mathcal{S}[x]\right\} \\ &= \int \mathcal{D}x \exp\left\{ i \int\limits_{t_0}^{t} {\rm d}t' \left[\frac{m}{2}\left(\dot{x}^2 - \omega^2 x^2\right) + F(t')x(t')\right]\right\}\,. \end{aligned} \tag{6.57}$$

As it stands, the integrand is purely oscillatory, to restore convergence one might add a term of the form

$$\exp\left\{-\frac{\epsilon}{2}\int\limits_{t_0}^{t} {\rm d}t' x^2(t')\right\} \tag{6.58}$$

with $\epsilon > 0$. Letting ϵ go to zero at the end of the calculation then ensures convergence (and, in fact, yields exactly the term $\sim i\epsilon$ in the denominators of the propagators). To calculate the transition amplitude, it is convenient to do a Fourier transform w.r.t. the time,

$$\begin{aligned}
&\left[\dot{x}^2(t) - (\omega^2 - i\epsilon)x^2(t)\right] \\
&\quad = \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}E}{\sqrt{2\pi}}\frac{\mathrm{d}E'}{\sqrt{2\pi}} e^{i(E+E')t} \left[-EE' - \omega^2 + i\epsilon\right] x(E)x(E')
\end{aligned} \tag{6.59}$$

and

$$F(t)x(t) = \frac{1}{2}\int\limits_{-\infty}^{\infty} \frac{\mathrm{d}E}{\sqrt{2\pi}}\frac{\mathrm{d}E'}{\sqrt{2\pi}} e^{i(E+E')t} \left[x(E)F(E') + x(E')F(E)\right] . \tag{6.60}$$

Using the integral representation of the δ-function and after a change of variables,

$$\begin{aligned}
x'(E) &= x(E) + \frac{F(E)}{E^2 - \omega^2 + i\epsilon} , \\
x'(t) &= x(t) + \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}E}{\sqrt{2\pi}} e^{iEt} \frac{F(E)}{E^2 - \omega^2 + i\epsilon} ,
\end{aligned} \tag{6.61}$$

which basically amounts to completing the squares, one finds

$$\begin{aligned}
G_F(x,\, +\infty;\, x_0,\, -\infty) &= \exp\left\{-\frac{i}{2}\int\limits_{-\infty}^{\infty} \mathrm{d}E \frac{F(E)F(-E)}{E^2 - \omega^2 + i\epsilon}\right\} \\
&\int \mathcal{D}x \exp\left\{\frac{i}{2}\int\limits_{-\infty}^{\infty} \mathrm{d}E \left[x'(E)(E^2 - \omega^2 + i\epsilon)x'(-E)\right]\right\} .
\end{aligned} \tag{6.62}$$

The magic of the path integral formulation is that the Jacobian of this transformation, i.e. of completing the squares is one, or, in other words, that

$$\mathcal{D}x = \mathcal{D}x' . \tag{6.63}$$

Then, one realizes immediately that the last term of the expression in Eq.(6.62) is nothing but the free oscillator transition amplitude. Therefore one can express the transition amplitude for the oscillator with an external force, G_F, through the free one, G, as

$$
\begin{aligned}
G_F(x,\, +\infty;\, x_0,\, -\infty) &= \langle x,\, +\infty | x_0,\, -\infty\rangle \\
&= \exp\left\{ -\frac{i}{2} \int\limits_{-\infty}^{\infty} \mathrm{d}E \frac{F(E)F(-E)}{E^2-\omega^2+i\epsilon} \right\} G(x,\, +\infty;\, x_0,\, -\infty)\,. \qquad (6.64)
\end{aligned}
$$

This can be massaged into a form that is a bit more intuitive,

$$
\exp\left\{ -\frac{i}{2} \int\limits_{-\infty}^{\infty} \mathrm{d}E \frac{F(E)F(-E)}{E^2-\omega^2+i\epsilon} \right\} = \exp\left\{ -\frac{i}{2} \int\limits_{-\infty}^{\infty} \mathrm{d}t\mathrm{d}t' F(t) g(t,\, t') F(-t') \right\}\,, \qquad (6.65)
$$

where, expressed through energies,

$$
g(t,\, t') = \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}E}{2\pi} \frac{e^{-iE(t-t')}}{E^2-\omega^2+i\epsilon}\,. \qquad (6.66)
$$

To understand the physical meaning of Eq.(6.65), assume that for $t \to \pm\infty$ there is no driving force F. Then the vacuum state will not depend on the existence of F. Let $|\phi_{\pm\infty}\rangle$ be the vacuum states in the infinite future and past. Then

$$
\begin{aligned}
&\langle x,\, +\infty | x_0,\, -\infty\rangle_F \\
&= \int \mathrm{d}\phi\mathrm{d}\phi' \left\langle x,\, +\infty | \phi'_{+\infty}\right\rangle \left\langle \phi'_{+\infty} | \phi_{-\infty}\right\rangle_F \langle \phi_{-\infty} | x_0,\, -\infty\rangle \\
&= \int \mathrm{d}\phi\mathrm{d}\phi' \left\langle x,\, +\infty | \phi'_{+\infty}\right\rangle \left\langle \phi'_{+\infty} | \phi_{-\infty}\right\rangle \langle \phi_{-\infty} | x_0,\, -\infty\rangle \\
&\qquad \exp\left\{ -\frac{i}{2} \int\limits_{-\infty}^{\infty} \mathrm{d}t\mathrm{d}t' F(t) g(t,\, t') F(-t') \right\}\,. \qquad (6.67)
\end{aligned}
$$

Identifying

$$
\left\langle \phi'_{+\infty} | \phi_{-\infty}\right\rangle_{F=0} = \langle x,\, +\infty | x_0,\, -\infty\rangle_{F=0} \qquad (6.68)
$$

allows to rewrite

$$
\begin{aligned}
&\langle x,\, +\infty | x_0,\, -\infty\rangle_F \\
&= \langle x,\, +\infty | x_0,\, -\infty\rangle_{F=0} \exp\left\{ -\frac{i}{2} \int\limits_{-\infty}^{\infty} \mathrm{d}t\mathrm{d}t' F(t) g(t,\, t') F(-t') \right\}. \qquad (6.69)
\end{aligned}
$$

In other words,

$$\exp\left\{-\frac{i}{2}\int\limits_{-\infty}^{\infty} \mathrm{d}t\mathrm{d}t' F(t)g(t,\, t')F(-t')\right\} \tag{6.70}$$

is the transition amplitude in the presence of an external driving force to go from the ground state in the far past to the ground state in the far future. Defining

$$Z[F] \;:=\; \exp\left\{-\frac{i}{2}\int\limits_{-\infty}^{\infty} \mathrm{d}t\mathrm{d}t' F(t)g(t,\, t')F(-t')\right\} := e^{i\mathcal{W}[F]} \tag{6.71}$$

one sees immediately that

$$g(t_1,\, t_2) = \frac{-i\delta}{\delta F(t_1)}\frac{-i\delta}{\delta F(t_2)}\mathcal{W}[F]\bigg|_{F=0}\,. \tag{6.72}$$

This provides a way to extract Green's functions from the vacuum-to-vacuum transition amplitude in the presence of a source: just take the according functional derivative w.r.t. the source.
It should be noted that

$$g(t_1,\, t_2) = \frac{1}{2i\omega}\left[\theta(t_1-t_2)e^{-i\omega(t_1-t_2)} + \theta(t_2-t_1)e^{i\omega(t_1-t_2)}\right]\,. \tag{6.73}$$

6.1.2 Perturbation theory

In a next step, a potential $V(x)$ can be added. Only in very few special cases such a case can be computed exactly, one of the most beloved examples is the harmonic oscillator. In other cases, a typical strategy is to separate the Hamiltonian $\mathcal{H}$ into two pieces: a part $\mathcal{H}_0$, which can be solved exactly, and an interaction part $\lambda V(x)$. The solution of the exactly known piece, like for instance, a free particle Hamiltonian, will be denoted as G_0 in the following. However, when λ is small, one can do a perturbative expansion in λ to find an approximate solution for G, the full Hamiltonians total amplitude, in terms of G_0. To do so, start with the path integral representation of G,

$$G(x,\, t;\, x_0,\, t_0) = \int \mathcal{D}\bar{x}\, \exp\left[\int\limits_{t_0}^{t} \mathrm{d}t'\left(\mathcal{S}_0[\bar{x}] - \lambda V(\bar{x})\right)\right]\,, \tag{6.74}$$

where $\mathcal{S}_0$ is the action of the system described by $\mathcal{H}_0$, i.e for $\lambda = 0$. Assuming that the Hamiltonian does not contain any unorthodox dependence on the momenta, one can carry out the corresponding Gaussian integrals. Also, one can then separate the unperturbed piece and the potential and do an expansion of the exponential containing it,

$$\begin{aligned}
&G(x,\, t;\, x_0,\, t_0)\\
&= \int \mathcal{D}\bar{x}\, \exp\left[\int\limits_{t_0}^{t} \mathrm{d}t' \mathcal{S}_0[\bar{x}]\right] \sum_{n=0}^{\infty} \left[\frac{(-i\lambda)^n}{n!} \left(\int\limits_{t_0}^{t} \mathrm{d}t' V(\bar{x})\right)^n\right]\\
&= \sum_{n=0}^{\infty} \left[\frac{(-i\lambda)^n}{n!} \int \mathcal{D}\bar{x} \left(\int\limits_{t_0}^{t} \mathrm{d}t' V(\bar{x})\right)^n \exp\left(\int\limits_{t_0}^{t} \mathrm{d}t' \mathcal{S}_0[\bar{x}]\right)\right]\\
&=: \sum_{n=0}^{\infty} G^{(n)}(x,\, t;\, x_0,\, t_0)\,, \qquad (6.75)
\end{aligned}$$

where n labels the number of interactions, and, obviously

$$G^{(0)}(x,\, t;\, x_0,\, t_0) = G(x,\, t;\, x_0,\, t_0)\,. \qquad (6.76)$$

The next term in this expansion yields

$$\begin{aligned}
G^{(1)}(x,\, t;\, x_0,\, t_0) &= -i\lambda \int \mathcal{D}\bar{x}(\bar{t}) \int\limits_{t_0}^{t} \mathrm{d}t' V(\bar{x}(t'))\, \exp\left(\int\limits_{t_0}^{t} \mathrm{d}t' \mathcal{S}_0[\bar{x}(t')]\right)\\
&= -i\lambda \int\limits_{t_0}^{t} \mathrm{d}t' \int \mathcal{D}\bar{x}(\bar{t}) V(\bar{x}(t'))\, \exp\left(\int\limits_{t_0}^{t} \mathrm{d}t' \mathcal{S}_0[\bar{x}(t')]\right)\,.
\end{aligned} \qquad (6.77)$$

Here the various time-dependences have been exhibited explicitly to clarify the situation. Now, the propagation of the particle between $(x_0,\, t_0)$ and $(x,\, t)$ can be broken up such that it passes $(x_1,\, t_1)$. Then, to first order in λ, the amplitude reads

$$-i\lambda G^{(0)}(x,\, t;\, x_1,\, t_1) V(x_1,\, t_1) G^{(0)}(x_1,\, t_1;\, x_0,\, t_0) \qquad (6.78)$$

To obtain the full amplitude, one has to sum over all possible $(x_1,\, t_1)$, where $t_1 \in [t_0,\, t]$. In other words

$$G^{(1)}(x,\, t;\, x_0,\, t_0)$$

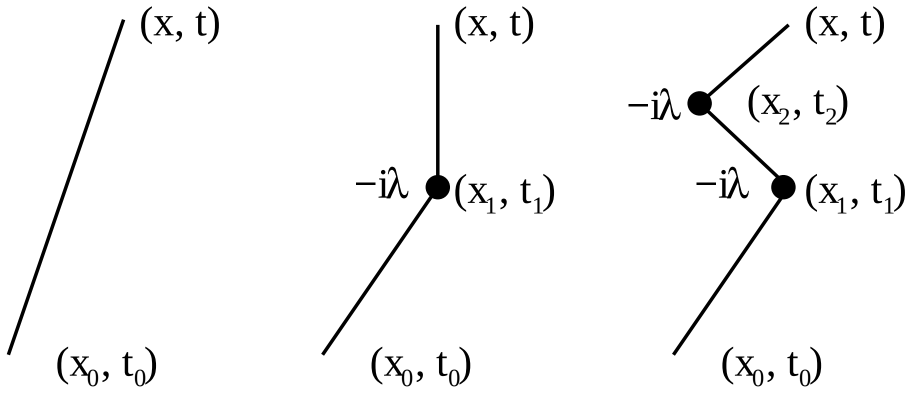


Figure 6.1: Diagrammatic representation of the first three terms of $G^{(n)}(x,\, t;\, x_0,\, t_0)$.

$$
\begin{aligned}
&= \int_{t_0}^{t} \mathrm{d}t_1 \int_{-\infty}^{\infty} \mathrm{d}x_1 (-i\lambda) G^{(0)}(x,\, t;\, x_1,\, t_1) V(x_1,\, t_1) G^{(0)}(x_1,\, t_1;\, x_0,\, t_0) \\
&= (-i\lambda) \int_{t_0}^{t} \mathrm{d}t_1 \int_{-\infty}^{\infty} \mathrm{d}x_1 \int \mathcal{D}\bar{x} \exp\left(\int_{t_0}^{t} \mathrm{d}t' \mathcal{S}_0[\bar{x}(t')] \right) V(x_1,\, t_1) \\
&\qquad \int \mathcal{D}\bar{y} \exp\left(\int_{t_0}^{t} \mathrm{d}t' \mathcal{S}_0[\bar{y}(t')] \right) ,
\end{aligned}
\tag{6.79}
$$

clarifying the notation above. Expanding the total amplitude in this fashion lends itself to a diagrammatic representation of the terms, cf. Fig.6.1. There, the free field total amplitudes are represented as straight lines and the interaction points are depicted as black blobs. To translate back from a diagram to an expression one starts at the top of a diagram, i.e. at its endpoint, and works the way back to its starting point. Each straight line between two points $(x_i,\, t_i)$ and $(x_{i-1},\, t_{i-1})$ then relates to a factor $G^{(0)}(x_i,\, t_i;\, x_{i-1},\, t_{i-1})$, and each blob at a position $(x_i,\, t_i)$ corresponds to a factor $-i\lambda V(x_i,\, t_i)$, where the position is to be integrated over. For the last diagram in the figure one then has

$$
\begin{aligned}
&G^{(2)}(x,\, t;\, x_0,\, t_0) \\
&\quad = (-i\lambda)^2 \int_{t_0}^{t} \mathrm{d}t_2 \int_{-\infty}^{\infty} \mathrm{d}x_2 G^{(0)}(x,\, t;\, x_2,\, t_2) V(x_2,\, t_2)
\end{aligned}
$$

$$\int_{t_0}^{t} \mathrm{d}t_1 \int_{-\infty}^{\infty} \mathrm{d}x_1 G^{(0)}(x_2,\, t_2;\, x_1,\, t_1) V(x_1,\, t_1) G^{(0)}(x_1,\, t_1;\, x_0,\, t_0) \tag{6.80}$$

for its contribution to the total amplitude $G^{(n)}$.

6.2 Free field theories

6.2.1 Free scalar fields

The concepts developed in the previous section are now translated to the case of quantum field theories. Instead of integrating over the position and momentum x and p of a single particle, now the integration is over the field ϕ and its conjugate momentum π. In this respect it is useful to remember that the quantum theory of a free field can be interpreted as a collection of infinitely many harmonic oscillators, where one wavefunction represents one mode of the field.

Exploring analogies

Let $|\Psi(t)\rangle$ be a state of the system, which evolves through the Schrödinger equation as

$$i\frac{\partial}{\partial t}|\Psi(t)\rangle = \mathcal{H}|\Psi(t)\rangle\,. \tag{6.81}$$

Here, the Hamiltonian is that of a free scalar field,

$$\mathcal{H} = \frac{1}{2}\int \mathrm{d}^3x \left(\pi^2 + \left|\vec{\nabla}\phi\right|^2 + m^2\phi^2\right)\,. \tag{6.82}$$

To construct a time-evolution operator, project the states on the ϕ-space and define corresponding wave functionals through

$$\Psi[\phi(\vec{x}),\, t] = \langle\phi|\Psi(t)\rangle\,. \tag{6.83}$$

Then, the evolution of such an object is given by

$$\begin{aligned}\Psi[\phi(\vec{x}),\, t] &= \int \mathcal{D}\phi_0 \left\langle \phi \left| e^{-i\mathcal{H}(t-t_0)} \right| \phi_0 \right\rangle \Psi[\phi_0(\vec{x}),\, t_0] \\ &=: \int \mathcal{D}\phi_0 G[\phi,\, t;\, \phi_0,\, t_0]\, \Psi[\phi_0(\vec{x}),\, t_0]\,. \end{aligned} \tag{6.84}$$

The functional integration

In the following paragraph some light will be shed on this nomenclature. Assuming real scalar fields, the field ϕ is a function connecting the real three-dimensional space with real numbers, $R^3 \to R$. Thus, the functional integral $\int \mathcal{D}\phi$ denotes an integral over a set of such functions. Therefore, to draw the evolution of $\Psi[\phi(\vec{x}),\, t]$, one would have to stack layers of such functions onto each other. These layers are labelled through the discretised time, i.e. through the t_i, where $t_i \in [t_0,\, t]$.
Hence, to obtain the path integral representation of $G[\phi,\, t;\, \phi_0,\, t_0]$, one again divides the interval $[t_0,\, t]$ into $N+1$ equal parts. Next, for each division, i.e. at each layer, a one is inserted in the form

$$1 = \int \mathcal{D}\phi_i \; |\phi_i\,\rangle\langle\,\phi_i| \quad \text{with} \quad \phi_i = \phi_i(\vec{x},\, t_i)\,. \tag{6.85}$$

Then, with $N \to \infty$,

$$G[\phi,\, t;\, \phi_0,\, t_0] = \lim_{N\to\infty} \left[\int \prod_{i=1}^{N} \mathcal{D}\phi_i\right] \left[\prod_{i=0}^{N} \left\langle \phi_{i+1} \left| e^{i\mathcal{H}(t_{i+1}-t_i)} \right| \phi_i \right\rangle\right]\,, \tag{6.86}$$

Again, with $N \to \infty$, the time intervals become small, allowing to expand the exponentials,

$$\begin{aligned} G[\phi_{i+1},\, t_{i+1};\, \phi_i,\, t_i] &= \left\langle \phi_{i+1} \left| e^{i\mathcal{H}(t_{i+1}-t_i)} \right| \phi_i \right\rangle \\ &\approx \langle \phi_{i+1} \left| 1 - i\mathcal{H}(t_{i+1}-t_i) \right| \phi_i \rangle \\ &= \delta\left[\phi_{i+1} - \phi_i\right] - i\epsilon \left\langle \phi_{i+1} \left| \mathcal{H} \right| \phi_i \right\rangle\,, \end{aligned} \tag{6.87}$$

where $\epsilon = t_{i+1} - t_i = (t_{N+1} - t_0)/(N+1)$ is assumed. In fact, the δ-functional can be understood as a product of infinitely many δ-functions, each of which is taken at one point $\vec{x}$ in space. Rewriting each of them as a Fourier-transform one has

$$\begin{aligned} \delta\left[\phi_{i+1} - \phi_i\right] &= \prod_{\vec{x}} \delta\left(\phi_{i+1}(\vec{x}) - \phi_i(\vec{x})\right) \\ &= \prod_{\vec{x}} \int_{-\infty}^{\infty} \frac{\mathrm{d}\pi_i}{2\pi} \exp\left\{i\pi_i(\vec{x})\left[\phi_{i+1}(\vec{x}) - \phi_i(\vec{x})\right]\right\} \\ &=: \int \mathcal{D}\pi_i \exp\left\{i \int \mathrm{d}^3\vec{x}\, \pi_i(\vec{x}) \left[\phi_{i+1}(\vec{x}) - \phi_i(\vec{x})\right]\right\}\,. \end{aligned} \tag{6.88}$$

In a similar way, and in full analogy to the path integral in quantum mechanics,

$$\langle\phi_{i+1}\,|\mathcal{H}|\,\phi_i\rangle = \int \mathcal{D}\pi_i\,\mathcal{H}[\pi_i,\,\bar{\phi}_i]\,\exp\left\{i\int \mathrm{d}^3\vec{x}\,\pi_i(\vec{x})\,[\phi_{i+1}(\vec{x})-\phi_i(\vec{x})]\right\}\,, \tag{6.89}$$

where, again, an average has been introduced. This time it is over the fields,

$$\bar{\phi}_i = \frac{\phi_{i+1}+\phi_i}{2}\,. \tag{6.90}$$

In the equation above, $\mathcal{H}[\pi_i,\,\bar{\phi}_i]$ is a functional, not an operator, whereas in the sandwich, $\mathcal{H}$ with no arguments whatsoever denotes an operator. Repeating the same tricks as above, i.e. re-exponentiating of the Hamiltonian functional and identifying derivatives of the fields, one finds

$$\begin{aligned}
&G[\phi,\,t;\,\phi_0,\,t_0]\\
&= \lim_{N\to\infty}\left[\int\prod_{i=1}^{N}\mathcal{D}\phi_i\right]\left[\int\prod_{j=0}^{N}\mathcal{D}\pi_j\right]\\
&\quad\exp\left\{i\epsilon\sum_{j=0}^{N}\int \mathrm{d}^3\vec{x}\left[\pi_j(\vec{x})\frac{\phi_{j+1}(\vec{x})-\phi_j(\vec{x})}{\epsilon}-H[\pi_j(\vec{x}),\,\bar{\phi}_j(\vec{x})]\right]\right\}\\
&= \int\mathcal{D}\bar{\phi}\mathcal{D}\pi\,\exp\left[i\int_{t_0}^{t}\mathrm{d}t'\int \mathrm{d}^3\vec{x}\left(\pi(\vec{x})\dot{\bar{\phi}}(\vec{x})-H[\pi,\,\bar{\phi}]\right)\right]\,.
\end{aligned} \tag{6.91}$$

Carrying out the Gaussian integration over π (again, $H[\pi_j(\vec{x}),\,\bar{\phi}_j(\vec{x})]$ is assumed to be quadratic in π), one ends up with

$$\begin{aligned}
G[\phi,\,t;\,\phi_0,\,t_0] &= \int\mathcal{D}\bar{\phi}\exp\left[i\int_{t_0}^{t}\mathrm{d}t'\int \mathrm{d}^3\vec{x}\mathcal{L}(\dot{\bar{\phi}},\,\bar{\phi},\,t')\right]\\
&= \int\mathcal{D}\bar{\phi}\exp\left[i\mathcal{S}(\bar{\phi})\right]
\end{aligned} \tag{6.92}$$

Again, $\mathcal{L}$ and $\mathcal{S}$ are the classical Lagrangian and action, respectively. It should be stressed that in the path formalism for the fields,

$$\mathcal{D}\bar{\phi} = \lim_{N\to\infty}\left[\int\prod_{i=1}^{N}\prod_{\vec{x}}d\phi(\vec{x},\,t_i)\left(\frac{1}{2\pi i\epsilon}\right)\right]\,. \tag{6.93}$$

In most cases, however, the normalisation will be put into the overall normalisation of the total amplitude and forgotten.
This total amplitude is over all paths in the space of field configurations, described by $\phi(\vec{x},\,t)$ and $\pi(\vec{x},\,t)$. the path is fixed at the two endpoints $\phi_0(\vec{x}_0,\,t_0)$ and $\phi(\vec{x},\,t)$. In quantum mechanics in contrast, the total amplitude is over all eigenvalues $x(t)$ and $p(t)$ of the position and momentum operators of the single particle.
Taking the Fourier transform of the fields in the total amplitude, Eq.(6.92), and of the action, one has

$$
\begin{aligned}
& G[\phi,\,t;\,\phi_0,\,t_0] \\
&= \int \mathcal{D}\bar{\phi} \exp\left\{ i \int\limits_{t_0}^{t} \mathrm{d}t' \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \frac{1}{2} \left[(\partial_t \bar{\phi}(-\vec{k}))(\partial_t \bar{\phi}(\vec{k})) - \omega_{\vec{k}} \bar{\phi}(-\vec{k})\bar{\phi}(\vec{k}) \right] \right\} .
\end{aligned}
\tag{6.94}
$$

This is just the infinite sum over actions of harmonic oscillators, one for each mode $\vec{k}$. It should be noted that this identification works so well, since the individual actions are diagonal in the modes, i.e. there is no mixing between two modes $\vec{k}_1$ and $\vec{k}_2$. Therefore, one has

$$
\begin{aligned}
& G[\phi,\,t;\,\phi_0,\,t_0] \\
&= \prod_{\vec{k}} \left(\frac{\omega_{\vec{k}}}{2\pi i \sin[\omega_{\vec{k}}(t-t_0)]} \right)^{\frac{1}{2}} \\
&\quad \exp\left\{ \frac{i}{2} \frac{1}{(2\pi)^3} \left(\frac{\omega_{\vec{k}}}{\sin[\omega_{\vec{k}}(t-t_0)]} \right) \left[\left(\phi_{\vec{k}}^2 + \phi_{0,\vec{k}}^2 \right) \cos[\omega_{\vec{k}}(t-t_0)] - 2\phi_{\vec{k}}\phi_{0,\vec{k}} \right] \right\} \\
&= \prod_{\vec{k}} \left(\frac{\omega_{\vec{k}}}{2\pi i \sin[\omega_{\vec{k}}(t-t_0)]} \right)^{\frac{1}{2}} \exp\left\{ \frac{i}{2} \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \frac{\omega_{\vec{k}}}{\sin[\omega_{\vec{k}}(t-t_0)]} \right. \\
&\qquad \left. \left[\left(\phi^2(\vec{k}) + \phi_0^2(\vec{k}) \right) \cos[\omega_{\vec{k}}(t-t_0)] - 2\phi(\vec{k})\phi_0(\vec{k}) \right] \right\} ,
\end{aligned}
\tag{6.95}
$$

where the difference between the first and second expression is that in the second expression the product over all $\vec{k}$ has been replaced by the integration. The ground state wave functional can be recovered by taking $T = t - t_0 \to \infty$ and considering the leading term only. Using

$$
\sin(\omega_k T) \to -\frac{i}{2} e^{i\omega_k T} \quad \text{and} \quad \frac{\cos(\omega_k T)}{\sin(\omega_k T)} \to i \tag{6.96}
$$

for $T \to \infty$ one finds

$$G[\phi,\, \phi_0,\, T] \longrightarrow \left[\prod_{\vec{k}} \left(\frac{\omega_k}{\pi}\right)^{\frac{1}{2}} e^{-\frac{i\omega_k T}{2}}\right] \exp\left[-\int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \frac{\omega_k}{2} \left(\phi^2(\vec{k}) + \phi_0^2(\vec{k})\right)\right] \tag{6.97}$$

From this one can read off the vacuum energy,

$$\mathcal{E}_0 = \frac{1}{2} \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \omega_k\,. \tag{6.98}$$

Adding sources

To add dynamics to the system, one couples the fields to external sources, each for one field. For a real scalar field this translates to coupling the field $\phi(x)$ locally to a source J by adding a term $J(t,\, \vec{x})\phi(t,\, \vec{x})$ to the action. For free fields this is not particularly interesting, but as will be seen later on, for the interacting case this will be the path towards vacuum expectation values of time-ordered products, i.e. n-point functions. In fact, as it will turn out, the so modified actions give rise to generating functionals of the n-point functions. When adding such a source, the action becomes

$$\begin{aligned}\mathcal{S} &= \int_{t_0}^{t} \mathrm{d}t' \int \mathrm{d}^3\vec{x} \left\{\frac{1}{2}\left[\partial_\mu\phi(t,\, \vec{x})\partial^\mu\phi(t,\, \vec{x}) - m^2\phi(t,\, \vec{x})^2\right] + J(t,\, \vec{x})\phi(t,\, \vec{x})\right\} \\ &= \int_{t_0}^{t} \mathrm{d}t' \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \left\{\frac{1}{2}\left[\partial_{t'}\phi(t,\, -\vec{k})\partial^t\phi(t,\, \vec{k}) - \omega_k^2\phi(t,\, -\vec{k})\phi(t,\, \vec{k})\right]\right. \\ &\qquad\qquad \left. + J(t,\, -\vec{k})\phi(t,\, \vec{k})\right\}\end{aligned} \tag{6.99}$$

Therefore, the total amplitude in presence of the sources reads

$$\begin{aligned}G_J[\phi,\, t;\, \phi_0,\, t_0] = \prod_{\vec{k}} \int \mathcal{D}\phi_k(t) \\ \exp\left\{\frac{i}{(2\pi)^3} \int_{t_0}^{t} \mathrm{d}t' \left[\frac{1}{2}\left(\partial_{t'}^2\phi_{\vec{k}}^2(t') - \omega_{\vec{k}}\phi_{\vec{k}}^2(t')\right) + J_{\vec{k}}(t')\phi_{\vec{k}}(t')\right]\right\}\end{aligned} \tag{6.100}$$

Completing the squares yields

$$\begin{aligned} G_J[\phi,\,t;\,\phi_0,\,t_0] &= G[\phi,\,t;\,\phi_0,\,t_0] \\ &\prod_{\vec{k}} \exp\left\{\frac{i}{2(2\pi)^3}\int\limits_{t_0}^{t}\mathrm{d}t'\int\limits_{t_0}^{t}\mathrm{d}\bar{t}\left[J(t',\,-\vec{k})g(t',\,\bar{t})J(\bar{t},\,\vec{k})\right]\right\} \end{aligned} \tag{6.101}$$

Here, again, $g(t',\,\bar{t})$ is the Green's function of an harmonic oscillator,

$$g(t',\,\bar{t}) = \begin{cases} \dfrac{\sin[\omega_k(t-t')]\sin[\omega_k(\bar{t}-t_0)]}{\omega_k\sin[\omega_k(t-t_0)]} & \text{for} \quad t'\le\bar{t} \\ \dfrac{\sin[\omega_k(t-\bar{t})]\sin[\omega_k(t'-t_0)]}{\omega_k\sin[\omega_k(t-t_0)]} & \text{for} \quad \bar{t}\le t'\,. \end{cases} \tag{6.102}$$

In a scattering experiment, as outlined before, two particles that are essentially free are collided and interact in a very tiny region. Then, the kinematics of the outgoing particles is measured, such that both the incoming and outgoing particles can be considered as asymptotic. In this situation, one is interested in the limit $t\to+\infty$ and $t_0\to-\infty$. Only the vacuum contributions to G_J will survive in this limits, therefore $G_J(\phi,\,+\infty,\,\phi_0,\,-\infty)$ is called the vacuum-to-vacuum amplitude in the presence of the source J, conventionally denoted by $Z[J]$. Taking this limit explicitly, i.e. considering $t=-t_0=T\to\infty$one finds for the $t'\le\bar{t}$-piece of the Green's function $g(\bar{t},\,t')$

$$\begin{aligned} g(\bar{t},\,t') &\overset{t'\le\bar{t}}{\equiv} \frac{\sin[\omega_k(t-\bar{t})]\sin[\omega_k(t'-t_0)]}{\omega_k\sin[\omega(t-t_0)]} \\ &= \frac{1}{\omega_k\sin(2\omega_kT)}\left[\sin(\omega_kT)\cos(\omega_k\bar{t})-\sin(\omega_k\bar{t})\cos(\omega_kT)\right] \\ &\qquad\qquad \left[\sin(\omega_kt')\cos(\omega_kT)-\sin(\omega_kT)\cos(\omega_kt')\right] \\ &= \frac{1}{2\omega_k}\Bigg[\sin(\omega_kt')\cos(\omega_k\bar{t})-\sin(\omega_k\bar{t})\cos(\omega_kt') \\ &\qquad +\cos(\omega_k\bar{t})\cos(\omega_kt')\frac{\sin(\omega_kT)}{\cos(\omega_kT)}-\sin(\omega_k\bar{t})\sin(\omega_kt')\frac{\cos(\omega_kT)}{\sin(\omega_kT)}\Bigg] \\ &\overset{iT\to\infty}{\longrightarrow} \frac{1}{2\omega_k}\left[\sin[\omega_k(\bar{t}-t')]-i\cos[\omega_k(\bar{t}-t')]\right] = \frac{i}{2\omega_k}e^{-i\omega_k(\bar{t}-t')}\,. \end{aligned} \tag{6.103}$$

The other piece of the Green's function, i.e for $\bar{t}\le t'$ then is given by

$$g(\bar{t},\,t') \overset{t'\ge\bar{t}}{\equiv} \frac{i}{2\omega_k}e^{i\omega_k(\bar{t}-t')}\,. \tag{6.104}$$

Taken together, therefore the Green's function reads

$$g(\bar{t},\, t') = \frac{i}{2\omega_k}\left[e^{-i\omega_k(\bar{t}-t')}\theta(\bar{t}-t') + e^{i\omega_k(\bar{t}-t')}\theta(t'-\bar{t})\right] = -\Delta_F(\vec{k},\, \bar{t}-t')\,. \tag{6.105}$$

This is nothing but the Feynman propagator which has been encountered before. Hence, for free field theories (denoted by the subscript 0), one finds

$$Z_0[J] = Z_0[0]\exp\left\{-\frac{i}{2}\int\frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\int\limits_{-\infty}^{\infty}\mathrm{d}\bar{t}\mathrm{d}t' J(-\vec{k},\, \bar{t})\Delta_F(\vec{k},\, \bar{t}-t')J(\vec{k},\, t')\right\}\,, \tag{6.106}$$

or, in position space,

$$Z_0[J] = Z_0[0]\exp\left\{-\frac{i}{2}\int\mathrm{d}^4\bar{x}\mathrm{d}^4x' J(\bar{x})\Delta_F(\bar{x}-x')J(x')\right\}\,. \tag{6.107}$$

Here, $Z_0[0]$ denotes the vacuum-to-vacuum amplitude, it contains the limiting value of $G(\phi,\, \infty;\, \phi_0,\, -\infty)$. In general, it is not necessary to know the exact form of $Z_0[0]$, since it cancels out in the construction of n-point functions. Therefore, it won't be discussed further.

Recall from previous considerations that the boundary conditions of the propagator, i.e. the selection of terms propagating forward and backward in time, was fixed through the $i\epsilon$-prescription. It was added to the path integral formulation of quantum mechanics by the addition of an extra term, which, in field theory, amounts to adding

$$G = \int\mathcal{D}\phi\,\exp\left\{i\mathcal{S}[\phi]\right\}\exp\left\{-\epsilon\int\mathrm{d}^4x\phi^2(x)\right\}\,. \tag{6.108}$$

As before, this term is interpreted as a suitable convergence factor. A heuristic argument of how this can be understood is as follows: the path integral is a sum over all paths, contributing as a sum of corresponding phases. These phases emerge due to the factor i in front of the action. This means that the integrand oscillates horribly when changing the path and there is no reason to assume that the full procedure then converges to anything meaningful. Only the addition of the real small convergence factor ensures proper behaviour.

n-point functions

In a previous section it has been shown that the path integral of the forced harmonic oscillator can be used as generating functional for Green's functions. By pretty much the same reasoning, the vacuum-to-vacuum amplitude in the presence of sources is the generating functional for time-ordered products of the fields. It is the desire to calculate such Green's functions and input them into the reduction formula that makes $Z[J]$ the object of central interest in the path integral representation. For illustration, a number of time-ordered products will be calculated, using the explicit form of $Z_0[J]$ as written in Eq.(6.107).

$$\langle 0\,|T\,[\phi(x_1),\,\phi(x_2)]|\,0\rangle = \left.\frac{-i\delta}{\delta J(x_1)}\frac{-i\delta}{\delta J(x_2)}\frac{Z_0[j]}{Z_0[0]}\right|_{J=0} = i\Delta_F(x_1-x_2)\,. \tag{6.109}$$

$$\begin{aligned}
&\langle 0\,|T\,[\phi(x_1),\,\phi(x_2),\,\phi(x_3),\,\phi(x_4)]|\,0\rangle \\
&= \left.\frac{-i\delta}{\delta J(x_1)}\frac{-i\delta}{\delta J(x_2)}\frac{-i\delta}{\delta J(x_3)}\frac{-i\delta}{\delta J(x_4)}\frac{Z_0[j]}{Z_0[0]}\right|_{J=0} \\
&= i\Delta_F(x_1-x_2)\Delta_F(x_3-x_4) + i\Delta_F(x_1-x_3)\Delta_F(x_2-x_4) \\
&\quad +i\Delta_F(x_1-x_4)\Delta_F(x_2-x_3)\,.
\end{aligned} \tag{6.110}$$

It is clear that without interactions, n-point functions with n higher than two just decompose into sums of products of two-point functions. Diagrammatically this is just a set of disconnected lines linking two points each. Defining a new functional, $\mathcal{W}_0[J]$ through

$$Z_0[J] = Z_0[0]\,\exp\{i\mathcal{W}_0[J]\} \tag{6.111}$$

one immediately understands that

$$\begin{aligned}
\left.\frac{i\delta}{\delta J(x_1)}\mathcal{W}_0[J]\right|_{J=0} &= 0\,, \\
\left.\frac{i\delta}{\delta J(x_1)}\frac{i\delta}{\delta J(x_2)}\mathcal{W}_0[J]\right|_{J=0} &= \Delta_F(x_1-x_2)\,, \\
\left.\prod_{j=1}^{n>2}\frac{i\delta}{\delta J(x_j)}\mathcal{W}_0[J]\right|_{J=0} &= 0\,,
\end{aligned} \tag{6.112}$$

i.e. that this new functional just generates the two-point function and, when introducing interactions later on, this functional generates the connected diagrams only.

6.2.2 Free vector fields

Outline of the procedure

In a next step, the vacuum-to-vacuum amplitudes of free vector fields will be calculated. To do so, one starts from the analogy of Eq.(6.92) and takes the limit $t = -t_0 \to \infty$. In addition, the correct boundary conditions for the propagators will be supplemented through the addition of a convergence factor in the spirit of Eq.(6.108). However, life is not that easy. To anticipate the difficulties in the construction of a path integral for vector (gauge) fields, one just has to recall the difficulties related to the choice of gauge encountered in the operator formulation of second quantisation of the photon field. Here this difficulty of fixing a gauge can be phrased in the following way: one would like to do a path integral over all field configuration A^μ, $\int \mathcal{D}A^\mu$. But the action, since it is gauge invariant, does not depend on all four components of the gauge field. This leads to a huge infinity when doing the integral over all four components, because only two of them are integrated over a function, the other two just lead to the full integration region as a result. To dwell a little bit on this consider an integral of the form

$$\int \mathrm{d}x\mathrm{d}y f(x)\,. \tag{6.113}$$

This is a divergent integral, due to the y-integration. For the path integral integration over all components of a gauge field, one encounters a very similar problem: one just integrates over non-dynamic degrees of freedom, yielding the (divergent) volume as a result. A straightforward way to get rid of this problem is to split off those degrees of freedom that are gauge dependent and to bury them in an overall normalisation which would maybe cancel out when taking the ratio $Z[J]/Z[0]$. For the example integral this would amount to split the integral into an integration over x, carrying the physics, and an integral over y, being buried in a normalisation. However, in general such a splitting may not be an easy task at all. This is especially true, when Yang-Mills fields are considered, which, due to their self-interactions, have the nasty feature of mixing different field components. This case will be dealt with later. In the meantime, for the photon field, gauge dependent and gauge invariant pieces will be separated, leading to the separation of an infinite volume from the physically relevant pieces. The price one has to pay for this ad-hoc procedure is, as usual, a loss of manifest Lorentz covariance.

Coulomb gauge

The simplest way to proceed is to construct a vacuum-to-vacuum amplitude with sources that couple in a gauge invariant way to the fields only, i.e. to their transverse components $\vec{A}_\perp$ only,

$$Z_0[J] = \int \mathcal{D}A^\mu \exp\left\{i\int \mathrm{d}^4x \left[\frac{1}{2}\left(\vec{E}^2 - \vec{B}^2\right) + \vec{J}\cdot\vec{A}_\perp\right]\right\} \tag{6.114}$$

Here, $\vec{E}$ and $\vec{B}$ are the usual electric and magnetic fields, respectively, with

$$\begin{aligned} \vec{E} &= -\vec{\nabla}A^0 - \partial_\perp\vec{A}\,, \\ \vec{B} &= \vec{\nabla}\times\vec{A}\,. \end{aligned} \tag{6.115}$$

Now, any part of the integral that does not depend on $\vec{J}$ will be put into $Z_0[0]$. Since the integrand is Gaussian, it is easy to see that the integrals over A^0 and $\vec{A}_\parallel$ will not depend on $\vec{J}$. Therefore, one may fix these field components and put the remaining integration into a gauge volume $\mathcal{N}$,

$$Z_0[J] = \mathcal{N}\int \mathcal{D}A^\mu \delta[A^0]\delta[\vec{A}_\parallel] \exp\left\{i\int \mathrm{d}^4x \left[\frac{1}{2}\left(\vec{E}^2 - \vec{B}^2\right) + \vec{J}\cdot\vec{A}\right]\right\}\,. \tag{6.116}$$

The appearance of the two δ-functionals is easy to interpret: they just reflect the gauge fixing, in this case for the Coulomb gauge. The infinite gauge volume, i.e. the effect of the superficial, unphysical degrees of freedom has been buried quietly in $\mathcal{N}$. The integrals over A^0 and $\vec{A}_\parallel$ are easily carried out, and one has

$$Z_0[J] = \mathcal{N}\int \mathcal{D}\vec{A}_\perp \exp\left\{i\int \mathrm{d}^4x \left[\frac{1}{2}\left(\vec{E}_\perp^2 - \vec{B}_\perp^2\right) + \vec{J}\cdot\vec{A}_\perp\right]\right\}\,, \tag{6.117}$$

where $E_\perp$ is the transverse electric field, depending on $\vec{A}_\perp$ only through

$$\vec{E}_\perp = -\partial_t\vec{A}_\perp\,. \tag{6.118}$$

To highlight the "transverseness" of the photon field, one employs its Fourier transform

$$\vec{A}_\perp(x) = \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\sum_{\lambda=1}^{2}\vec{\varepsilon}(\vec{k},\,\lambda)A(\vec{k},\,\lambda,\,t)e^{i\vec{k}\cdot\vec{x}}\,, \tag{6.119}$$

where the polarisation vectors have been defined before and satisfy

$$\vec{k}\cdot\vec{\varepsilon}(\vec{k},\,\lambda)=0 \quad\text{and}\quad \vec{\varepsilon}(\vec{k},\,\lambda)\cdot\vec{\varepsilon}(\vec{k},\,\lambda')=\delta_{\lambda\lambda'}\,. \tag{6.120}$$

Written in these terms the vacuum-to-vacuum amplitude reads

$$Z_0[J]=\mathcal{N}\int\mathcal{D}A(\vec{k},\,\lambda,\,t)\,\exp\left\{i\sum_{\lambda=1}^{2}\int\mathrm{d}t'\int\frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\right.$$
$$\left.\left[\left(\partial_{t'}A(\vec{k},\,\lambda,\,t')\right)^2-\vec{k}^2A^2(\vec{k},\,\lambda,\,t')+J(-\vec{k},\,\lambda,\,t')A(\vec{k},\,\lambda,\,t')\right]\right\} \tag{6.121}$$

Again, $Z_0[J]$ is a sum over infinitely many harmonic oscillator path integrals, one for each mode $\vec{k}$ and polarisation λ. Completing the squares in the exponent and doing the (Gaussian) integral then yields

$$\frac{Z_0[J]}{Z_0[0]}$$
$$=\exp\left\{-\frac{i}{2}\sum_{\lambda=1}^{2}\int\frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\int\limits_{-\infty}^{\infty}\mathrm{d}\bar{t}\int\limits_{-\infty}^{\infty}\mathrm{d}t'\left[J(-\vec{k},\,\lambda,\,\bar{t})\Delta_F(\vec{k},\,\bar{t}-t')J(\vec{k},\,\lambda,\,t')\right]\right\} \tag{6.122}$$

Inserting some factors of unity through relations like

$$J(\vec{k},\,\lambda,\,t')=\sum_{\lambda'=1}^{2}\vec{\varepsilon}(\vec{k},\,\lambda)\vec{\varepsilon}(\vec{k},\,\lambda')J(\vec{k},\,\lambda',\,t')=\varepsilon_i(\vec{k},\,\lambda)J_i(\vec{k},\,t')\,, \tag{6.123}$$

where the summation over vector indices i is understood, and identifying

$$\begin{aligned}D^{\mathrm{tr.}}_{ij}(x,\,y) &= \int\frac{\mathrm{d}^3\vec{k}}{(2\pi)^4}\frac{e^{-i\vec{k}\cdot(\vec{x}-\vec{y})}}{k^2+i\epsilon}\sum_{\lambda=1,\,2}\varepsilon_i(k,\,\lambda)\varepsilon_j(k,\,\lambda)\\ &= \int\frac{\mathrm{d}^4k}{(2\pi)^4}\Delta_F(\vec{k},\,\bar{t}-t')\sum_{\lambda=1,\,2}\varepsilon_i(k,\,\lambda)\varepsilon_j(k,\,\lambda)\\ &\quad ,\end{aligned} \tag{6.124}$$

one can cast Eq.(6.122) back into position space through a Fourier transform

$$Z_0[J]=Z_0[0]\exp\left\{-\frac{i}{2}\int\mathrm{d}^4x\mathrm{d}^4x'\,J_i(x)D^{\mathrm{tr.}}_{ij}(x-x')J_j(x')\right\}\,, \tag{6.125}$$

where the individual components of the sources are summed over the contributions from both polarisations, cf. Eq.(6.123).

6.2.3 Free spinor fields

Functional calculus for Grassmann numbers

Coordinate representations of spinor fields are anticommuting or Grassmann numbers. This is bad news, since the experiences gained so far are for bosonic structures only. In addition, the path integral of bosonic fields could be interpreted as an infinite sum over harmonic oscillators, and this interpretation is not available for Grassmann numbers. Also, for fermionic variables the integration is more a kind of formal manipulation. Assume x and η to be an ordinary, commuting, and a Grassmann, anticommuting number, respectively. Then

$$[x,\, x] = \{\eta,\, \eta\} = 0\,, \tag{6.126}$$

and, in full analogy,

$$\left[\frac{\mathrm{d}}{\mathrm{d}x},\, x\right] = \left\{\frac{\mathrm{d}}{\mathrm{d}\eta},\, \eta\right\} = 1\,. \tag{6.127}$$

For any function $f(\eta)$, the power series expansion in η must terminate in the linear term, i.e.

$$f(\eta) = a_0 + a_1\eta\,. \tag{6.128}$$

Since $\mathrm{d}^2 f/\mathrm{d}\eta^2 = 0$,

$$\left\{\frac{\mathrm{d}}{\mathrm{d}\eta},\, \frac{\mathrm{d}}{\mathrm{d}\eta}\right\} = 0\,. \tag{6.129}$$

So far, nothing really shocking has happened. But now, consider this. Assume that the a_i in the series expansion of f are ordinary numbers. Then

$$\frac{df}{\mathrm{d}\eta} = a_1\,. \tag{6.130}$$

However, if the a's are anticommuting numbers, the differential operator has to be swapped with a_1 before it can act on η. In such a case,

$$\frac{df}{\mathrm{d}\eta} = -a_1\,. \tag{6.131}$$

In other words: some care has to be spent to ensure the correct statistics in differentiation with respect to Grassmann numbers. For such numbers one has to fix integration through definition. To this end, one defines

$$\begin{aligned}\int \mathrm{d}\eta &:= 0\\ \int \mathrm{d}\eta\, \eta &:= 1\,,\end{aligned}\tag{6.132}$$

chosen such that the operation $\int \mathrm{d}\eta$ is translationally invariant and linear. It has to be stressed that integration is not the inverse operation of differentiation, in fact, integration acts just like differentiation. Again, the order of the operations and variables is important. To exemplify this consider

$$\int \mathrm{d}\eta_1 \mathrm{d}\eta_2\, \eta_1\eta_2 = -\int \mathrm{d}\eta_1 \mathrm{d}\eta_2\, \eta_2\eta_1 = -\int \mathrm{d}\eta_2 \mathrm{d}\eta_1\, \eta_1\eta_2 = -1\,. \tag{6.133}$$

To proceed, switch to the infinite-dimensional limit, i.e. to functions $\xi(x)$ and $\eta(x)$ in the space of ordinary function or in the space of Grassmann-valued functions. There, the analogy of bosonic and fermionic functions, differing only by the usage of commutators and anti-commutators, respectively, is further employed to define the corresponding functional derivatives as

$$\left[\frac{\delta}{\delta\xi(x)},\, \xi(y)\right] = \left\{\frac{\delta}{\delta\eta(x)},\, \eta(y)\right\} = \delta(x-y)\,. \tag{6.134}$$

The functional integrals are given by

$$\begin{aligned}\mathcal{D}\xi &= \prod_x \mathrm{d}\xi(x)\\ \mathcal{D}\eta &= \prod_x \mathrm{d}\eta(x)\,,\end{aligned}\tag{6.135}$$

but for the fermionic variables the ordering of the $\mathrm{d}\eta$ is assumed to be opposite to the ordering in the functional integral. This will ensure that no extra minus signs will show up when the integral is evaluated. Similarly, the δ-functionals are defined such that

$$\int \mathcal{D}\xi\delta[\xi] = \int \mathcal{D}\eta\delta[\eta] = 1 \tag{6.136}$$

leading to the following representations of the δ-functional

$$\begin{aligned}\delta[\xi - a] &= \prod_x \delta(\xi(x) - a(x))\\ \delta[\eta - \sigma] &= \prod_x \delta(\eta(x) - \sigma(x))\,.\end{aligned}\tag{6.137}$$

Then,

$$
\begin{aligned}
\int \mathcal{D}\xi F(\xi)\delta[\xi - a] &= \int \prod_x \mathrm{d}\xi(x) F[\xi] \prod_y \delta(\xi(y) - a(y)) \\
&= \prod_x \int \mathrm{d}\xi(x)\delta(\xi(x) - a(x)) = F[a] \\
\int \mathcal{D}\eta F(\eta)\delta[\eta - \sigma] &= \int \prod_x \mathrm{d}\eta(x) F[\eta] \prod_y \delta(\eta(y) - \sigma(y)) \\
&= \prod_x \int \mathrm{d}\eta(x)\delta(\eta(x) - \sigma(x)) F[\eta] = F[\sigma] \,.
\end{aligned}
\tag{6.138}
$$

Finally, consider the Gaussian integral, first for bosonic variables. From

$$
\begin{aligned}
\int \mathcal{D}\xi \exp\left[-\int \mathrm{d}x \xi^2(x)\right] &\longrightarrow \int \prod_x \mathrm{d}\xi(x) \exp\left[-\sum_x \xi^2(x)\right] \\
&= \int \prod_x \mathrm{d}\xi(x) \prod_x \exp\left[-\xi^2(x)\right] \\
&= \prod_x \int \mathrm{d}\xi(x) \exp\left[-\xi^2(x)\right] = \prod_x \sqrt{\pi}
\end{aligned}
\tag{6.139}
$$

it follows

$$
\begin{aligned}
\int \mathcal{D}\xi \exp\left[-\int \mathrm{d}x \alpha \xi^2(x)\right] &= \prod_x \sqrt{\frac{\pi}{\alpha}} \,, \\
\int \mathcal{D}\xi \exp\left[-\int \mathrm{d}x \alpha(x) \xi^2(x)\right] &= \prod_x \sqrt{\frac{\pi}{\alpha(x)}} \,.
\end{aligned}
\tag{6.140}
$$

Extend the consideration above to a case, where α is a function that depends on two variables, $\alpha(x) \to \alpha(x,\, y)$. This allows to evaluate

$$
\begin{aligned}
&\int \mathcal{D}\xi \exp\left[-\int \mathrm{d}x \mathrm{d}y \xi(x)\alpha(x,\, y)\xi(y)\right] \\
&= \prod_x \sqrt{\frac{\pi}{\alpha(x,\, y)}} = \frac{1}{\sqrt{\det\left[\alpha(x,\, y)\right]}} \prod_x \sqrt{\pi} \,.
\end{aligned}
\tag{6.141}
$$

The result can be easily justified by considering the finite-dimensional case, for instance on a lattice, and taking the continuum limit. To extend this even further, add a source term to the Gaussian. By completing the squares one finds

$$\int \mathcal{D}\xi \exp\left[-\int \mathrm{d}x\mathrm{d}y\xi(x)\alpha(x,\, y)\xi(y) - \int \mathrm{d}xJ(x)\xi(x)\right]$$
$$= \frac{\prod\limits_x \sqrt{\pi}}{\sqrt{\det\left[\alpha(x,\, y)\right]}} \exp\left[\frac{1}{4}\int \mathrm{d}x\mathrm{d}yJ(x)\alpha^{-1}(x,\, y)J(y)\right] , \qquad (6.142)$$

where $\alpha^{-1}(x,\, y)$ denotes the inverse operator matrix of $\alpha(x,\, y)$, defined through

$$\int \mathrm{d}z\alpha^{-1}(x,\, z)\alpha(z,\, y) = \delta(x-y)\,. \qquad (6.143)$$

The source term in the Gaussian allows to compute the functional integral of any moment of the Gaussian through differentiation w.r.t. the sources. Especially,

$$\int \mathcal{D}\xi\xi(x_1)\xi(x_2)\ldots\xi(x_n) \exp\left[-\int \mathrm{d}x\mathrm{d}y\xi(x)\alpha(x,\, y)\xi(y)\right]$$
$$= \int \mathcal{D}\xi \frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_2)}\cdots\frac{\delta}{\delta J(x_n)}$$
$$\exp\left[-\int \mathrm{d}x\mathrm{d}y\xi(x)\alpha(x,\, y)\xi(y) - \int \mathrm{d}xJ(x)\xi(x)\right]\Big|_{J=0}\,. \qquad (6.144)$$

For Grassmann numbers the Gaussian integral reads

$$\int \mathcal{D}\eta \exp\left[-\int \mathrm{d}x\mathrm{d}y\eta(x)\alpha(x,\, y)\eta(y)\right] , \qquad (6.145)$$

where α should be antisymmetric in its arguments, $\alpha(x,\, y) = \alpha(y,\, x)$ in order to yield a non-vanishing result. To gain some insight into how such integrals are evaluated, consider the finite-dimensional case first. There, due to the antisymmetry of α

$$\int \mathrm{d}\eta_1\mathrm{d}\eta_2\, e^{-\eta_1\alpha_{12}\eta_2-\eta_2\alpha_{21}\eta_1} = \int \mathrm{d}\eta_1\mathrm{d}\eta_2\, e^{-2\eta_1\alpha_{12}\eta_2}$$
$$= \int \mathrm{d}\eta_1\mathrm{d}\eta_2\, (1-2\eta_1\alpha_{12}\eta_2) = 2\alpha_{12}\,. \qquad (6.146)$$

The series terminates due to $\eta_1^2 = \eta_2^2 = 0$. For an antisymmetric 2×2 matrix,

$$\alpha_{12}^2 = -\alpha_{12}\alpha_{21} = \det\alpha \qquad (6.147)$$

and therefore

$$\int \mathrm{d}\eta_1 \mathrm{d}\eta_2 \, e^{-2\eta_1\alpha_{12}\eta_2} = 2\sqrt{\det[\alpha]}\,, \tag{6.148}$$

just opposite to the bosonic case. It is easy to check that for higher dimensions

$$\int \left[\prod_{j=1}^{N} \mathrm{d}\eta_j\right] \exp\left[-2\sum_{j=1}^{N-1}\sum_{i=j+1}^{N} \eta_j\alpha_{ji}\eta_i\right] = 2^{\frac{N}{2}}\sqrt{\det[\alpha]}\,. \tag{6.149}$$

This can be simply generalised to infinite-dimensional integrals, yielding

$$\int \mathcal{D}\eta \exp\left[-\int \mathrm{d}x\mathrm{d}y\, \eta(x)\alpha(x,\, y)\eta(y)\right] = \left(\prod_x \sqrt{2}\right)\sqrt{\det[\alpha]}\,. \tag{6.150}$$

As a last extension, consider complex fields. For commuting, ordinary numbers this generalisation is nothing but straightforward. To underline this consider $\xi(x) = [\xi_1(x) + i\xi_2(x)]/\sqrt{2}$ to be such an ordinary complex function with ξ_1 and ξ_2 real-valued. Switching from integration variables ξ^* and ξ to ξ_1 and ξ_2, the Gaussian integral

$$\int \mathcal{D}\xi^*\mathcal{D}\xi\, \exp\left[-\int \mathrm{d}x\mathrm{d}y\, \xi*(x)a(x,\, y)\xi(y)\right] \tag{6.151}$$

with symmetric function $a(x,\, y) = a(y,\, x)$ transforms into two copies of Eq.(6.141). Thus

$$\int \mathcal{D}\xi^*\mathcal{D}\xi\, \exp\left[-\int \mathrm{d}x\mathrm{d}y\, \xi*(x)a(x,\, y)\xi(y)\right] = \frac{(i\pi)^\infty}{\det[a]}\,, \tag{6.152}$$

where the numerator hindsights the infinite normalisation. The Gaussian integral over complex Grassmann numbers is more complicated. To understand this, define the complex anticommuting numbers η

$$\eta = \frac{\chi_1 + i\chi_2}{\sqrt{2}} \quad \text{and} \quad \eta^* = \frac{\chi_1 - i\chi_2}{\sqrt{2}}\,, \tag{6.153}$$

where $\chi_{1,\,2}$ are real-valued Grassmann numbers. Demanding that integration is a linear and translationally invariant operation one has

$$\begin{aligned} 1 &= \int \mathrm{d}\eta\eta = \int \mathrm{d}\eta^*\eta^* \\ 0 &= \int \mathrm{d}\eta = \int \mathrm{d}\eta^* = \int \mathrm{d}\eta\eta^* = \int \mathrm{d}\eta^*\eta\,, \end{aligned} \tag{6.154}$$

translating into

$$\mathrm{d}\eta = \frac{\mathrm{d}\chi_1 - i\mathrm{d}\chi_2}{\sqrt{2}}\,, \quad \mathrm{d}\eta^* = \frac{\mathrm{d}\chi_1 + i\mathrm{d}\chi_2}{\sqrt{2}} \quad \text{and} \quad \mathrm{d}\eta\mathrm{d}\eta^* = -i\mathrm{d}\chi_1\mathrm{d}\chi_2\,. \tag{6.155}$$

Using these integration rules and the expansion of the exponential one easily finds

$$\int \mathrm{d}\eta^*\mathrm{d}\eta\ \exp\left[-\eta^* a\eta\right] = \int \mathrm{d}\eta^*\mathrm{d}\eta\ \left[1 - \eta^* a\eta\right] = a = \det[a]\,, \tag{6.156}$$

where, this time, a is an ordinary commuting number. Adding one more dimension,

$$\begin{aligned}\int \mathrm{d}\eta_1^*\mathrm{d}\eta_2^*\mathrm{d}\eta_1\mathrm{d}\eta_2\ \exp\left[-\eta_i^* a_{ij}\eta_j\right] &= \frac{1}{2!}\int \mathrm{d}\eta_1^*\mathrm{d}\eta_2^*\mathrm{d}\eta_1\mathrm{d}\eta_2\ \left[\eta_i^* a_{ij}\eta_j\right]^2 \\ = \int \mathrm{d}\eta_1^*\mathrm{d}\eta_2^*\mathrm{d}\eta_1\mathrm{d}\eta_2\ \left[\eta_2^*\eta_1^*\eta_2\eta_1\left(a_{11}a_{22} - a_{21}a_{12}\right)\right] &= -\det[a]\,. \end{aligned} \tag{6.157}$$

This looks very good, apart from the sign-difference in front of the determinant. To resolve this one can add yet another integration variable, leading to a 3 instead of a 2×2-matrix a. Then one finds

$$\begin{aligned}&\int \mathrm{d}\eta_1^*\mathrm{d}\eta_2^*\mathrm{d}\eta_3^*\mathrm{d}\eta_1\mathrm{d}\eta_2\mathrm{d}\eta_3\ \exp\left[-\eta_i^* a_{ij}\eta_j\right] \\ &= -\frac{1}{3!}\int\int \mathrm{d}\eta_1^*\mathrm{d}\eta_2^*\mathrm{d}\eta_3^*\mathrm{d}\eta_1\mathrm{d}\eta_2\mathrm{d}\eta_3\ \left[\eta_i^* a_{ij}\eta_j\right]^3 = -\det[a]\,. \end{aligned} \tag{6.158}$$

For the general, N-dimensional case the result reads

$$\int \mathrm{d}\eta_1^*\ldots\mathrm{d}\eta_N^*\mathrm{d}\eta_1\ldots\mathrm{d}\eta_N\ \exp\left[-\eta_i^* a_{ij}\eta_j\right] = (-1)^{\frac{N(N+3)}{2}}\ \det[a]\,. \tag{6.159}$$

Due to this oscillating sign, the limit $N\to\infty$ and the corresponding functional integral are ill-defined. In principle, one could try to bury this sign in some overall normalisation, but this might not always lead to unambiguous definitions of objects derived from path integrals. To get all signs correct there, one has to keep track of all factors -1, possibly an unsurmountable task. The resolution to this apparent problem lies in the fact that the overall sign is defined by the ordering of the integration variables w.r.t. their ordering in the exponential. Since the ordering in the exponential seems to be somewhat more fixed one could therefore try to fiddle around with the ordering of the integration parameters hoping to gain a reasonable result in the end. It can be shown that, in order to fix the overall sign to be positive, the ordering of choice in the integration variables is $d\eta_1^* d\eta_1 d\eta_2^* d\eta_2 \dots d\eta_N^* d\eta_N$. Defining the functional integration in the same fashion, i.e. defining

$$\mathcal{D}\eta^*\mathcal{D}\eta := \prod_x [d\eta^*(x) d\eta(x)] \tag{6.160}$$

one has, for the functional integral,

$$\int \mathcal{D}\eta^*\mathcal{D}\eta \exp\left[-\int dx dy\, \eta^*(x) a(x,\, y) \eta(y)\right] = \det[a]\,. \tag{6.161}$$

In fact, applying the same trick, just with reversed sign, leads to

$$\int \mathcal{D}\eta^*\mathcal{D}\eta \exp\left[+\int dx dy\, \eta^*(x) a(x,\, y) \eta(y)\right] = \det[a]\,, \tag{6.162}$$

for the hyperbolic fermionic integral. To gain this result, however, one has to define

$$\mathcal{D}\eta^*\mathcal{D}\eta := \prod_x [d\eta(x) d\eta^*(x)] \tag{6.163}$$

in contrast to the ordering before.

Transition amplitude for spinor fields

For the construction of the transition amplitude for spinor fields one cannot rely entirely on the experiences gained so far for bosonic fields. First of all, as outlined above, integration proceeds in a different fashion; on top of that, the interpretation through the harmonic oscillator path integral won't work. Nevertheless, one might start by simply copying the structures used for the bosonic case and see what happens. A good guess for the path integral for fermionic fields can be obtained by taking the exponential of the free spinor action, as described by the Dirac equation. Thus, one has for field configurations ψ and ψ_0 at times t and t_0, respectively

$$G[\psi,\, t;\, \psi_0,\, t_0] = \int \mathcal{D}\psi^\dagger \mathcal{D}\psi \exp\left[i \int_{t_0}^{t} dt' \int d^3\vec{x}\, \bar{\psi}(t',\, \vec{x})\, (i\partial\!\!\!/ - m)\, \psi(t',\, \vec{x})\right] \tag{6.164}$$

where the ψ are, as expected, four-component (Dirac) spinors, out of which each component represents an anti-commuting variable. Obviously this makes up for the big difference between the bosonic examples encountered so far and this case: the integration is over the space of anticommuting functions. However, in a first step, it proves to be useful to decouple the spin information by switching to the momentum representation, Eq.(3.75),

$$\psi(t,\,\vec{x}) = \sum_{r=1}^{4}\int \frac{\mathrm{d}^3\vec{p}}{\sqrt{(2\pi)^3(2E_p)}} b(t,\,\vec{p},\,r)w(\vec{p},\,r)e^{i\vec{p}\vec{x}}\,, \tag{6.165}$$

where the w denote the momentum spinors u and v, cf. Eqs.(3.62) and (3.64) and $E_p^2 = \vec{p}^2 + m^2$. Here, also the spin-functions b collectively stand for both functions b and d from Eq.(3.75). Plugging this expansion into the path integral and making use of the spinor identities, one obtains

$$\begin{aligned}&\int \mathrm{d}^3\vec{x}\,\bar{\psi}(t',\,\vec{x})\,(i\not{\partial} - m)\,\psi(t',\,\vec{x})\\ =\;& \sum_{r=1}^{4}\int \mathrm{d}^3\vec{p}\,b^*(t',\,\vec{p},\,r)\,(i\partial_{t'} - \delta_r E_p)\,b(t',\,\vec{p},\,r)\,,\end{aligned} \tag{6.166}$$

where the sign factor $\delta_r = 1$ for $r = 1,\,2$ and $\delta_r = -1$ for $r = 3,\,4$. Under this transformation, the path integral is cast into a manageable form, i.e. factorised into an infinite product of single-particle fermionic path integrals, one for each spin component r. One can therefore write

$$\begin{aligned}&G[b_f,\,t;\,b_i,\,t_0]\\ =\;& \prod_{r=1}^{4}\prod_{\vec{p}}\int \mathcal{D}b^*\mathcal{D}b\exp\left[i\int_{t_0}^{t}\mathrm{d}t' b^*(t',\,\vec{p},\,r)\,(i\partial_{t'} - \delta_r E_p)\,b(t',\,\vec{p},\,r)\right]\,.\end{aligned} \tag{6.167}$$

Concentrating on one mode $\vec{p}$ for positive energy E_p only and denoting it by η one has

$$G[\eta,\,t;\,\eta_0,\,t_0] = \int \mathcal{D}\eta^*(t')\mathcal{D}\eta(t')\exp\left[i\int_{t_0}^{t}\mathrm{d}t'\eta^*(t')\,(i\partial_{t'} - E_p)\,\eta(t')\right] \tag{6.168}$$

Using the definitions above [1], this yields

$$G[\eta,\,t;\,\eta_0,\,t_0] = -\eta + \exp\left[-iE_p(t-t_0)\right]\eta_0\,. \tag{6.176}$$

[1] Alternatively, one could start with the constructive definition of the path integral and divide the integration interval between t_0 and t into $N+1$ equal pieces of size $\epsilon = t_{i+1} - t_i$.

Then one would calculate each of these slices before taking the limit $N \to \infty$. In so doing, one finds

$$\begin{aligned}
&G[\eta,\, t;\, \eta_0,\, t_0] \\
&= \lim_{N\to\infty} \int \mathrm{d}\eta_0^* \prod_{i=1}^{N} (\mathrm{d}\eta_i \mathrm{d}\eta_i^*) \prod_{j=0}^{N} \exp\left[\eta_j^* \left(-\frac{\eta_{j+1}-\eta_j}{\epsilon}\epsilon - i\epsilon E_p \eta_j\right)\right] \\
&= \lim_{N\to\infty} \int \mathrm{d}\eta_0^* \prod_{i=1}^{N} (\mathrm{d}\eta_i \mathrm{d}\eta_i^*) \prod_{j=0}^{N} \left\{1 - \eta_j^* \left[(\eta_{j+1}-\eta_j) + i\epsilon E_p \eta_j\right]\right\} , \qquad (6.169)
\end{aligned}$$

where the ordering of the exponentials is such that earlier time slices are to the right of later slices. The choice of ordering in the differentials is made in accordance with Eq.(6.163) because the overall sign in the exponential is $+$. Using the rules of integration over Grassmann numbers discussed in the previous section, the η_j^* integrals are straightforward and eliminate the term $1-$ from each factor. Therefore,

$$\begin{aligned}
&G[\eta,\, t;\, \eta_0,\, t_0] \\
&= \lim_{N\to\infty} (-1)^{N+1} \int \mathrm{d}\eta_1 \ldots \mathrm{d}\eta_N \left[(\eta - \eta_N) + i\epsilon E_p \eta_N\right] \ldots \left[(\eta_1 - \eta_0) + i\epsilon E_p \eta_0\right] \qquad (6.170)
\end{aligned}$$

The integral over, say, η_N yields

$$\begin{aligned}
&\int \mathrm{d}\eta_N \left[\eta - (1 - i\epsilon E_p)\eta_N\right] \left[\eta_N - (1 - i\epsilon E_p)\eta_{N-1}\right] \\
&= -\eta + (1 - i\epsilon E_p)^2 \eta_{N-1} \,. \qquad (6.171)
\end{aligned}$$

Continuing with the η_{N-1} integration, one has

$$\begin{aligned}
&\int \mathrm{d}\eta_{N-1} \left[-\eta + (1 - i\epsilon E_p)^2 \eta_{N-1}\right] \left[\eta_{N-1} - (1 - i\epsilon E_p)\eta_{N-2}\right] \\
&= \eta - (1 - i\epsilon E_p)^3 \eta_{N-2} \,, \qquad (6.172)
\end{aligned}$$

and so on. Taken together, one finds

$$\begin{aligned}
&\int \mathrm{d}\eta_1 \ldots \mathrm{d}\eta_N \left[(\eta - \eta_N) + i\epsilon E_p \eta_N\right] \ldots \left[(\eta_1 - \eta_0) + i\epsilon E_p \eta_0\right] \\
&= (-1)^N \eta + (-1)^{N+1} (1 - i\epsilon E_p)^{N+1} \eta_0 \,. \qquad (6.173)
\end{aligned}$$

Thus

$$\begin{aligned}
G[\eta,\, t;\, \eta_0,\, t_0] &= \lim_{N\to\infty} \left[-\eta + (1 - i\epsilon E_p)^{N+1} \eta_0\right] \\
&= -\eta + \lim_{N\to\infty} \left[\left(1 - \frac{i(t-t_0)E_p}{N+1}\right)^{N+1} \eta_0\right] . \qquad (6.174)
\end{aligned}$$

Using

$$\lim_{n\to\infty} (1 - x/n)^n = \exp(-x) \qquad (6.175)$$

gives the desired result.

In doing the path integral a specific time-ordering was assumed implicitly, namely $t > t_0$ which could be represented by a step function $\theta(t-t_0)$ attached to the result. For a single negative energy mode, i.e. for $r = 3$ or $r = 4$ one has

$$
\begin{aligned}
G[\xi,\, t;\, \xi_0,\, t_0] &= \int \mathcal{D}\xi^*(t')\mathcal{D}\xi(t') \exp\left[i\int\limits_{t_0}^{t} \mathrm{d}t'\xi^*(t')\left(i\partial_{t'} + E_p\right)\xi(t')\right] \\
&= -\xi + \exp\left[iE_p(t-t_0)\right]\xi_0\,, \qquad (6.177)
\end{aligned}
$$

where again the step function is implicit. Combination of both a positive and a negative mode with the same $\vec{p}$ leaves

$$
G = \eta\xi - \exp\left[-iE_p(t-t_0)\right]\eta_0\xi - \exp\left[iE_p(t-t_0)\right]\eta\xi_0 + \eta_0\xi_0\,. \quad (6.178)
$$

In the limit $i(t-t_0) \to \infty$ one realizes that the term $\exp\left[iE_p(t-t_0)\right]\eta\xi_0$ dominates. This immediately implies that in this limit the dominant state is $\psi_0(\xi,\, \eta) = \eta$, in other words, in the vacuum state the positive energy mode is empty and only the negative energy mode is occupied. This is in accordance to what has been found out before, doing the second quantisation: the vacuum of a fermionic field theory is described by a filled Dirac sea of negative energy modes.

Adding sources

In order to construct a generating functional, sources have to be added to the vacuum-to-vacuum amplitude. It should be easy to accept that such a fermionic generating functional is given by

$$
Z_0[\bar{\chi},\, \chi] = \int \mathcal{D}\psi^*(x)\mathcal{D}\psi(x) \exp\left[i\int \mathrm{d}^4x \left(\bar{\psi}(i\partial\!\!\!/ - m)\psi + \bar{\chi}\psi + \bar{\psi}\chi\right)\right] (6.179)
$$

where the sources are χ and $\bar{\chi}$. They can be Fourier-transformed, much in the same fashion as the spinors before,

$$
\chi(t\,\vec{x}) = \sum_{r=1}^{4}\int \frac{\mathrm{d}^3\vec{p}}{\sqrt{(2\pi)^3 2E_p}}\chi(t,\, \vec{p},\, r)w(\vec{p},\, r)e^{i\vec{p}\cdot\vec{x}}\,. \quad (6.180)
$$

Then, in terms of the Fourier transformed quantities, the generating functional reads

$$
Z_0[\chi^*,\, \chi] = \int \mathcal{D}b^*\mathcal{D}b \exp\left\{i\int\limits_{-\infty}^{\infty}\mathrm{d}t' \int \mathrm{d}^3\vec{p}\left[b^*(t',\, \vec{p},\, r)\left(i\partial_{t'} - \epsilon_r E_p\right)b(t',\, \vec{p},\, r)\right.\right.
$$

$$+\frac{i\epsilon_r}{2E_p}\left(\chi^*(t',\,\vec{p},\,r)b(t',\,\vec{p},\,r)+b^*(t',\,\vec{p},\,r)\chi(t',\,\vec{p},\,r)\right)\Bigg]\Bigg\}$$

(6.181)

Concentrating on a single, positive energy mode $\vec{p}$, one has

$$\begin{aligned}
&G[\zeta^*,\,\zeta]\\
&= \int \mathcal{D}\eta^*\mathcal{D}\eta \exp\left\{i\int\limits_{-\infty}^{\infty}\mathrm{d}t'\left[\eta^*(t')\left(i\partial_{t'}-E_p\right)\eta(t)+\frac{1}{2E_p}\left(\zeta^*\eta+\eta^*\zeta\right)\right]\right\}
\end{aligned} \tag{6.182}$$

One could try to follow the same procedure as in the case without sources; this, however, leads to a rather involved and intransparent calculation. Instead, one may use the fact that the fermionic integration is designed to be translationally invariant, i.e. that

$$\int \mathrm{d}\eta = \int \mathrm{d}(\eta+\zeta) = 0 \quad \text{and} \quad \int \mathrm{d}\eta(\eta+\zeta) = \int \mathrm{d}(\eta+\zeta)(\eta+\zeta) = 1\,. \tag{6.183}$$

After Fourier transforming sources and fields w.r.t. time, e.g.

$$\eta(t) = \int \frac{\mathrm{d}\omega}{\sqrt{2\pi}}\,e^{-i\omega t}\eta\omega \tag{6.184}$$

one has

$$\begin{aligned}
&G[\zeta^*,\,\zeta]\\
&= \int \mathcal{D}\eta^*\mathcal{D}\eta \exp\Bigg\{i\int \mathrm{d}\omega\,\Bigg[\eta^*(\omega)\left(\omega-E_p\right)\eta(\omega)\\
&\qquad\qquad +\frac{1}{2E_p}\left(\zeta^*(\omega)\eta(\omega)+\eta^*(\omega)\zeta(\omega)\right)\Bigg]\Bigg\}\\
&= \int \mathcal{D}\eta^*\mathcal{D}\eta \exp\Bigg\{-i\int \mathrm{d}\omega\,\Bigg[\left(\eta^*+\frac{1}{\sqrt{2E_p}}\frac{1}{\omega-E_p}\zeta^*\right)\\
&\qquad\qquad (\omega-E_p)\left(\eta+\frac{1}{\sqrt{2E_p}}\frac{1}{\omega-E_p}\zeta\right)
\end{aligned}$$

$$
-\frac{1}{(2E_p)^2}\zeta^*(\omega)\frac{1}{\omega-E_p}\zeta(\omega)\Big]\Big]\Big\}
$$

$$
= \int \mathcal{D}\eta^*\mathcal{D}\eta \exp\left\{\frac{i}{(2E_p)^2}\int \mathrm{d}\omega\,\left[\zeta^*(\omega)\frac{1}{\omega-E_p}\zeta(\omega)\right]\right\}\cdot G[\zeta^*=0,\,\zeta=0]\,, \tag{6.185}
$$

where, in order to gain this result, some squares have been completed.

Fixing the poles

To cast this result back into a form where the sources depend on time instead of energy, i.e. for a Fourier back-transformation, a prescription is needed for handling the poles at $\omega = E_p$. Put in other words, boundary conditions for the Green's function $g_-(\bar{t},\,t')$ have to be specified - they have been lost in the procedure above because there the limits $t,\,t_0 \to \pm\infty$ have been taken for the initial and final times. In the bosonic case, the boundary conditions came into play in quite a natural fashion, namely as convergence factor $i\epsilon$ being added to the exponential. However, for the fermionic case, no such thing is in sight, the integration here seems to converge anyway.
So one has to side step this and one may hope for some physics to resolve this. Recalling that the positive energy modes are supposed to exclusively propagate forward in time, one has

$$
g_-(t,\,t') = -i\theta(t-t')e^{-iE_p(t-t')} \tag{6.186}
$$

for the Green's function, fixing the poles. Without any memory about things that have been discussed before, one has to find another solution. To do so, one may go back a step and consider the Green's function above as calculated through the path integral and take the limit only then. To do so, consider

$$
G_2[\eta,\,t;\,\eta_0,\,t_0] = \int \mathcal{D}\eta^*\mathcal{D}\eta\,\eta^*(t_1)\eta(t_2)\exp\left[i\int\limits_{t_0}^{t}\mathrm{d}t'\,\eta^*(t')\left(i\partial_{t'}-E_p\right)\eta(t')\right] \tag{6.187}
$$

and compute it directly. Following the same steps as before, divide the interval $[t_0,\,t]$ into $N+1$ equal pieces, such that $t_{1,\,2}$ falls on the Jth and Kth slice, respectively. Then

$$
G_2[\eta,\,t;\,\eta_0,\,t_0]
$$

$$
\begin{aligned}
&= \lim_{N\to\infty} \int \mathrm{d}\eta_0^* \prod_{i=1}^{N} (\mathrm{d}\eta_i \mathrm{d}\eta_i^*) \eta_J^* \eta_K \prod_{j=0}^{N} \exp\left[-\eta_j^* \left(\eta_{j+1} - (1 - i\epsilon E_p)\eta_j\right)\right] \\
&= \lim_{N\to\infty} \int \mathrm{d}\eta_0^* \prod_{i=1}^{N} (\mathrm{d}\eta_i \mathrm{d}\eta_i^*) \eta_J^* \eta_K \prod_{j=0}^{N} \left[1 - \eta_j^* \left(\eta_{j+1} - (1 - i\epsilon E_p)\eta_j\right)\right] \\
&= \lim_{N\to\infty} (-1)^J \int \mathrm{d}\eta_1 \ldots \mathrm{d}\eta_N \, \eta_K \left[(\eta - (1 - i\epsilon E_p)\eta_N) \ldots \right. \\
&\quad (\eta_{J+2} - (1 - i\epsilon E_p)\eta_{J+1}) \cdot 1 \cdot (\eta_J - (1 - i\epsilon E_p)\eta_{J-1}) \ldots \\
&\quad \left. (\eta_1 - (1 - i\epsilon E_p)\eta_0) \right] . \qquad (6.188)
\end{aligned}
$$

The integration over the η^* has been simple. For all $j \neq J$, each integral replaces "its" factor $1 - \eta_j^* \left(\eta_{j+1} - (1 - i\epsilon E_p)\eta_j\right)$ by $-\eta_{j+1} - (1 - i\epsilon E_p)\eta_j$, for $j = J$, in contrast, the factor is replaced by 1, since $\eta_J^* \eta_J^* = 0$. To continue, a time ordering has to be specified. To begin with, consider $K > J$, or, $t_2 > t_1$. Then the first $N - K + 1$ integrals are identical to what has been done before, Eq.(6.173), and one finds

$$
\begin{aligned}
G_2 &= \lim_{N\to\infty} (-1)^J (-1)^{2(N-K)+1} \int \mathrm{d}\eta_1 \ldots \mathrm{d}\eta_K \\
&\quad \left[\left(\eta - (1 - i\epsilon E_p)^{N-K+1} \eta_K \right) \eta_K \left(\eta_K - (1 - i\epsilon E_p)\eta_{K-1}\right) \ldots \right. \\
&\quad (\eta_{J+2} - (1 - i\epsilon E_p)\eta_{J+1}) \cdot 1 \cdot (\eta_J - (1 - i\epsilon E_p)\eta_{J-1}) \ldots \\
&\quad \left. (\eta_1 - (1 - i\epsilon E_p)\eta_0) \right] . \qquad (6.189)
\end{aligned}
$$

The integral over η_K picks out the factor proportional to η, and leaving a factor of η_{K+1} out in front of the remaining product,

$$
\begin{aligned}
G_2 &= \lim_{N\to\infty} (-1)^J (-1)^{2(N-K)+1} \int \mathrm{d}\eta_1 \ldots \mathrm{d}\eta_{K-1} \\
&\quad \left[\eta (1 - i\epsilon E_p) \eta_{K-1} \left(\eta_{K-1} - (1 - i\epsilon E_p)\eta_{K-2}\right) \ldots \right. \\
&\quad (\eta_{J+2} - (1 - i\epsilon E_p)\eta_{J+1}) \cdot 1 \cdot (\eta_J - (1 - i\epsilon E_p)\eta_{J-1}) \ldots \\
&\quad \left. (\eta_1 - (1 - i\epsilon E_p)\eta_0) \right] . \qquad (6.190)
\end{aligned}
$$

Now, each integral over η_i with $i > J + 1$ just results in another factor of

$(-1)^2(1-i\epsilon E_p)$, and the integral over η_{J+1} just yields a 1. Hence,

$$G_2 = \lim_{N\to\infty}(-1)^J(-1)^{2(N-K)+1}(-1)^{2(K-J-1)}\int \mathrm{d}\eta_1\ldots\mathrm{d}\eta_J$$
$$\left[\eta(1-i\epsilon E_p)^{K-J-1}\left(\eta_J-(1-i\epsilon E_p)\eta_{J-1}\right)\ldots\left(\eta_1-(1-i\epsilon E_p)\eta_0\right)\right]. \tag{6.191}$$

The remaining integrals just contribute factors of (-1) each. Therefore, the total result for $K>J$ is

$$G_2^{K>J} = \lim_{N\to\infty} -\eta(1-i\epsilon E_p)^{K-J-1} = -\theta(t_2-t_1)e^{-iE_p(t_2-t_1)}\,\eta\,, \tag{6.192}$$

whereas for $J>K$ the result, obtained by the same methods, reads

$$G_2^{J>K} = \lim_{N\to\infty}(1-i\epsilon E_p)^{N-(J-K-1)}\eta_0 = \theta(t_1-t_2)e^{-iE_p(t-t_0)}e^{-iE_p(t_2-t_1)}\,\eta_0\,. \tag{6.193}$$

Combining both, one finds

$$G_2[\eta,\,t;\,\eta_0,\,t_0] = e^{-iE_p(t_2-t_1)}\left[-\eta\theta(t_2-t_1)+\eta_0\theta(t_1-t_2)e^{-iE_p(t-t_0)}\right] \tag{6.194}$$

and, taking the limit $t,\,t_0\to\pm\infty$ the factor proportional to η_0 vanishes leaving Eq.(6.192). Recalling that

$$\langle\eta^*(t_1)\eta(t_2)\rangle \sim (-i)^2\frac{1}{(2E_p)^2}\frac{\delta}{\delta\zeta(t_1)}\frac{\delta}{\delta\zeta^*(t_2)}\,G[\zeta^*,\,\zeta] \tag{6.195}$$

and comparing with the results above allows to fix the time-dependent operators between the sources. For positive and negative energy modes one therefore has

$$G^{\pm}[\zeta^*,\,\zeta] = \exp\left[-\frac{i}{(2E_p)^2}\int\limits_{-\infty}^{\infty}\mathrm{d}t\mathrm{d}t'\zeta^*(t)g_{\mp}(t,\,t')\zeta(t')\right], \tag{6.196}$$

where, see above,

$$\begin{aligned} g_-(t,\,t') &= \theta(t-t')e^{-iE_p(t-t')}\\ g_+(t,\,t') &= \theta(t'-t)e^{iE_p(t-t')}\,. \end{aligned} \tag{6.197}$$

Putting things together, one finds

$$Z_0[\chi^*,\, \chi] = \\ Z_0[0,\, 0] \prod_{\vec{p}} \prod_{r=1}^{4} \exp\left[-\frac{i}{(2E_p)^2} \int_{-\infty}^{\infty} \mathrm{d}t\mathrm{d}t' \chi^*(t,\, \vec{p},\, r) g(t,\, t';\, r) \chi(t',\, \vec{p},\, r) \right] , \tag{6.198}$$

where $g(t,\, t';\, r)$ equals $g_\pm$ for the corresponding modes labeled by r. This is to be transformed back from momentum to position space. Using the ortho-normality of the basic w spinors,

$$\epsilon_r \bar{w}(\vec{p},\, r) w(\vec{p},\, r') = \delta_{rr'} \tag{6.199}$$

allows to rewrite the equation above as

$$\begin{aligned} Z_0[\chi^*,\, \chi] = \exp \Bigg\{ & \int_{-\infty}^{\infty} \mathrm{d}t\mathrm{d}t' \int \mathrm{d}^3\vec{p} \frac{i}{(2E_p)^2} \\ & \sum_{r,\, r'=1}^{2} \left[\chi^*(t,\, \vec{p},\, r) \bar{w}(\vec{p},\, r) g(t,\, t';\, r) w(\vec{p},\, r') \chi(t',\, \vec{p},\, r') \right] \\ & - \sum_{r,\, r'=3}^{4} \left[\chi^*(t,\, \vec{p},\, r) \bar{w}(\vec{p},\, r) g(t,\, t';\, r) w(\vec{p},\, r') \chi(t',\, \vec{p},\, r') \right] \Bigg\} \, Z_0[0,\, 0] \, . \end{aligned} \tag{6.200}$$

Next, one makes use of the Dirac equation, the equation of motion for the w-spinors,

$$(\not{p} - \epsilon_r m)\, w(\vec{p},\, r) = 0 \tag{6.201}$$

to suitably insert $(\not{p} \pm m)$,

$$\begin{aligned} \sum_{r,\, r'=1}^{2} \bar{w}(\vec{p},\, r) w(\vec{p},\, r') &= \sum_{r,\, r'=1}^{2} \bar{w}(\vec{p},\, r) (\not{p} + m) w(\vec{p},\, r') \\ \sum_{r,\, r'=3}^{4} \bar{w}(\vec{p},\, r) w(\vec{p},\, r') &= \sum_{r,\, r'=3}^{4} \bar{w}(\vec{p},\, r) (\not{p} - m) w(\vec{p},\, r') \, . \end{aligned} \tag{6.202}$$

Then

$$
\begin{aligned}
Z_0[\chi^*,\,\chi] = \exp\Bigg\{ & \int \mathrm{d}^4x\mathrm{d}^4x' \int \mathrm{d}^3\vec{p}\frac{i}{(2E_p)^2}\sum_{r,\,r'=1}^{4} \\
& \Bigg[\chi^*(t,\,\vec{p},\,r)\bar{w}(\vec{p},\,r)e^{-i\vec{p}\cdot\vec{x}} \\
& \Big(\theta(t-t')e^{-ip\cdot(x-x')}(\not{p}+m) - \theta(t'-t)e^{ip\cdot(x-x')}(\not{p}-m)\Big) \\
& e^{i\vec{p}\cdot\vec{x}'}w(\vec{p},\,r')\chi(t',\,\vec{p},\,r')\Bigg]\Bigg\}\, Z_0[0,\,0] \\
= \;\; & \exp\left\{-\int \mathrm{d}^4x\mathrm{d}^4x'\ [\bar{\chi}(x)S_F(x-x')\chi(x')]\right\} Z_0[0,\,0]\,,
\end{aligned}
\tag{6.203}
$$

where $S_F(x-x')$ is the free spinor field propagator of Eq.(3.99).

6.3 Interacting field theories

So far the focus of the discussion was on the calculation of vacuum-to-vacuum amplitudes of free field theories. Such amplitudes act, as has been demonstrated, as generating functionals for vacuum expectation values of time-ordered products of free fields. It was also shown that for bosonic fields the path integral representation factorised into infinite products of harmonic oscillators for the various polarisation modes of the fields.
For interacting fields, however, it was shown that vacuum expectation values of time-ordered products can be expanded as a perturbative series, where the first term was the corresponding free field result. The question addressed in this section is how one can construct higher order terms of such a perturbative expansion from the path integral formulation. The objects of interest there are, as mentioned before, time-ordered products of the fields, the Green's functions. From the considerations in chapter 4 it is known that a reduction formula is employed to break n-point functions down into Green's functions with fewer legs and to construct S-matrix elements accordingly. In the following, using two examples, namely $\lambda\phi^4$ theory and QED, such n-point functions will be constructed from the path integral.

6.3.1 $\lambda\phi^4$ theory

Basic trick

In a first step, define the path integral for interacting fields by replacing the free field action with the interacting field action in the exponential. This leads to

$$Z[J] = \int \mathcal{D}\phi \exp\left\{i\int \mathrm{d}^4x \left[\frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m_0^2\phi^2\right) - \frac{\lambda_0}{4!}\phi^4 + J(x)\phi(x)\right]\right\}. \tag{6.204}$$

It is expected that this is the generating functional for the vacuum expectation values of time-ordered products of the fields. At first sight, this sounds good but it has to be taken with a grain of salt: $Z[J]$ cannot be calculated in closed form. Even more, due to the quartic potential it does definitely not resemble anything that can be calculated - only Gaussian integrals can be treated in functional integration. So the question is how to proceed. If λ_0 is "small", one can expect to do a perturbative expansion in λ_0. Extracting it, one finds

$$\begin{aligned}
Z[J] &= \int \mathcal{D}\phi \exp\left\{-i\int \mathrm{d}^4x \left[\frac{\lambda_0}{4!}\phi^4\right]\right\} \\
&\qquad \exp\left\{i\int \mathrm{d}^4x \left[\frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m_0^2\phi^2\right) + J(x)\phi(x)\right]\right\}. \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{-i\lambda_0}{4!}\right)^n \int \mathcal{D}\phi \left[\int \mathrm{d}^4x\,\phi^4\right]^n \\
&\qquad \exp\left\{i\int \mathrm{d}^4x \left[\frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m_0^2\phi^2\right) + J(x)\phi(x)\right]\right\}.
\end{aligned} \tag{6.205}$$

The key step in simplifying this expression is to notice that fields occuring can be replaced by differentiation w.r.t. the source J. For the quartic term this translates to

$$\left[\int \mathrm{d}^4x\,\phi^4(x)\right]^n \longrightarrow \left[\int \mathrm{d}^4x \left(\frac{-i\delta}{\delta J(x)}\right)^4\right]^n. \tag{6.206}$$

These functional derivatives can be brought outside the functional integral,

$$Z[J] = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{-i\lambda_0}{4!}\right)^n \left[\int \mathrm{d}^4x \left(\frac{-i\delta}{\delta J(x)}\right)^4\right]^n$$

$$
\begin{aligned}
& \int \mathcal{D}\phi \, \exp\left\{ i \int \mathrm{d}^4x \, \left[\frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m_0^2\phi^2\right) + J(x)\phi(x) \right] \right\} . \\
= \; & \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{-i\lambda_0}{4!} \right)^n \left[\int \mathrm{d}^4x \, \left(\frac{-i\delta}{\delta J(x)} \right)^4 \right]^n Z_0[J] \\
= \; & Z_0[0] \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{-i\lambda_0}{4!} \right)^n \left[\int \mathrm{d}^4x \, \left(\frac{-i\delta}{\delta J(x)} \right)^4 \right]^n \\
& \exp\left\{ -\frac{i}{2} \int \mathrm{d}^4x \mathrm{d}^4x' \, [J(x)\Delta_F(x-x')J(x')] \right\} , \qquad (6.207)
\end{aligned}
$$

making the computation of vacuum expectation values of time-ordered products in terms of a perturbative series incredibly simple: just functionally differentiate. In other words,

$$
\langle | T\phi(x_1)\phi(x_2)\ldots\phi(x_N) | \, 0 \rangle = \frac{-i\delta}{\delta J(x_1)} \frac{-i\delta}{\delta J(x_2)} \cdots \frac{-i\delta}{\delta J(x_N)} \left. \frac{Z[J]}{Z[0]} \right|_{J=0} . \qquad (6.208)
$$

Example: Two-point function to first order

To exemplify these findings consider the two-point function. Writing

$$
\langle | T\phi(x_1)\phi(x_2) | \, 0 \rangle = G_2^{(0)}(x_1,\, x_2) + \lambda_0 G_2^{(1)}(x_1,\, x_2) + \lambda_0^2 G_2^{(2)}(x_1,\, x_2) + \ldots \qquad (6.209)
$$

one has

$$
\begin{aligned}
G_2^{(0)}(x_1,\, x_2) = \; & \frac{-i\delta}{\delta J(x_1)} \frac{-i\delta}{\delta J(x_2)} \left. \frac{Z[J]}{Z[0]} \right|_{J=0} \\
= \; & \frac{-i\delta}{\delta J(x_1)} \frac{-i\delta}{\delta J(x_2)} \exp\left\{ -\frac{i}{2} \int \mathrm{d}^4x \mathrm{d}^4x' \, [J(x)\Delta_F(x-x')J(x')] \right\} \Bigg|_{J=0} \\
= \; & i\Delta_F(x_1 - x_2) \, . \qquad (6.210)
\end{aligned}
$$

This is, not surprisingly, the free field result. Going to first order, one finds

$$
\begin{aligned}
G_2^{(1)}(x_1,\, x_2) \; = \; & \frac{-i\delta}{\delta J(x_1)} \frac{-i\delta}{\delta J(x_2)} \left[-i \int \mathrm{d}^4y \, \left(\frac{-i\delta}{\delta J(y)} \right)^4 \right] \\
& \exp\left\{ -\frac{i}{2} \int \mathrm{d}^4x \mathrm{d}^4x' \, [J(x)\Delta_F(x-x')J(x')] \right\} \Bigg|_{J=0} \\
= \; & -\frac{1}{2} \int \mathrm{d}^4y \Delta_F(x_1 - y)\Delta_F(y-y)\Delta_F(y-x_2) \, , \qquad (6.211)
\end{aligned}
$$

which is nothing but the “tadpole” diagram encountered before.

Generating functional for connected diagrams

$Z[J]$, as in the free field case, generates all diagrams, connected and disconnected ones. Similarly to the free field case, one can define another functional $W[J]$ such that it generates the connected diagrams only. In fact the respective definition is identical,

$$Z[J] = Z[0]\,\exp(iW[J])\,. \tag{6.212}$$

In terms of the free field result,

$$W_0[J] = -\frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4x'\, J(x)\Delta_F(x-x')J(x') \tag{6.213}$$

one has for interacting fields

$$\exp(iW[J]) \;=\; \frac{1}{Z[0]}\exp\left[-\frac{i\lambda_0}{4!}\int \mathrm{d}^4y\left(\frac{-i\delta}{\delta J(y)}\right)^4\right]\exp\left[iW_0[J]\right] \tag{6.214}$$

6.3.2 QED

Vacuum-to-vacuum amplitude

The methods applied to the case of $\lambda\phi^4$ theory can be equally applied to spinor quantum electrodynamics, QED. In this section, QED in the Coulomb gauge will be discussed. In previous sections it was noted that working in the Coulomb gauge translates into coupling the source to the gauge-invariant part of the vector potential only. Under an unimportant redefinition of the normalisation in front of $Z_0[J]$ this was equivalent to adding $\delta[\vec{A}_L]\delta[A_0]$ into the free field functional integral, cf. Eq.(6.116). Intuitively this is very appealing, since in the absence of charges classical electrodynamic fields satisfy $\vec{A}_L = A_0 = 0$ in coulomb gauge. In the presence of charges, however, life is not that simple. Still, $\vec{A}_L$, the longitudinal component of the vector potential can be fixed to be zero; the zero component, A_0, no longer satisfies $A_0 = 0$ due to Gauss' law. Thus, in the path integral for interacting fields one has to also couple A_0 to the source. Equivalently, one only keeps the term $\delta[\vec{A}_L]$. Recalling the action

$$\mathcal{S} = \int \mathrm{d}^4x\left[\frac{1}{2}\left(\vec{E}^2-\vec{B}^2\right)+\bar{\psi}\left(i\not{D}-m_0\right)\psi\right]\,, \tag{6.215}$$

where $D^\mu = \partial^\mu + ie_0 A^\mu$ is the covariant derivative one has for the vacuum-to-vacuum amplitude

$$Z[J,\, \bar{\chi},\, \chi] = \mathcal{N} \int \mathcal{D}A^\mu \mathcal{D}\bar{\psi} \mathcal{D}\psi \, \delta[\vec{A}_L] \\ \exp\left\{ i \int \mathrm{d}^4x \left[\frac{1}{2}\left(\vec{E}^2 - \vec{B}^2\right) + \bar{\psi}\,(i\not{D} - m_0)\,\psi + J_\mu A^\mu + \bar{\chi}\psi + \bar{\psi}\chi \right] \right\} , \tag{6.216}$$

where

$$\vec{E} = -\vec{\nabla} A_0 - \partial_\perp \vec{A}_\perp \,. \tag{6.217}$$

Breaking up the covariant derivative and repeating the same trick of Eq.(6.206), one finds

$$Z[J,\, \bar{\chi},\, \chi] = \mathcal{N} \exp\left\{ -ie_0 \int \mathrm{d}^4y \frac{\delta}{\delta J^\mu(y)} \frac{\delta}{\delta \chi(y)} \frac{\delta}{\delta \bar{\chi}(y)} \right\} \int \mathcal{D}A^\mu \mathcal{D}\bar{\psi} \mathcal{D}\psi \, \delta[\vec{A}_L] \\ \exp\left\{ i \int \mathrm{d}^4x \left[\frac{1}{2}\left(\vec{E}^2 - \vec{B}^2\right) + \bar{\psi}\,(i\not{\partial} - m_0)\,\psi + J_\mu A^\mu + \bar{\chi}\psi + \bar{\psi}\chi \right] \right\} , \tag{6.218}$$

This expression includes the free path integral for spinor fields, with the solution given by Eq.(6.203). Using this result, i.e. doing the integrals over $\mathcal{D}\bar{\psi}\mathcal{D}\psi$ yields

$$\begin{aligned} Z[J,\, \bar{\chi},\, \chi] \;&=\; \mathcal{N}' \exp\left\{ -ie_0 \int \mathrm{d}^4y \frac{\delta}{\delta J^\mu(y)} \frac{\delta}{\delta \chi(y)} \frac{\delta}{\delta \bar{\chi}(y)} \right\} \\ &\quad \exp\left\{ - \int \mathrm{d}^4x \mathrm{d}^4x' \; [\bar{\chi}(x) S_F(x - x') \chi(x')] \right\} \\ &\quad \int \mathcal{D}A^\mu \, \delta[\vec{A}_L] \exp\left\{ i \int \mathrm{d}^4x \left[\frac{1}{2}\left(\vec{E}^2 - \vec{B}^2\right) + J_\mu A^\mu \right] \right\} , \end{aligned} \tag{6.219}$$

The $\delta[\vec{A}_L]$ term guarantees that

$$\vec{\nabla} \cdot \vec{A} = 0 \,, \tag{6.220}$$

and that therefore the integral over A_0 decouples from the integral over $\vec{A}_\perp$, the physically sensible transversal components of the vector potential. The only coupling between these components then is in the $\vec{E}^2$-term. There we have

$$
\begin{aligned}
\int \mathrm{d}^4x\vec{E}^2 &= \int \mathrm{d}^4x \left(\vec{\nabla}A_0 + \partial_t\vec{A}_\perp\right)^2 \\
&= \int \mathrm{d}^4x \left[\left(\vec{\nabla}A_0\right)^2 + 2\left(\vec{\nabla}A_0\right)\cdot\left(\partial_t\vec{A}_\perp\right) + \left(\partial_t\vec{A}_\perp\right)^2\right] \\
&= \int \mathrm{d}^4x \left[\left(\vec{\nabla}A_0\right)^2 - 2A_0\partial_t\left(\vec{\nabla}\vec{A}_\perp\right) + \left(\partial_t\vec{A}_\perp\right)^2\right] \\
&= \int \mathrm{d}^4x \left[\left(\vec{\nabla}A_0\right)^2 + \left(\partial_t\vec{A}_\perp\right)^2\right], \qquad (6.221)
\end{aligned}
$$

and also there the two components decouple. This allows to rewrite the remaining integral as a product of two integrals, namely

$$
\begin{aligned}
&\int \mathcal{D}A^\mu\,\delta[\vec{A}_L]\exp\left\{i\int \mathrm{d}^4x\left[\frac{1}{2}\left(\vec{E}^2-\vec{B}^2\right)+J_\mu A^\mu\right]\right\} \\
&= \int \mathcal{D}A^0\exp\left\{i\int \mathrm{d}^4x\left[-\frac{1}{2}\left(A^0\nabla^2A^0\right)+J_0A^0\right]\right\} \\
&\quad\times\int \mathcal{D}\vec{A}_\perp\exp\left\{i\int \mathrm{d}^4x\left[\frac{1}{2}\left(\vec{E}_\perp^2-\vec{B}_\perp^2\right)+\vec{J}_\perp\cdot\vec{A}_\perp\right]\right\}. \qquad (6.222)
\end{aligned}
$$

The second term is, not surprisingly, identical to the one that has been computed previously, Eq.(6.117), the free photon propagator in Coulomb gauge. To compute the first term, some by now familiar and straightforward manipulations are to be done: Fourier transformation and completion of the squares. Then the vacuum-to-vacuum amplitude reads

$$
\begin{aligned}
Z[J,\,\bar{\chi},\,\chi] = Z[0,\,0,\,0]\exp&\left\{-ie_0\gamma^\mu\int \mathrm{d}^4y\frac{\delta}{\delta J^\mu(y)}\frac{\delta}{\delta\chi(y)}\frac{\delta}{\delta\bar{\chi}(y)}\right\} \\
\exp\Bigg\{-\int \mathrm{d}^4x\int \mathrm{d}^4x'&\Big[\bar{\chi}(x)S_F(x-x')\chi(x')+ \\
&\frac{i}{2}J^j(x)D^{\text{tr.}}_{jk}(x-x')J^k(x')+\frac{i}{2}J^0(x)\Delta_0(x-x')J^0(x')\Big]\Bigg\}
\end{aligned}
\qquad (6.223)
$$

with

$$
\Delta_0(x-x') = \int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{e^{-ik\cdot(x-x')}}{\left|\vec{k}\right|^2} = \frac{\delta(t-t')}{4\pi\left|\vec{x}-\vec{x}'\right|}\,. \qquad (6.224)
$$

Of course, since A_0 couples to $J_0 = \gamma^0\bar{\psi}\psi \equiv \rho$, the charge density, this is nothing but the instantaneous Coulomb energy. The full result, however, is not exactly in the form one would like it to be - this Coulomb energy piece is a bit disturbing, and the transversal propagator still contains some non-covariant pieces. In the following this will be taken care of.
Recall that the transversal propagator can be cast into a form which, separates the covariant and the non-covariant pieces by introducing and using the coordinates of the orthonormal frame of quantisation. The result then reads, cf. Eq.(3.174),

$$D^{\rm tr.}_{\mu\nu}(x,\,y) = \int\frac{{\rm d}^4k}{(2\pi)^4}\left[\frac{e^{-ik\cdot(x-y)}}{k^2+i\epsilon}\left(-g_{\mu\nu}-\frac{k^2n_\mu n_\nu+(k\cdot n)(k_\mu n_\nu+k_\nu n_\mu)k_\mu k_\nu}{(k\cdot n)^2-k^2}\right)\right]\,, \tag{6.225}$$

where n^μ is a time-like vector with unit length which, in the frame of quantisation, is $n^\mu = (1,\,\vec{0})$. The result of the A_0-integration is written in this frame. Going to the more general case with free n^μ one finds

$$J^0(x)\Delta_0(x-x')J^0(x') = J^\mu(x)n_\mu\Delta_0(x-x')n_\nu J^\nu(x')\,. \tag{6.226}$$

However, in the frame of quantisation,

$$(k\cdot n)^2 = k_0^2 \implies (k\cdot n)^2 - k^2 = \left|\vec{k}\right|^2\,. \tag{6.227}$$

Then, in the limit $\epsilon \to 0^+$, the $n_\mu n_\nu$ term in $D^{\rm tr.}_{\mu\nu}$ becomes

$$-\int\frac{{\rm d}^4k}{(2\pi)^4}\frac{n_\mu n_\nu e^{-ik\cdot(x-x')}}{\left|\vec{k}\right|^2} = -\Delta_0(x-x')n_\mu n_\nu\,. \tag{6.228}$$

In other words, the A_0 integration just leads to a term that cancels the $n_\mu n_\nu$ term in $D^{\rm tr.}_{\mu\nu}$. Performing this cancellation and defining

$$\bar{D}^{\rm tr.}_{\mu\nu}(x,\,y) = \int\frac{{\rm d}^4k}{(2\pi)^4}\left[\frac{e^{-ik\cdot(x-y)}}{k^2+i\epsilon}\left(-g_{\mu\nu}+\frac{(k\cdot n)(k_\mu n_\nu+k_\nu n_\mu)k_\mu k_\nu}{(k\cdot n)^2-k^2}\right)\right]\,, \tag{6.229}$$

one finally has

$$Z[J,\,\bar{\chi},\,\chi] = Z[0,\,0,\,0]\exp\left\{-ie_0\gamma^\mu\int \mathrm{d}^4y\frac{\delta}{\delta J^\mu(y)}\frac{\delta}{\delta\chi(y)}\frac{\delta}{\delta\bar{\chi}(y)}\right\}$$
$$\exp\left\{-\int\mathrm{d}^4x\int\mathrm{d}^4x'\left[\bar{\chi}(x)S_F(x-x')\chi(x')+\frac{i}{2}J^\mu(x)\bar{D}^{\mathrm{tr.}}_{\mu\nu}(x-x')J^\nu(x')\right]\right\} \tag{6.230}$$

6.4 Yang-Mills and Fadeev-Popov

As already noted a few times, special difficulties in quantising a field theory arise when internal local symmetries such as gauge symmetries are present in the classical action already. This is true also for the path integral representation of such theories. There, the path integral is even more divergent than usual. The problem in such cases is the following: suppose $A^\mu(x)$ is the generic classical gauge dependent field, where the action $\mathcal{S}[A^\mu(x)]$ is gauge invariant. Naively, one would write for the path integral $Z[J] = \int\mathcal{D}A^\mu\exp[i\mathcal{S}[A^\mu]+J\cdot A]$. This, however, means that one is integrating over all degrees of freedom of the gauge dependent field and, hence, over all configurations of this field. The problem in so doing is that there are subsets of equivalent configurations, each of which may contain an infinite number of field configurations that are related through gauge transformations and, thus, lead to the same integrand in the path integral. Phrased in other words: when doing the integral over $A^\mu(x)$ one integrates over both gauge invariant and gauge dependent components, the latter ones playing no role for the value of the integrand. For them, the integrand basically equals unity and so, the integral diverges. This is the divergence one needs to isolate to kick it out of the game.

One solution, cheap but workable, to this was already presented for the free photon field. There one could play with the sources and couple them to the gauge invariant part of the vector potential only. The integral over the gauge dependent components could then be thrown into some overall normalisation and forgotten, since it would cancel in the ratios. There are three reason why this is not the ultima ratio: first of all, free photon fields, i.e. QED, has a simple gauge structure, such that it is quite easy to identify gauge dependent and invariant pieces. Second, the method used there automatically leads to some type of Coulomb gauge for the gauge field. Finally, it is not clear how to switch gauges and check results of physical quantities for their gauge independence. Altogether this urges to use another method for quantising gauge field theories. The method of choice presented here was invented by Fadeev and Popov, the underlying idea is to pick a gauge and then change coordinates. In doing so one gets a handle on the "gauge volume", i.e. some overall normalisation related to this extra divergence discussed before. In the following this procedure will be developed and tested for QED before going to non-Abelian theories such as QCD.

6.4.1 A simple example

To highlight the basic ideas of the Fadeev-Popov procedure, one may start with a toy example. Consider the following integral

$$Z = \int\limits_{-\infty}^{\infty} \mathrm{d}x\mathrm{d}y\, e^{-(x-y)^2} \,. \tag{6.231}$$

The integrand depends only on the difference of x and y, and in doing the integral one picks up a divergence. Its size is easily determined by changing to new variables, $z_- = x-y$ and $z_+ = (x+y)/2$. The Jacobian of this change is unity and hence one finds

$$Z = \int\limits_{-\infty}^{\infty} \mathrm{d}z_+\mathrm{d}z_- e^{-z_-^2} = \sqrt{\pi} \int\limits_{-\infty}^{\infty} \mathrm{d}z_+ \,. \tag{6.232}$$

The size of the divergence is the volume of the reals. However, one can cast this into a somewhat different setting. The integrand is translationally invariant, i.e. the transformation

$$x \to x' = x + a \quad \text{and} \quad y \to y' = y + a \tag{6.233}$$

with $a \in |R$ leaves the "action" $(x-y)^2$ and thus the exponential invariant. For the fun of it and to stress similarities to what follows one can call this transformation a gauge transformation. Then a is an element of the gauge group, in this case the reals. The gauge volume is $\int \mathrm{d}a$, a gauge "orbit", i.e. the path of equivalent configurations of x and y in the space of $|R^2$ under gauge transformations is a line of constant $(x-y)$. Then, by integrating over all configurations of x and y in the entire (x, y)-plane, one is redundantly adding an infinite number of copies of one finite Gaussian integral. In the language of gauge theories, it is clear that the "gauge" transformation of this example just expresses the separation of configurations in $(x,\, y)$-space into gauge dependent and invariant configurations, the latter one integrated over by $\mathrm{d}z_-$. To stress this even more, do just another change of variables, namely from $(z_+,\, z_-)$ to $(a,\, z_-)$. Then, by identifying a as the gauge transformation that takes a point on the line $x+y=0$ to another one but with $x-y$ fixed. By fixing $x+y=0$ one has a "slice" of configurations, i.e. a one-to-one mapping of x and $y=y(x)$, each of which represents a different starting point for an "orbit". Again, the Jacobian of this transformation is unity, and one has

$$Z = \int\limits_{-\infty}^{\infty} \mathrm{d}a \mathrm{d}z_- e^{-z_-^2} = \sqrt{\pi} \int\limits_{-\infty}^{\infty} \mathrm{d}a\,, \tag{6.234}$$

where the "gauge" volume appears explicitly now. Now, rewrite the integral once more, by introducing a "gauge fixing term",

$$Z = \int\limits_{-\infty}^{\infty} \mathrm{d}a \int\limits_{-\infty}^{\infty} \mathrm{d}x \mathrm{d}y\, 2\delta(x+y) e^{-(x-y)^2} = \sqrt{\pi} \int\limits_{-\infty}^{\infty} \mathrm{d}a\,. \tag{6.235}$$

In this form the gauge fixing is such that $x+y=0$, something analogous to the Coulomb gauge in QED. The factor of two just reflects the change of $x+y$ under the action of the gauge transformation.
So far so good. But now suppose, instead of the simple gauge before one chooses to make a different choice, say $f(x,\, y)=0$. To pursue the analogy, suppose that s is a variable that runs on the corresponding gauge slice defined by the function f. Then $\partial/\partial s$ is a vector tangent to f and $\mathrm{d}f/\mathrm{d}s = 0$. Again, let $a(x,\, y)$ be a gauge transformation that takes points on the slice to equivalent points off the gauge slice; in other words, for $a=0$ one is on the point where a particular gauge orbit crosses the slice. Changing variables from $(x,\, y)$ to $(s,\, a)$ leads to

$$Z = \int\limits_{-\infty}^{\infty} \mathrm{d}a \mathrm{d}s \, \det \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial a} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial a} \end{vmatrix} e^{-h(s)} = \int\limits_{-\infty}^{\infty} \mathrm{d}a \mathrm{d}s\, J\, e^{-h(s)}\,. \tag{6.236}$$

So far, $\mathrm{d}s$ was chosen arbitrarily, i.e. the rate of change of s along the slice is not yet fixed. To fix it, assume that

$$\int \mathrm{d}x \mathrm{d}y \delta[f(x,\, y)] = \int \mathrm{d}s\,, \tag{6.237}$$

a convenient choice. Now one can substitute this back into the expression of Z, making the choice of gauge explicit in the integral. The Jacobian will then take care that the final result is independent of the choice of gauge.
What remains to be done is to calculate the Jacobian J. To do so, insert a one into the basic equation:

$$Z = \int_{-\infty}^{\infty} \mathrm{d}x\mathrm{d}y\, e^{-(x-y)^2} . = \int_{-\infty}^{\infty} \mathrm{d}x\mathrm{d}y \int \mathrm{d}f\, \delta(f)\, e^{-(x-y)^2} . \tag{6.238}$$

Starting on a slice $(x',\, y')$ where $f(x',\, y') = 0$ and moving off the slice through gauge transformation, the value of f will change; f better is not constant under gauge transformations otherwise it would be useless for fixing the gauge. Therefore, one can understand f as a function of the gauge transformation a, $f = f(a)$. Then, changing variables from f to a one has

$$Z = \int_{-\infty}^{\infty} \mathrm{d}x\mathrm{d}y \int \mathrm{d}a\, \left.\frac{\mathrm{d}f}{\mathrm{d}a}\right|_{f=0} \delta[f(a)]\, e^{-(x-y)^2} . \tag{6.239}$$

From the gauge transformations above one finds

$$\left.\frac{\mathrm{d}f}{\mathrm{d}a}\right|_{f=0} = \left.\left(\frac{\partial f}{\partial x'}\frac{\partial x'}{\partial a} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial a}\right)\right|_{f=0} = \left.\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\right)\right|_{f=0} \tag{6.240}$$

leading to

$$Z = \int \mathrm{d}a \int_{-\infty}^{\infty} \mathrm{d}x\mathrm{d}y \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\right) \delta[f(x,\, y)]\, e^{-(x-y)^2} , \tag{6.241}$$

and again the gauge volume has been singled out.
To prepare the stage for the real thing rewrite this into a form that is more standard for gauge theories. Define

$$\Delta_f^{-1} = \int \mathrm{d}a\delta[f(x(a),\, y(a))] = \int \mathrm{d}f \det\left|\frac{\mathrm{d}a}{\mathrm{d}f}\right| \delta(f) = \det\left.\left|\frac{\mathrm{d}a}{\mathrm{d}f}\right|\right|_{f=0} \tag{6.242}$$

leading to

$$\Delta_f = \det\left.\left|\frac{\mathrm{d}f}{\mathrm{d}a}\right|\right|_{f=0} . \tag{6.243}$$

Inserting $1 = \Delta_f \Delta_f^{-1}$ into the integral yields

$$
\begin{aligned}
Z &= \int\limits_{-\infty}^{\infty} \mathrm{d}x\mathrm{d}y\, \Delta_f \Delta_f^{-1} e^{-(x-y)^2} \\
&= \int\limits_{-\infty}^{\infty} \mathrm{d}x\mathrm{d}y \det \left|\frac{\mathrm{d}f}{\mathrm{d}a}\right|_{f=0} \int \mathrm{d}a \delta(f) e^{-(x-y)^2} . \qquad (6.244)
\end{aligned}
$$

So far the gauge slices have been chosen such that they satisfy $f = 0$. But since the result should be independent of the specific gauge slice and in preparation for what follows one could have chosen them equally well such that $f = c$ with c a real constant. Also, so far we have defined the interesting part of Z to be non-zero only on the gauge slice. But, again, this is not the only option. After all, the goal is to find a definition of Z that is finite and independent of the choice of the gauge slice. One way to achieve this is to add up all slices and weighting them such that the integral along one gauge orbit is finite. For example, one could add up all slices by integrating over the constant c weighting each slice $f = c$ by $\exp(-c^2)$. This can be seen by inspecting

$$
\begin{aligned}
Z &= \int \mathrm{d}a \int\limits_{-\infty}^{\infty} \mathrm{d}x\mathrm{d}y \left|\frac{\mathrm{d}f}{\mathrm{d}a}\right|_{f=0} \delta[f(x,\, y) - c]\, e^{-(x-y)^2} \\
&= \int \mathrm{d}a \int\limits_{-\infty}^{\infty} \mathrm{d}x\mathrm{d}y \int \mathrm{d}c \delta(f-c) \frac{e^{-c^2}}{\sqrt{\pi}} \left|\frac{\mathrm{d}f}{\mathrm{d}a}\right|_{f=0} e^{-(x-y)^2} \\
&= \frac{1}{\sqrt{\pi}} \int \mathrm{d}a \int\limits_{-\infty}^{\infty} \mathrm{d}x\mathrm{d}y e^{-f^2} \left|\frac{\mathrm{d}f}{\mathrm{d}a}\right|_{f=0} e^{-(x-y)^2} . \qquad (6.245)
\end{aligned}
$$

Here, the emerging term $\exp(-f^2)$ is called the gauge-fixing term.

6.4.2 QED

The QED vacuum-to-vacuum amplitude $Z[J]$ expressed as a path integral in a naive way is expected to be

$$
\begin{aligned}
Z[J] &= \int \mathcal{D}A^\mu \exp\left\{i \int \mathrm{d}^4x \left[\frac{1}{2}\left(\vec{E}^2 - \vec{B}^2\right) + J_\mu A^\mu\right]\right\} \\
&= \int \mathcal{D}A^\mu \exp\left\{i\mathcal{S}[A,\, J]\right\} , \qquad (6.246)
\end{aligned}
$$

where for the moment the fermions have been left out and

$$\vec{E} = -\vec{\nabla} A_0 - \partial_t \vec{A} \quad \text{and} \quad \vec{B} = \vec{\nabla} \times \vec{A} . \tag{6.247}$$

This action is invariant under gauge transformations

$$A^\mu(x) \longrightarrow A^\mu(x) - \partial^\mu \Lambda(x) , \tag{6.248}$$

provided the source is conserved, i.e. $\partial_\mu J^\mu = 0$. In the naive path integral the integration is over all components of A, i.e. over all configurations of the vector potential. But since the action in invariant under the gauge transformation above, any two configurations related through such a transformation will contribute the same amount, a unphysical redundancy. Since for any gauge field configuration one can find an infinite number of equivalent configurations, the path integral is wildly diverging. To cure this one has to constrain the A^μ by forming equivalence classes and integrate only over these classes. To do so one has to divide the path integral by an infinite gauge volume, $V_g = V_{\rm gauge}$. To apply the same procedure developed in the previous section by the toy example, one has to pick a gauge first. This will be done in the following, first for the Coulomb and then for the Lorentz gauge.

Coulomb gauge

In the Coulomb gauge the gauge condition $F[A]$ is

$$F[A] = \vec{\nabla} \cdot \vec{A} = 0 . \tag{6.249}$$

On gauge slices defined by this condition the vector potential is transverse only. Anticipating that Λ in Eq.(6.248) plays the same role as a in the toy example one can write

$$Z[J] = \frac{1}{V_g} \int \mathcal{D}A^\mu \int \mathcal{D}\Lambda \det \left| \frac{\delta F[A]}{\delta \Lambda} \right|_{F=0} \delta[F[A]] \exp\left\{ i\mathcal{S}[A,\, J] \right\} . \tag{6.250}$$

Now, let $\vec{A}_\perp$ be the transverse vector potential, analogous to the coordinates on the gauge slice. Then, any $\vec{A}$ off the slice can be written as

$$\vec{A} = \vec{A}_\perp - \vec{\nabla}\Lambda , \tag{6.251}$$

spanning the gauge orbits. Therefore

$$\frac{\delta F[A]}{\delta \Lambda} = \frac{\delta(\vec{\nabla} \cdot \vec{A})}{\delta \Lambda} = \frac{\delta(\vec{\nabla} \cdot (\vec{A}_\perp - \vec{\nabla}\Lambda))}{\delta \Lambda} = -\vec{\nabla}^2 . \tag{6.252}$$

Plugging this into the path integral it becomes obvious that nothing in the integrand depends on Λ, so this integral can be used to cancel V_g. Also, by simple Fourier transformation, it is clear that the Jacobian, $\det[\vec{\nabla}^2]$ does not depend on A^μ, therefore it can be taken out of the integral and absorbed in the normalisation. Since this Jacobian is the same for both $Z[J]$ and $Z[0]$, it will cancel anyhow when according ratios are built. This yields

$$Z[J] = \mathcal{N}\int \mathcal{D}A^\mu \delta[\vec{\nabla}\cdot\vec{A}]\exp\{i\mathcal{S}[A,\, J]\}\ . \tag{6.253}$$

The δ-functional again decouples the A_0 piece from the $\vec{A}_\perp$ piece of the integration. The resulting integral is identical to what has been obtained before, Eq.(6.223),

$$\begin{aligned} Z[J] &= Z[0]\exp\left\{-\frac{i}{2}\int \mathrm{d}^4x\int \mathrm{d}^4x'\left[J^j(x)D^{\mathrm{tr.}}_{jk}(x-x')J^k(x')+\right.\right.\\ &\qquad\qquad\left.\left. J^0(x)\Delta_0(x-x')J^0(x')\right]\right\}\ . \end{aligned} \tag{6.254}$$

Again the A_0 piece guarantees that Gauss' law is satisfied, i.e.

$$\left\langle\vec{\nabla}\cdot\vec{E}(y)\right\rangle = \int \mathcal{D}A^\mu\delta[\vec{\nabla}\cdot\vec{A}]\left(\vec{\nabla}\cdot\vec{E}(y)\right)\exp\{i\mathcal{S}[A,\, J]\} = J^0(y)\,. \tag{6.255}$$

Lorentz gauge

To repeat the same procedure for the Lorentz gauge, consider first the gauge condition

$$F[A] = \partial_\mu A^\mu = 0\,. \tag{6.256}$$

Again, this choice of gauge is linear in A, the Jacobian therefore will remain independent of A and, again, it can be taken outside the integral and thrown into the normalisation:

$$\det\left|\frac{\delta F[A]}{\delta\Lambda}\right|_{F=0} = \det\left|\frac{\delta(\partial_\mu A^\mu)}{\delta\Lambda}\right|_{F=0} = \det\left|-\partial_\mu\partial^\mu\right|\ . \tag{6.257}$$

This leads to

$$Z[J] = \int \mathcal{D}A^\mu\delta[\partial_\mu A^\mu]\exp\{i\mathcal{S}[A,\, J]\}\ , \tag{6.258}$$

a solution which is perfectly acceptable but, unfortunately, not particularly easy to deal with. This is due to the fact that the term $\partial \cdot A$ does not appear explicitly in the action. A form that is by far better suited for further manipulations is the gauge condition

$$F[A] = \partial_\mu A^\mu(x) - c(x)\,, \tag{6.259}$$

where $c(x)$ is an arbitrary, real-valued function of the coordinates x. Then, following the discussion in the toy example, it is advantageous to sum the contributions from each gauge slice as defined by $c(x)$ and weighting each contribution by

$$\exp\left[\frac{i\alpha}{2}\int \mathrm{d}^4x c^2(x)\right]\,. \tag{6.260}$$

The Jacobian of all slices is identical to the one of Eq.(6.257). After absorbing it in the normalisation and, thus, cancelling the gauge volume the generating functional reads

$$\begin{aligned} Z[J] &= \mathcal{N}\int \mathcal{D}A^\mu \int \mathcal{D}c\,\delta[\partial_\mu A^\mu - c]\,\exp\left\{\frac{i\alpha}{2}\int \mathrm{d}^4x c^2(x)\right\}\,\exp\left\{i\mathcal{S}[A,\,J]\right\}\\ &= \mathcal{N}\int \mathcal{D}A^\mu \exp\left\{\frac{i\alpha}{2}\int \mathrm{d}^4x\,(\partial_\mu A^\mu)^2 + i\mathcal{S}[A,\,J]\right\}\\ &= \mathcal{N}\int \mathcal{D}A^\mu \exp\left\{i\mathcal{S}'[A,\,J]\right\} \end{aligned} \tag{6.261}$$

with a modified action, which is identical to the one of Eq.(3.178), defining the Lagrangian that has been used in the quantisation of the electromagnetic field in the Lorentz gauge through the operator formalism. Of course, this new, additional term

$$\frac{i\alpha}{2}\int \mathrm{d}^4x\,(\partial_\mu A^\mu)^2 \tag{6.262}$$

in the exponential is the gauge fixing term. The resulting integral is now simple to compute, it is a simple Gaussian. Setting $\alpha = 1$ one has the usual covariant propagator

$$D_{\mu\nu}(x-x') = -g_{\mu\nu}\Delta_F^{m=0}(x-x')\,, \tag{6.263}$$

and

$$Z[J] = Z[0]\,\exp\left\{-\frac{i}{2}\int \mathrm{d}^4x\mathrm{d}^4x'\left[J^\mu(x)D_{\mu\nu}(x-x')J^\nu(x')\right]\right\}\,. \tag{6.264}$$

6.4.3 Yang-Mills

Finally, the focus is shifted to the really interesting case, namely to Yang-Mills theories. These are gauge theories with a non-Abelian group structure. The Yang-Mills action is given by

$$S[A_a^\mu] = -\frac{1}{4}\int \mathrm{d}^4x \mathrm{Tr}\left[F_{\mu\nu}F^{\mu\nu}\right] , \tag{6.265}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig\left[A_\mu,\, A_\nu\right] = \frac{i}{g}\left[D_\mu,\, D_\nu\right] \tag{6.266}$$

and

$$D^\mu = (D^\mu)_{ab} = \partial^\mu \delta_{ab} + igf_{abc}A^{\mu c} . \tag{6.267}$$

The vector potential is a tensor,

$$\left[A_\mu(x)\right]_i^{\,j} = \sum A_\mu^a(x)\left[T_a\right]_i^{\,j} , \tag{6.268}$$

with the T_a the generators of the group, cf. Eqs.(2.219), (2.216), and (2.212). The generators satisfy the algebra

$$\left[T^a,\, T^b\right] = if^{abc}T_c . \tag{6.269}$$

The field A_μ transforms under the local gauge transformation

$$\left[U(x)\right]_i^{\,j} = \left[e^{ig\theta_a(x)T_a}\right]_i^{\,j} \tag{6.270}$$

as

$$A_\mu(x) \longrightarrow A'_\mu(x) = U(x)A_\mu(x)U^\dagger(x) + \frac{i}{g}\left(\partial_\mu U(x)\right)\, U^\dagger(x) , \tag{6.271}$$

and, correspondingly, the covariant derivative transforms like

$$D_\mu(x) \longrightarrow D'_\mu(x) = U(x)D_\mu(x)U^\dagger(x) . \tag{6.272}$$

Under these transformations the action of Eq.(6.265) remains invariant.

Fixing the gauge

Since in the path integral representation one is integrating over all configurations of the gauge field one must again form equivalence classes and integrate only over these classes A^μ/U. This means that Z is divided by the gauge volume, and, again, that one single representative A_a^μ is picked from each equivalence class by fixing a gauge slice. The coordinate changes necessary to cancel the gauge volume are defined by the changes U that take a configuration A_a^μ on the gauge slice to a general configuration A'^μ_a off the slice. The coordinates used to characterise the U are, not surprisingly, the angles $\theta_a(x)$ from Eq.(6.270), and the gauge slice is defined by $\theta_a(x) = 0$. It should be stressed that this construction is adequate only for a perturbative evaluation of Z, since only those transformations $U(x)$ can be generated that are continously connected to unity δ_{ij}.
Working in the covariant Lorentz gauge, the gauge choice becomes

$$F[A] = F_a T^a = \partial_\mu A_a^\mu(x) T^a = \partial_\mu A^\mu(x)\,. \tag{6.273}$$

In turn, Z reads

$$Z = \frac{N}{V_g}\int \mathcal{D}A_a^\mu \int \mathcal{D}\theta^b(x)\delta[F[A]]\det\left|\frac{\delta F}{\delta\theta}\right|_{F=0} \exp\left\{i\mathcal{S}[A]\right\}\,. \tag{6.274}$$

In a first step, the Jacobian has to be computed,

$$\Delta^F = \det\left|\frac{\delta F}{\delta\theta}\right|_{F=0} = \det\left|M_{ab}\right|_{F=0}\,, \tag{6.275}$$

where

$$\begin{aligned} M_{ab} := \frac{\delta F_a(x)}{\delta\theta^b(y)} &= \int \mathrm{d}^4 z \left[\frac{\delta F_a(x)}{\delta A_c^\mu(z)}\right]\left[\frac{\delta A_c^\mu(z)}{\delta\theta^b(y)}\right]\Bigg|_{\theta^b=0} \\ &= \int \mathrm{d}^4 z \left[\partial_\mu \delta_{ac}\delta^4(x-z)\right]\left[(D^\mu)_{cb}\delta^4(z-y)\right] \\ &= (\partial_\mu D^\mu)_{ab}\delta^4(x-y)\,. \end{aligned} \tag{6.276}$$

Δ^F is a gauge invariant quantity, therefore it is possible to separate the gauge volume $\mathcal{D}\theta^b(x)$ from the integral over the fields, $\mathcal{D}A_a^\mu$, and to cancel the term $1/V_g$. Hence, in Lorentz gauge, the remaining integral is

$$Z = \int \mathcal{D}A_a^\mu \delta\left[\partial_\mu A^\mu\right]\det\left|(\partial_\mu D^\mu)_{ab}\right|\exp\left\{i\mathcal{S}[A]\right\}\,. \tag{6.277}$$

Again it is more suitable to add all gauge slices $c^a(x)$, each of which is weighted by the Gaussian of Eq.(6.260),

$$\exp\left[\frac{i\alpha}{2}\int \mathrm{d}^4x c^2(x)\right] . \tag{6.278}$$

Following the same steps as in QED one finds

$$Z = \int \mathcal{D}A_a^\mu \det\left|(\partial_\mu D^\mu)_{ab}\right| \exp\left\{i\mathcal{S}[A] + \frac{i\alpha}{2}\int \mathrm{d}^4x \mathrm{Tr}\left[\partial_\mu A^\mu\right]\right\} . \tag{6.279}$$

But now, for Yang-Mills theories, the Jacobian is no longer independent of the fields due to the occurrence of the covariant derivative. Therefore it cannot be hidden in the normalisation - it is just not clear what the difference of this term with and without sources is. To compute Z perturbatively then leads to non-local interactions, which are far beyond the grasp of what the tools of quantum field theory developed so far can deal with.

Introducing ghosts

The solution to this problem is to find a representation of the determinant that does lead to local interactions. Recall from Eq.(6.161),

$$\int \mathcal{D}\eta^*\mathcal{D}\eta \exp\left[-\int \mathrm{d}x\mathrm{d}x'\, \eta^*(x)a(x-x')\eta(y)\right] = \det[a]\,, \tag{6.280}$$

that the result of a Gaussian integral over complex anticommuting fields is the determinant of the operator sandwiched between them. Therefore the Jacobian Δ^F can be represented through such an integral,

$$\begin{aligned}
&\det\left|M_{ab}\right|_{F=0} \\
&= \int \mathcal{D}\eta^{*a}\mathcal{D}\eta^a \exp\left\{-\int \mathrm{d}x\mathrm{d}x'\, \eta^{a*}(x)M_{ab}(x-x')\eta^b(y)\right\} \\
&= \int \mathcal{D}\eta^{*a}\mathcal{D}\eta^a \exp\left\{-\int \mathrm{d}x\mathrm{d}x'\, \eta^{a*}(x)\partial_\mu D^\mu_{ab}\delta^4(x-x')\eta^b(y)\right\} \\
&= \int \mathcal{D}\eta^{*a}\mathcal{D}\eta^a \exp\left\{-\int \mathrm{d}x\left[\mathrm{Tr}\left(\eta^*(x)\partial_\mu\partial^\mu\eta(x)\right) + ig\eta^{*a}f_{abc}\partial_\mu\left(A^{\mu c}\eta^b\right)\right]\right\} .
\end{aligned} \tag{6.281}$$

This is fun. The result above resembles a vacuum-to-vacuum amplitude for fermionic fields η^a coupled to an external field $A^{\mu c}$ and with a bosonic kinetic

term. These fictitious fields are called Fadeev-Popov ghosts. Note that the term "ghost" is used in two different connotations in quantum field theory: one way, as above, is to describe states that represent a Jacobian - they are fermions of canonical dimension 1 but with kinetic terms like bosons, something weird. The other way is for states whose norm have the wrong sign, like the scalar photons of the Gupta-Bleuler metric.
When sources are added to Z, the Fadeev-Popov ghosts are not to be forgotten. Thus the Yang-Mills vacuum-to-vacuum amplitude in the presence of sources and in Lorentz gauge is

$$\begin{aligned} Z[J,\,\chi^*,\,\chi] &= \mathcal{N}\int \mathcal{D}A^\mu_a \mathcal{D}\eta^{*b}\mathcal{D}\eta^b \exp\left\{ i\int \mathrm{d}^4x \left[\frac{1}{4}\mathrm{Tr}\,(F_{\mu\nu}F^{\mu\nu}) \right.\right. \\ &\qquad + \frac{1}{2}\mathrm{Tr}\,(\partial_\mu A^\mu)^2 - i\mathrm{Tr}\,(\partial_\mu \eta^* \partial^\mu \eta) - igf_{abc}\eta^{*a}A^{\mu b}\partial_\mu \eta^c \\ &\qquad \left.\left. + \mathrm{Tr}\,(J^\mu A_\mu) + \mathrm{Tr}\,(\chi^*\eta) + \mathrm{Tr}\,(\eta^*\chi) \right]\right\}. \end{aligned} \tag{6.282}$$

To evaluate this perturbatively, all terms in the action that are not linear or quadratic in the fields are replaced by functional derivatives w.r.t. the appropriate source. These terms are, as usual, pulled outside the now Gaussian integral, which can be done exactly.
Ghost fields could have been introduced equally well for QED. There, however, since $\delta A^\mu/\delta\Lambda$ is independent of A^μ, they would never couple to any field, so they would remain invisible in any calculation. In contrast, for Yang-Mills theories, since $\delta A^\mu_a/\delta\theta^b$ does depend on the vector potential, does this mean that one has to introduce ghosts in every gauge? The answer is no. To underline this, consider the axial gauge defined by

$$F[A] = A^a_z = 0\,. \tag{6.283}$$

In this gauge,

$$\begin{aligned} \int \mathrm{d}^4z \left[\frac{\delta F_a(x)}{\delta A^\mu_c(z)}\right]\left[\frac{\delta A^\mu_c(z)}{\delta\theta^b(y)}\right]\Bigg|_{\theta^b=0} &= \int \mathrm{d}^4z \left[\delta_{\mu z}\delta_{ac}\delta^4(x-z)(D^\mu)_{cb}\delta^4(z-y)\right] \\ &= (D^z)_{ab}\delta^4(x-y)\,. \end{aligned} \tag{6.284}$$

However, since $A^a_z = 0$ for all fields a, then

$$(D^z)_{ab} = \partial^z\delta_{ab} \tag{6.285}$$

and the Jacobian is independent of A^a. This is nice, if one wishes to calculate without ghosts. However, since there is always a price to pay, in such "ghost-free" gauges the propagators become quite complicated objects.

Chapter 7

Calculation of simple processes at tree-level

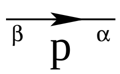
7.1 QED

7.1.1 Feynman rules

To construct Feynman amplitudes for QED processes in Lorentz gauge directly, use the following steps:

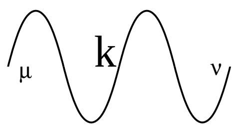
1. Draw all distinct, connected diagrams for a given process. As discussed before, the disconnected diagrams will vanish. Since the fermion lines have an orientation (the spin-flow), the direction of internal fermion lines matters!

2. For each internal fermion line, carrying momentum p from Dirac index β to Dirac index α, assign a fermion propagator



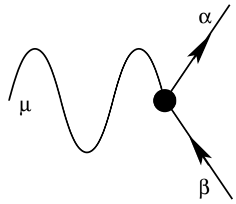
$$= iS_F(p)_{\alpha\beta} = \left(i\frac{\not{p}+m}{p^2-m^2+i\epsilon}\right)_{\alpha\beta}. \tag{7.1}$$

For each internal photon line with momentum k, assign a photon propagator



$$= iD_{\mu\nu}(k) = -\frac{ig_{\mu\nu}}{k^2+i\epsilon}. \tag{7.2}$$

3. For each vertex, where a spinor line for a particle with charge e_f with index α is outgoing and a spinor line with index β is incoming, assign a factor



$$= -iee_f\,(\gamma^\mu)_{\alpha\beta}\ , \tag{7.3}$$

where e is the electromagnetic coupling constant and e_f is the fractional charge of the fermion in units of the proton charge. Conserve momentum locally at each vertex by adding a factor

$$(2\pi)^4\delta^4\left(\sum p_{\mathrm{in}} - \sum p_{\mathrm{out}}\right)\ , \tag{7.4}$$

where the sums are over the momenta entering and leaving the vertex.

4. Contract all spinor and Lorentz indices along the corresponding internal lines. Integrate over all internal momenta through

$$\int \frac{\mathrm{d}^4 p_i}{(2\pi)^4}\ . \tag{7.5}$$

Many of these integrations will cancel due to the δ functions attached to the vertices.

5. Assign a global sign to odd permutations of external spinor indices. Also, attach a factor (-1) to each closed fermion loop.

6. Ensure global momentum conservation by a global factor

$$(2\pi)^4\delta^4\left(\sum p_{\mathrm{in}} - \sum p_{\mathrm{out}}\right)\ , \tag{7.6}$$

where the sums are over the momenta entering and leaving the diagram.

7. For each external photon line with Lorentz index μ, add a polarisation vector $\epsilon^\mu(k)$ with $k^2 = 0$. External spinors are represented by $u(p)$ or $\bar{u}(p)$ for incoming or outgoing particles and by $\bar{v}(-p)$ or $v(-p)$ for incoming or outgoing anti-particles, respectively. There, p and $-p$ are the momenta entering or leaving the diagram, fulfilling $p^2 = m^2$.

7.1.2 Coulomb scattering

Transition amplitude for a static potential

To highlight how the Feynman rules above come in, consider the scattering of an incoming electron with a static Coulomb potential. The corresponding transition amplitude in position space reads:

$$\mathcal{S}_{fi} = -iee_e\bar{\psi}_f(x)\gamma^\mu\psi_i(x)A_\mu(x)\,. \tag{7.7}$$

By Fourier transformation the wave functions of the incoming and outgoing electron can be cast into

$$\begin{aligned}\psi_i(x) &= \sqrt{\frac{1}{2E_i}}\,u(p_i,\,s_i)e^{-ip_i\cdot x}\\ \bar{\psi}_f(x) &= \sqrt{\frac{1}{2E_f}}\,\bar{u}(p_f,\,s_f)e^{ip_f\cdot x}\end{aligned} \tag{7.8}$$

and the static Coulomb potential for a point-like source with charge $-Ze > 0$ reads

$$A_0(x) = \frac{-Ze}{4\pi|\vec{x}|} \quad \text{and} \quad \vec{A}(x) = 0\,. \tag{7.9}$$

Taken together and integrating over all x this yields

$$\begin{aligned}\mathcal{S}_{fi} &= \frac{iZe^2}{4\pi}\sqrt{\frac{1}{4E_iE_f}}\bar{u}(p_f,\,s_f)\gamma_0u(p_i,\,s_i)\int\frac{\mathrm{d}^4x}{|\vec{x}|}e^{-i(p_i-p_f)\cdot x}\\ &= \frac{iZe^2}{4\pi}\sqrt{\frac{1}{4E_iE_f}}\bar{u}(p_f,\,s_f)\gamma_0u(p_i,\,s_i)\frac{4\pi}{|\vec{p}_f-\vec{p}_i|^2}(2\pi)\delta(E_f-E_i)\,.\end{aligned} \tag{7.10}$$

The number of final states in the momentum interval $\mathrm{d}^3\vec{p}_f$ equals $\mathrm{d}^3\vec{p}_f/(2\pi)^3$, therefore the transition probability for the scattering into such states is given by

$$|\mathcal{S}_{fi}|^2\,\frac{\mathrm{d}^3\vec{p}_f}{(2\pi)^3} = \frac{Z^2e^4}{2E_i}\frac{|\bar{u}(p_f,\,s_f)\gamma_0u(p_i,\,s_i)|^2}{|\vec{p}_f-\vec{p}_i|^4}(2\pi)^2\delta^2(E_f-E_i)\frac{\mathrm{d}^3\vec{p}_f}{(2\pi)^32E_f} \tag{7.11}$$

Similar to the four-dimensional case of Eq.(4.142), the square of the δ-function leads to

$$(2\pi)^2\delta^2(E_f - E_i) = (2\pi)^2\delta(0)\delta(E_f - E_i) = 2\pi T\delta(E_f - E_i) \quad (7.12)$$

where T is some large but finite time. To see, how this is realized consider

$$\begin{aligned}
2\pi\delta(E_f - E_i) &= \int\limits_{-T/2}^{T/2} \mathrm{d}t e^{i(E_f-E_i)t} = \frac{2\sin[(T/2)(E_f - E_i)]}{E_f - E_i} \\
(2\pi)^2\delta^2(E_f - E_i) &= \frac{4\sin^2[(T/2)(E_f - E_i)]}{(E_f - E_i)^2}\,.
\end{aligned} \quad (7.13)$$

Interpreting this as a function of $E_f - E_i$, one realizes that this is just an area $2\pi T$ leading to the above identification. However, returning to the transition probability above and dividing by the time one obtains the rate for the transition in question $\mathcal{R}$ as

$$\mathcal{R} = \frac{Z^2e^4}{4E_i}\frac{|\bar{u}(p_f,\, s_f)\gamma_0 u(p_i,\, s_i)|^2}{|\vec{p}_f - \vec{p}_i|^4}(2\pi)\delta(E_f - E_i)\frac{\mathrm{d}^3\vec{p}_f}{(2\pi)^3}\,. \quad (7.14)$$

The cross section is defined by the transition rate divided by the current of incoming particles, in this case

$$J^a_{\mathrm{in}} = \bar{\psi}_i(x)\gamma^a\psi_i(x)\,, \quad (7.15)$$

where a is the component pointing into the direction of the incoming velocity v, $\vec{v} = \vec{p}_i/E_i$. With the normalisation above,

$$J^a_{\mathrm{in}} = v^a \quad (7.16)$$

and the differential cross section per angle $\mathrm{d}\Omega$ is

$$\begin{aligned}
\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \int \frac{Z^2e^4}{4E_i|v|}\frac{|\bar{u}(p_f,\, s_f)\gamma_0 u(p_i,\, s_i)|^2}{|\vec{p}_f - \vec{p}_i|^4}(2\pi)\delta(E_f - E_i)\frac{p_f^2\mathrm{d}p_f}{(2\pi)^3} \\
&= \frac{Z^2e^4}{4(2\pi)^2}\frac{|\bar{u}(p_f,\, s_f)\gamma_0 u(p_i,\, s_i)|^2}{|\vec{p}_f - \vec{p}_i|^4}\,,
\end{aligned} \quad (7.17)$$

where the identities

$$p_f\mathrm{d}p_f = E_f\mathrm{d}E_f \quad \text{and} \quad |\vec{p}_f| = |\vec{p}_i| = |\vec{p}| = \sqrt{E^2 - m^2} \quad (7.18)$$

have been used. In fact, in the non-relativistic limit the expression for the cross section is identical to the Rutherford result, as will be seen in the next few paragraphs.

Traces over γ matrices

What remains to be done is the evaluation of the spinor product

$$|\bar{u}(p_f,\, s_f)\gamma_0 u(p_i,\, s_i)|^2 \,. \tag{7.19}$$

Since in many cases polarisation effects are of no relevance, it is appropriate to sum over all outgoing spins and to average over all incoming spins. Apart from a factor $1/N_s$ for the number of possible spin orientations for the incoming electron, one can therefore write

$$\begin{aligned}\left|\bar{u}(p_f,\, s_f)\gamma^0 u(p_i,\, s_i)\right|^2 &\longrightarrow \frac{1}{2}\sum_{s_f,\, s_i}\left[\bar{u}(p_f,\, s_f)\gamma^0 u(p_i,\, s_i)\right]^2 \\ &= \frac{1}{2}\sum_{s_f,\, s_i}\left[\bar{u}_\alpha(p_f,\, s_f)\gamma^0_{\alpha\beta}u_\beta(p_i,\, s_i)\bar{u}_\gamma(p_i,\, s_i)\gamma^0_{\gamma\delta}u_\delta(p_f,\, s_f)\right] \,. \end{aligned} \tag{7.20}$$

In so doing, the following relations have been used

$$\begin{aligned} \bar{u}_\alpha(p,\, s) &= u^\dagger_\alpha(p,\, s)\gamma^{0\dagger}_{\alpha\beta} \\ \bar{\Gamma} &= \gamma^0\Gamma^\dagger\gamma^0 \\ \bar{\gamma}^\mu &= \gamma^\mu \\ \overline{i\gamma_5} &= i\gamma_5 \\ \overline{\gamma_\mu\gamma_5} &= \gamma_\mu\gamma_5 \\ \overline{\not{a}\not{b}\dots\not{z}} &= \not{z}\dots\not{a}\,. \end{aligned} \tag{7.21}$$

cf.Eq.(2.170). Using the completeness relations of Eq.(3.71)

$$\begin{aligned} \sum_{s=\pm} u_\alpha(p,\, s)\bar{u}_\beta(p,\, s) &= (\not{p}+m)_{\alpha\beta} \\ \sum_{s=\pm} v_\alpha(p,\, s)\bar{v}_\beta(p,\, s) &= (\not{p}-m)_{\alpha\beta} \end{aligned} \tag{7.22}$$

for the spinors yields

$$\begin{aligned} &\sum_{s_f,\, s_i}\left[\bar{u}_\alpha(p_f,\, s_f)\gamma^0_{\alpha\beta}u_\beta(p_i,\, s_i)\bar{u}_\gamma(p_i,\, s_i)\gamma^0_{\gamma\delta}u_\delta(p_f,\, s_f)\right] \\ &= \left[(\not{p}_f+m)_{\delta\alpha}\gamma^0_{\alpha\beta}(\not{p}_i+m)_{\beta\gamma}\gamma^0_{\gamma\delta}\right] = \mathrm{Tr}\left[(\not{p}_f+m)\gamma^0(\not{p}_i+m)\gamma^0\right] \end{aligned} \tag{7.23}$$

This finally leaves the evaluation of the trace as the last task. To do so, a detour to some identities for such traces is in order; they are listed in the appendix.

Using Eq.(A.2) yields

$$\begin{aligned}
\mathrm{Tr}\left[(\not{p}_f+m)\gamma^0(\not{p}_i+m)\gamma^0\right] &= \mathrm{Tr}\left[\not{p}_f\gamma^0\not{p}_i\gamma^0\right]+m^2\mathrm{Tr}\left[\gamma^0\gamma^0\right] \\
&= 4[p_f^0p_i^0-g^{00}p_f\cdot p_i+p_f^0p_i^0+m^2] \\
&= 8E_fE_i-4(p_f\cdot p_i-m^2)\,. \qquad (7.24)
\end{aligned}$$

To cast the differential cross section in a nicer form, one notes that with the scattering angle θ one has

$$\begin{aligned}
2p_f\cdot p_i &= p_f^2+p_i^2-(p_f-p_i)^2=2m^2-(E_f-E_i)^2+(\vec{p}_f-\vec{p}_i)^2 \\
&= 2m^2+(\vec{p}_f-\vec{p}_i)^2=2m^2+2|\vec{p}|^2(1-\cos\theta)\,. \qquad (7.25)
\end{aligned}$$

Plugging this into the expression for the cross section one finds

$$\begin{aligned}
\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \frac{Z^2e^4}{2\cdot4(2\pi)^2}\frac{|\bar{u}(p_f,\,s_f)\gamma_0u(p_i,\,s_i)|^2}{|\vec{p}_f-\vec{p}_i|^4} \\
&= \frac{Z^2(4\pi\alpha)^2}{8(2\pi)^2}\frac{8E^2-4|\vec{p}|^2(1-\cos\theta)}{\left[2|\vec{p}|^2(1-\cos\theta)\right]^2}=\frac{4Z^2\alpha^2(2E^2-2|\vec{p}|^2\sin^2\frac{\theta}{2})}{32|\vec{p}|^4\sin^4\frac{\theta}{2}} \\
&= \frac{Z^2\alpha^2(E^2-|\vec{p}|^2\sin^2\frac{\theta}{2})}{4|\vec{p}|^4\sin^4\frac{\theta}{2}}=\frac{Z^2\alpha^2}{4|\vec{p}|^2\beta^2\sin^4\frac{\theta}{2}}\left(1-\beta^2\sin^2\frac{\theta}{2}\right)\,, \qquad (7.26)
\end{aligned}$$

where the electromagnetic fine structure constant

$$\alpha=\frac{e^2}{4\pi} \qquad (7.27)$$

has been identified. This is Mott's cross section for an electron scattering off a static potential, for $\beta\to0$, i.e. for $E\to m$ (or $|\vec{p}|/m\to0$), one recovers the Rutherford formula,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{Z^2\alpha^2m^2}{4|\vec{p}|^4\sin^4\frac{\theta}{2}} \qquad (7.28)$$

Coulomb scattering with a Dirac proton

Assuming that the proton has no intrinsic structure (which is quite an unfounded assumption at higher energies and in view of its magnetic moment)

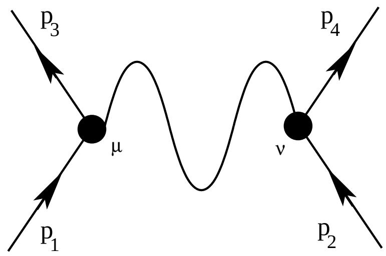


Figure 7.1: The Feynman diagram for elastic e^-p-scattering.

Coulomb scattering with a proton can be described by replacing the potential by a photon propagator coupled to a proton current,

$$A^\mu(x) \longrightarrow \int \mathrm{d}^4 y D^{\mu\nu}(x-y) J_\nu(y)\,, \tag{7.29}$$

where, in analogy to the electron current, the proton current is given by

$$J_\nu(y) = -iee_p \bar{\psi}^p_f(y)\gamma_\nu \psi^p_i(y)\,. \tag{7.30}$$

The corresponding Feynman diagram is exhibited in Fig.7.1. Using the Feynman rules in momentum space, one finds for the transition amplitude for $e^-(p_1) + p(p_2) \to e^-(p_3) + p(p_4)$

$$\begin{aligned}
\mathcal{S}_{fi} &= \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \Big\{ (2\pi)^4\delta^4(p_1 - p_3 - q)(2\pi)^4\delta^4(p_2 - p_4 + q) \\
&\qquad (2\pi)^4\delta^4(p_1 + p_2 - p_3 - p_4) \\
&\qquad (-ie)^2(e_p e_e)[\bar{u}(p_3)\gamma^\mu u(p_1)][\bar{u}(p_4)\gamma^\nu u(p_2)]\frac{-ig_{\mu\nu}}{q^2} \Big\} \\
&= -ie^2 \frac{[\bar{u}(p_3)\gamma^\mu u(p_1)][\bar{u}(p_4)\gamma_\mu u(p_2)]}{(p_1 - p_3)^2}(2\pi)^4\delta^4(p_1 + p_2 - p_3 - p_4)
\end{aligned} \tag{7.31}$$

Squaring of the spinor part, taking summing and taking averages over spins in due order results in

$$\begin{aligned}
&|[\bar{u}(p_3)\gamma^\mu u(p_1)][\bar{u}(p_4)\gamma_\mu u(p_2)]|^2 \\
&\quad\longrightarrow \frac{1}{4}\sum [\bar{u}(p_3)\gamma^\mu u(p_1)]^2 \, [\bar{u}(p_4)\gamma_\mu u(p_2)]^2
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4}\mathrm{Tr}\left[(\not{p}_3+m_e)\gamma^\mu(\not{p}_1+m_e)\gamma^\nu\right]\mathrm{Tr}\left[(\not{p}_4+m_p)\gamma_\mu(\not{p}_2+m_p)\gamma_\nu\right] \\
&= 4\left[p_3^\mu p_1^\nu + p_1^\mu p_3^\nu - g^{\mu\nu}(p_3p_1-m_e^2)\right]\left[p_{4\mu}p_{2\nu}+p_{2\mu}p_{4\nu}-g_{\mu\nu}(p_4p_2-m_p^2)\right] \\
&= 4\Big[2(p_3p_4)(p_1p_2)+2(p_3p_2)(p_4p_1)+4(p_3p_1-m_e^2)(p_4p_2-m_p^2) \\
&\qquad -2(p_4p_2)(p_3p_1-m_e^2)-2(p_3p_1)(p_4p_2-m_p^2)\Big] \\
&= 8\left[(p_3p_4)(p_1p_2)+(p_3p_2)(p_4p_1)-(p_3p_1)m_p^2-(p_4p_2)m_e^2+2m_p^2m_e^2\right]
\end{aligned}
\tag{7.32}
$$

Therefore, the fully differential cross section in Lorentz covariant form reads

$$
\mathrm{d}\sigma = \frac{(2\pi)^4\delta^4(p_1+p_2-p_3-p_4)}{4\sqrt{(p_1p_2)^2-m_1^2m_2^2}}\frac{\mathrm{d}^3\vec{p}_3}{(2\pi)^3 2E_3}\frac{\mathrm{d}^3\vec{p}_4}{(2\pi)^3 2E_4}\left|\tilde{\mathcal{S}}_{fi}\right|^2 ,
\tag{7.33}
$$

where all constant factors due to the various normalisations, the incoming current, and due to the square of the δ-function responsible for total four momentum conservation have been absorbed and mutually cancelled. Also, $\tilde{\mathcal{S}}_{fi}$ is the transition matrix element without the total four momentum conservation. A simple check for the characteristic dimensions of this expression reveals that

$$
[m^{-2}] = \frac{[m^{-4}]}{[m^2]}\frac{[m^3]}{[m]}\frac{[m^3]}{[m]}[1]
\tag{7.34}
$$

as expected.
To evaluate the emerging cross section, one has to fix kinematics. A good choice to do so is the lab system. In the case considered here one may assume that the proton is at rest. Then the initial vectors are given by

$$
p_1 = (E_1,\,\vec{p}_1)^T = (E_1,\,0,\,0,\,\bar{p}_1)^T \quad\text{and}\quad p_2 = (M,\,\vec{0})^T
\tag{7.35}
$$

and the final state vectors read

$$
\begin{aligned}
p_3 &= (E_3,\,\vec{p}_3)^T = (E_3,\,\bar{p}_3\sin\theta,\,0,\,\bar{p}_3\cos\theta)^T \\
p_4 &= (E_4,\,\vec{p}_4)^T = (E_4,\,\bar{p}_4\sin\theta,\,0,\,\bar{p}_4\cos\theta)^T .
\end{aligned}
\tag{7.36}
$$

With

$$
\frac{\mathrm{d}^3\vec{p}_4}{(2\pi)^3 2E_4} = \frac{\mathrm{d}^4 p_4}{(2\pi)^4}(2\pi)\delta(p_4^4-m_4^2)\theta(p_4^0)
\tag{7.37}
$$

one finds

$$\begin{aligned}
\mathrm{d}\sigma = & \frac{(2\pi)^4\delta^4(p_1+p_2-p_3-p_4)}{4\sqrt{(p_1p_2)^2-m_1^2m_2^2}}\frac{\mathrm{d}^3\vec{p}_3}{(2\pi)^3 2E_3}\frac{\mathrm{d}^3\vec{p}_4}{(2\pi)^3 2E_4}\left|\tilde{\mathcal{S}}_{fi}\right|^2 \\
= & \left.\frac{(2\pi)\delta(p_4^2-m_4^2)\theta(p_4^0)}{4\sqrt{(p_1p_2)^2-m_1^2m_2^2}}\frac{\bar{p}_3^2\mathrm{d}\bar{p}_3\mathrm{d}^2\Omega}{(2\pi)^3 2E_3}\left|\tilde{\mathcal{S}}_{fi}\right|^2\right|_{p_4=p_1+p_2-p_3} \\
= & \frac{\delta[(p_1+p_2-p_3)^2-m_p^2)\theta(E_1+m_p-E_3)}{8m_p\bar{p}_1}\frac{\bar{p}_3\mathrm{d}E_3\mathrm{d}^2\Omega}{4\pi^2}\left|\tilde{\mathcal{S}}_{fi}\right|^2 , \qquad (7.38)
\end{aligned}$$

where again

$$\frac{\mathrm{d}\bar{p}}{\mathrm{d}E} = \frac{\mathrm{d}}{\mathrm{d}E}\sqrt{E^2-m^2} = \frac{E}{\sqrt{E^2-m^2}} = \frac{E}{\bar{p}} \qquad (7.39)$$

has been used. From the δ-function one has

$$\delta[(p_1+p_2-p_3)^2-m_p^2) = \delta[2m_e^2-2m_p(E_3-E_1)-2E_3E_1+2\bar{p}_3\bar{p}_1\cos\theta] . \qquad (7.40)$$

After convincing oneself that the solution for E_3 described by

$$E_3(m_p+E_1)-\sqrt{E_3^2-m_e^2}\sqrt{E_1^2-m_e^2}\cos\theta = E_1m_p+m_e^2 \qquad (7.41)$$

is inside its region of integration the differential cross section with respect to the angle reads

$$\frac{\mathrm{d}\sigma}{\mathrm{d}^2\Omega} = \frac{\left|\tilde{\mathcal{S}}_{fi}\right|^2}{4\pi^2\cdot 8m_p}\cdot\frac{\bar{p}_3}{\bar{p}_1}\cdot\frac{1}{2\left(m_p+E_1-\frac{\bar{p}_1E_3}{\bar{p}_3}\cos\theta\right)} \qquad (7.42)$$

In the limit where the incoming electrons energy is small compared to the proton mass, the proton won't recoil, i.e. $E_4 = E_2 = m_p$. This means that $E_3 = E_1 = E$ and that one can neglect all terms $E_1/m_p \ll 1$ and $m_e/m_p \ll 1$. One then finds for the squared transition amplitude

$$\left|\tilde{\mathcal{S}}_{fi}\right|^2 = \frac{8e^4m_p^2(2E^2+m_e^2-p_3\cdot p_1)}{(p_3-p_1)^4} \qquad (7.43)$$

Adding things, one recovers the result for scattering at a static Coulomb potential.

In the ultrarelativistic limit, i.e. with high energies of the electron, the proton will recoil, but one can neglect all occurrences of the electron mass. Then the electron momenta are proportional to their energy and the transition matrix element squared reads

$$\begin{aligned}
\left|\tilde{\mathcal{S}}_{fi}\right|^2 &= \frac{8(4\pi\alpha)^2}{q^4}\left[(p_3p_4)(p_1p_2)+(p_3p_2)(p_4p_1)-(p_3p_1)m_p^2\right] \\
&\to \frac{8(4\pi\alpha)^2}{(p_1-p_3)^4}\left[p_3\cdot(p_1+p_2)(p_1p_2)+(p_3p_2)p_1\cdot(p_2-p_3)-(p_3p_1)m_p^2\right] \\
&= \frac{8(4\pi\alpha)^2}{(2p_1p_3)^2}\left[2(p_3p_2)(p_1p_2)+(p_3p_1)(p_1p_2-p_3p_2-m_p^2)\right] \\
&= \frac{2(4\pi\alpha)^2m_p^2E_1E_3}{E_1^2E_3^2\left(2\sin^2\frac{\theta}{2}\right)^2}\left[2-2\sin^2\frac{\theta}{2}\left(\frac{E_3-E_1}{m_p}+1\right)\right] \\
&= \frac{(4\pi\alpha)^2m_p^2}{E_1E_3\sin^4\frac{\theta}{2}}\left[\cos^2\frac{\theta}{2}-\frac{q^2}{2m_p^2}\sin^2\frac{\theta}{2}\right], \qquad (7.44)
\end{aligned}$$

where Eq.(7.41) in the ultrarelativistic limit

$$E_3E_1(1-\cos\theta)=(E_1-E_3)m_p=-\frac{(p_3-p_1)^2}{2}=\frac{q^2}{2} \qquad (7.45)$$

has been used. Then the differential cross section becomes

$$\begin{aligned}
\frac{\mathrm{d}\sigma}{\mathrm{d}^2\Omega} &= \frac{\left|\tilde{\mathcal{S}}_{fi}\right|^2}{4\pi^2\cdot 8m_p}\cdot\frac{\bar{p}_3}{\bar{p}_1}\cdot\frac{1}{2\left(m_p+E_1-\frac{\bar{p}_1E_3}{\bar{p}_3}\cos\theta\right)} \\
&\to \frac{\alpha^2}{4E_1^2\sin^4\frac{\theta}{2}}\cdot\frac{\cos^2\frac{\theta}{2}-\frac{q^2}{2m_p^2}\sin^2\frac{\theta}{2}}{\left(1+2\frac{E_1}{m_p}\sin^2\frac{\theta}{2}\right)}. \qquad (7.46)
\end{aligned}$$

7.1.3 Electron scattering

Next, consider the scattering of two electrons, i.e. the process $e^-(p_1)+e^-(p_2)\to e^-(p_3)+e^-(p_4)$. Due to the identical particles in the final state, two diagrams interfere, where they are interchanged, see Fig.7.2. Because they are fermions, the interference is destructive. Applying the Feynman rules and using momentum conservation at the vertices, the two amplitudes

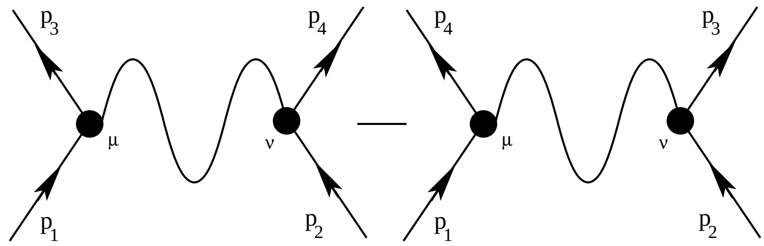


Figure 7.2: The Feynman diagrams for elastic e^-e^--scattering. Note that the momenta and thus the identical particles in the final state are interchanged, since they are fermions the interference is destructive.

read

$$\begin{aligned}
\mathcal{M}^{(a)} &= \frac{(-ie)\bar{u}(p_3)\gamma^\mu u(p_1)\cdot(-ie)\bar{u}(p_4)\gamma^\nu u(p_2)(-ig_{\mu\nu})}{(p_1-p_3)^2} \\
\mathcal{M}^{(b)} &= -\frac{(-ie)\bar{u}(p_3)\gamma^\mu u(p_2)\cdot(-ie)\bar{u}(p_4)\gamma^\nu u(p_1)(-ig_{\mu\nu})}{(p_1-p_4)^2}\,,
\end{aligned} \tag{7.47}$$

where the minus-sign reflecting the Fermi statistics is attached to diagram (b). The transition matrix element squared is given by

$$\left|\tilde{\mathcal{S}}_{fi}\right|^2 = \left|\mathcal{M}^{(a)}+\mathcal{M}^{(b)}\right|^2 = \left|\mathcal{M}^{(a)}\right|^2 + \left|\mathcal{M}^{(b)}\right|^2 + 2\left|\mathcal{M}^{(a)}\overline{\mathcal{M}}^{(b)}\right|\,, \tag{7.48}$$

leading to three contributions to be computed. To start with, consider the first one. Summing and averaging over the spins outgoing and incoming particles one recovers the result from the proton-electron scattering, but with the replacement $m_p \to m_e = m$,

$$\begin{aligned}
&\left|\mathcal{M}^{(a)}\right|^2 = \left|\frac{(-ie)\bar{u}(p_3)\gamma^\mu u(p_1)\cdot(-ie)\bar{u}(p_4)\gamma^\nu u(p_2)(-ig_{\mu\nu})}{4(p_1-p_3)^2}\right|^2 \\
&= \frac{e^4}{4(p_1-p_3)^4}\mathrm{Tr}[(\not{p}_3+m)\gamma^\mu(\not{p}_1)\gamma^\nu]\mathrm{Tr}[(\not{p}_4+m)\gamma_\mu(\not{p}_2)\gamma_\nu] \\
&= \frac{8e^4}{(p_1-p_3)^4}[(p_3p_4)(p_2p_1)+(p_3p_2)(p_4p_1)-m^2(p_4p_2+p_3p_1)+2m^4]\,.
\end{aligned} \tag{7.49}$$

Noting that the contributions from diagrams (a) and (b) differ just by an exchange of p_3 and p_4 allows to write

$$\begin{aligned}\left|\mathcal{M}^{(b)}\right|^2 &= \left|\mathcal{M}^{(a)}\right|^2_{3\leftrightarrow 4} \\ &= \frac{8e^4}{(p_1-p_4)^4}[(p_3p_4)(p_2p_1)+(p_4p_2)(p_3p_1)-m^2(p_3p_2+p_4p_1)+2m^4]\,.\end{aligned} \tag{7.50}$$

What remains to be done is to compute the interference term. Summing and averaging again leads to

$$\begin{aligned}&\left|\mathcal{M}^{(a)}\overline{\mathcal{M}}^{(b)}\right| \\ &= \left|-\frac{e^4\bar{u}(p_3)\gamma^\mu u(p_1)\cdot\bar{u}(p_4)\gamma_\mu u(p_2)\bar{u}(p_1)\gamma^\nu u(p_4)\cdot\bar{u}(p_2)\gamma_\nu u(p_3)}{4(p_1-p_3)^2(p_1-p_4)^2}\right| \\ &= -\frac{e^4\mathrm{Tr}[(\not{p}_3+m)\gamma^\mu(\not{p}_1+m)\gamma^\nu(\not{p}_4+m)\gamma_\mu(\not{p}_2+m)\gamma_\nu]}{4(p_1-p_3)^2(p_1-p_4)^2} \\ &= -\frac{e^4\mathrm{Tr}[(\not{p}_3+m)(-2m^2\gamma^\nu+4mp_1^\nu+4mp_4^\nu-2\not{p}_4\gamma^\nu\not{p}_1)(\not{p}_2+m)\gamma_\nu]}{4(p_1-p_3)^2(p_1-p_4)^2} \\ &= -\frac{e^4}{4(p_1-p_3)^2(p_1-p_4)^2}\Big\{-\mathrm{Tr}[8m^4]-\mathrm{Tr}[8\not{p}_3\not{p}_4(p_1p_2)] \\ &\qquad +\mathrm{Tr}[4m^2(\not{p}_1\not{p}_2+\not{p}_1\not{p}_3+\not{p}_1\not{p}_4+\not{p}_2\not{p}_3+\not{p}_2\not{p}_4+\not{p}_3\not{p}_4)]\Big\} \\ &= -\frac{4e^4\,[4m^2(p_1p_2)-2(p_1p_2)^2]}{(p_1-p_3)^2(p_1-p_4)^2}\end{aligned} \tag{7.51}$$

where in intermediate steps the identities of γ matrices from Eq.(A.4) for $D=4$ as well as the identities $p_1+p_2=p_3+p_4$ and $p_3p_4=p_1p_2$ stemming from total four momentum conservation and $p_i^2=m^2$ stemming from the on-shell condition on external particles have been employed.
Taking the ultrarelativistic limit, i.e. $m\to 0$, one can fix the four momenta in the centre-of mass frame as

$$p_{1,2}=(E,\,0,\,0,\,\pm E)^T \quad\text{and}\quad p_{3,4}=(E,\,\pm E\sin\theta,\,0,\,\pm E\cos\theta)^T \tag{7.52}$$

and compute the transition matrix element,

$$\left|\tilde{\mathcal{S}}_{fi}\right|^2 = 8e^4\left[\frac{(p_1p_2)^2+(p_1p_4)^2}{(p_1p_3)^2}+\frac{(p_1p_2)^2+(p_1p_3)^2}{(p_1p_4)^2}+\frac{2(p_1p_2)^2}{(p_1p_3)(p_1p_4)}\right]$$

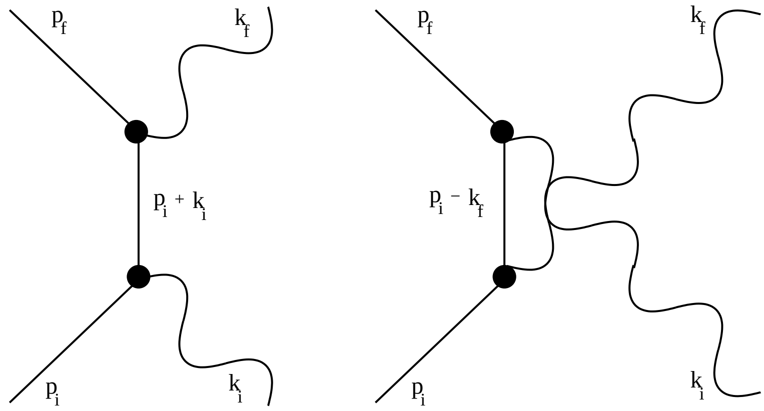


Figure 7.3: The Feynman diagrams for Compton-scattering.

$$
\begin{aligned}
&= 8e^4\left[\frac{4+(1-\cos\theta)^2}{(1+\cos\theta)^2}+\frac{4+(1+\cos\theta)^2}{(1-\cos\theta)^2}+\frac{8}{1-\cos^2\theta}\right]\\
&= 8e^4\left[\frac{1+\sin^4\frac{\theta}{2}}{\cos^4\frac{\theta}{2}}+\frac{1+\cos^4\frac{\theta}{2}}{\sin^4\frac{\theta}{2}}+\frac{2}{\cos^2\frac{\theta}{2}\sin^2\frac{\theta}{2}}\right]. \qquad (7.53)
\end{aligned}
$$

From this, the same manipulations as above yield the differential cross section, the high-energy limit of the Moller-formula

$$
\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{8E^2}\left[\frac{1+\cos^4\frac{\theta}{2}}{\sin^4\frac{\theta}{2}}+\frac{1+\sin^4\frac{\theta}{2}}{\cos^4\frac{\theta}{2}}+\frac{2}{\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}}\right]. \qquad (7.54)
$$

7.1.4 Compton scattering

Amplitudes

Consider now Compton scattering, i.e. the absorption and reemission of a photon when colliding with an electron, or $e^-(p_i)+\gamma(k_i)\to e^-(p_f)+\gamma(k_f)$ There are two Feynman diagrams that contribute, cf. Fig.7.3, giving rise to two amplitudes, namely

$$
\begin{aligned}
\mathcal{M}^{(a)} &= \bar{u}(p_f)(-ie\gamma^\mu)\frac{i(\not{p}_i+\not{k}_i+m)}{(p_i+k_i)^2-m^2}(-ie\gamma^\nu)u(p_i)\epsilon_\nu(k_i)\epsilon^*_\mu(k_f)\\
\mathcal{M}^{(b)} &= \bar{u}(p_f)(-ie\gamma^\nu)\frac{i(\not{p}_i-\not{k}_f+m)}{(p_i-k_f)^2-m^2}(-ie\gamma^\mu)u(p_i)\epsilon_\nu(k_i)\epsilon^*_\mu(k_f)
\end{aligned}
\qquad (7.55)
$$

In a first step recall the Dirac equation of motion leading to the replacement

$$(\not{p}_i + m)\gamma^\nu u(p_i) = \left[2p_i^\nu - \gamma^\nu(\not{p}_i - m)\right] u(p_i) = 2p_i^\nu u(p_i) \,. \tag{7.56}$$

Using also the on-shell condition for the external particles, i.e. $p_i^2 = p_f^2 = m^2$ and $k_i^2 = k_f^2 = 0$ one finds

$$\begin{aligned} \mathcal{M}^{(a)} &= -ie^2\,\bar{u}(p_f)\left[\frac{\not{\epsilon}_f[2p_i\epsilon_i + \not{k}_i\not{\epsilon}_i]}{2p_ik_i}\right]u(p_i) = -ie^2\,\bar{u}(p_f)\left[\frac{\not{\epsilon}_f\not{\epsilon}_i\not{k}_i}{2p_ik_i}\right]u(p_i) \\ \mathcal{M}^{(b)} &= ie^2\,\bar{u}(p_f)\left[\frac{\not{\epsilon}_i[2p_i\epsilon_f - \not{k}_f\not{\epsilon}_f]}{2p_ik_f}\right]u(p_i) = -ie^2\,\bar{u}(p_f)\left[\frac{\not{\epsilon}_i\not{\epsilon}_f\not{k}_f}{2p_ik_f}\right]u(p_i)\,, \end{aligned} \tag{7.57}$$

where the polarisations are assumed to be real have been identified as

$$\epsilon^\mu(k_i) = \epsilon_i^\mu \quad \text{and} \quad \epsilon^{\mu *}(k_f) = \epsilon_f^\mu \,. \tag{7.58}$$

and have been chosen such that they are orthogonal to the “time”-axis p_i, i.e. they satisfy

$$\begin{aligned} \epsilon_ik_i &= \epsilon_ip_i = \epsilon_fk_f = \epsilon_fp_i = 0 \\ \epsilon_i^2 &= \epsilon_f^2 = -1\,. \end{aligned} \tag{7.59}$$

This translates basically into working in a physical gauge like the Coulomb gauge, and one can then assume the initial electron to be at rest, waiting for the incoming photon to scatter off it. From momentum conservation one also finds

$$p_f = p_i + k_i - k_f \quad \longrightarrow \quad p_f\cdot\epsilon_f = k_i\cdot\epsilon_f\,. \tag{7.60}$$

Note that the amplitude is symmetric under the exchange of

$$\epsilon_i,\, k_i \leftrightarrow \epsilon_f,\, -k_f\,. \tag{7.61}$$

In fact this symmetry is a manifestation of a more general one, called “crossing symmetry”, which plays a fundamental role in the theory of the S matrix. Squaring and summing and averaging over the electron polarisations yields

$$\left|\tilde{\mathcal{S}}_{fi}\right|^2 = \frac{e^4}{2}\mathrm{Tr}\left[(\not{p}_f + m)\left(\frac{\not{\epsilon}_f\not{\epsilon}_i\not{k}_i}{2p_ik_i} + \frac{\not{\epsilon}_i\not{\epsilon}_f\not{k}_f}{2p_ik_f}\right)(\not{p}_i + m)\left(\frac{\not{k}_i\not{\epsilon}_i\not{\epsilon}_f}{2p_ik_i} + \frac{\not{k}_f\not{\epsilon}_f\not{\epsilon}_i}{2p_ik_f}\right)\right]\,. \tag{7.62}$$

Using the symmetry relation of Eq.(7.61) there are two cumbersome terms that remain to be computed, namely

$$
\begin{aligned}
T_1 &= \mathrm{Tr}\left[\not\epsilon_f \not\epsilon_i \not k_i (\not p_i + m) \not k_i \not\epsilon_i \not\epsilon_f (\not p_f + m)\right] \\
&= \mathrm{Tr}\left[\not\epsilon_f \not\epsilon_i \not k_i \not p_i \not k_i \not\epsilon_i \not\epsilon_f \not p_f\right] = 2p_i k_i \mathrm{Tr}\left[\not\epsilon_f \not\epsilon_i \not k_i \not\epsilon_i \not\epsilon_f \not p_f\right] = 2p_i k_i \mathrm{Tr}\left[\not\epsilon_f \not k_i \not\epsilon_f \not p_f\right] \\
&= 8p_i k_i \left[2(\epsilon_f k_i)(\epsilon_f p_f) + (k_i p_f)\right] = 8p_i k_i \left[2(\epsilon_f k_i)^2 + (k_i p_f)\right] , \qquad (7.63)
\end{aligned}
$$

where the terms $\sim m$ disappear due to $\not k_i m \not k_i = m k_i^2 = 0$. The second term reads

$$
\begin{aligned}
T_2 &= \mathrm{Tr}\left[\not\epsilon_f \not\epsilon_i \not k_i (\not p_i + m) \not k_f \not\epsilon_f \not\epsilon_i (\not p_f + m)\right] \\
&= \mathrm{Tr}\left[\not\epsilon_f \not\epsilon_i \not k_i (\not p_i + m) \not k_f \not\epsilon_f \not\epsilon_i (\not p_i + m)\right] + \mathrm{Tr}\left[\not\epsilon_f \not\epsilon_i \not k_i (\not p_i + m) \not k_f \not\epsilon_f \not\epsilon_i (\not k_i - \not k_f)\right] \\
&= \mathrm{Tr}\left[(\not p_i + m) \not k_i (\not p_i + m) \not k_f \not\epsilon_f \not\epsilon_i \not\epsilon_f \not\epsilon_i\right] \\
&\quad -2(k_i \epsilon_f)\mathrm{Tr}\left[\not k_i \not p_i \not k_f \not\epsilon_f\right] + 2(k_f \epsilon_i)\mathrm{Tr}\left[\not\epsilon_i \not k_i \not p_i \not k_f\right] \\
&= 2(k_i p_i)\mathrm{Tr}\left[\not p_i \not k_f \not\epsilon_f \not\epsilon_i \not\epsilon_f \not\epsilon_i\right] - 8(k_i \epsilon_f)^2 (p_i k_f) + 8(k_f \epsilon_i)^2 (p_i k_i) \\
&= 2(k_i p_i)(k_f p_i)\mathrm{Tr}\left[\not\epsilon_f \not\epsilon_i \not\epsilon_f \not\epsilon_i\right] - 8(k_i \epsilon_f)^2 (p_i k_f) + 8(k_f \epsilon_i)^2 (p_i k_i) \\
&= 8(k_i p_i)(k_f p_i)\left[2(\epsilon_f \epsilon_i)^2 - 1\right] - 8(k_i \epsilon_f)^2 (p_i k_f) + 8(k_f \epsilon_i)^2 (p_i k_i) \qquad (7.64)
\end{aligned}
$$

Generating the other contributions through the symmetry of Eq.(7.61) the result for the matrix element squared reads

$$
\begin{aligned}
\left|\tilde{\mathcal{S}}_{fi}\right|^2 &= \frac{e^4}{2}\left\{\frac{8p_i k_i\left[(k_i p_f) + 2(\epsilon_f k_i)^2\right]}{[2p_i k_i]^2} + \frac{8p_i k_f\left[(k_f p_f) - 2(\epsilon_i k_f)^2\right]}{[2p_i k_f]^2}\right. \\
&\qquad \left. +2\frac{8(k_i p_i)(k_f p_i)\left[2(\epsilon_f \epsilon_i)^2 - 1\right] - 8(k_i \epsilon_f)^2 (p_i k_f) + 8(k_f \epsilon_i)^2 (p_i k_i)}{(2p_i k_i)(2p_i k_f)}\right\} \\
&= (4\pi\alpha)^2\left\{\frac{k_f p_i}{k_i p_i} + \frac{k_i p_i}{k_f p_i} + 4(\epsilon_f \epsilon_i)^2 - 2\right\} , \qquad (7.65)
\end{aligned}
$$

where for the transition to the last line total four-momentum conservation has been employed in the form

$$
p_i - k_f = k_i - p_f \quad \longrightarrow \quad (p_i - k_f)^2 = p_i^2 - 2p_i k_f = p_f^2 - 2p_f k_i = (k_i - p_f)^2 \quad (7.66)
$$

By choosing the lab system to be the one, in which the initial electron is at rest and by identifying the photon energies as $\omega_{i,f}$ one has

$$
\left|\tilde{\mathcal{S}}_{fi}\right|^2 = (4\pi\alpha)^2\left\{\frac{\omega_f}{\omega_i} + \frac{\omega_i}{\omega_f} + 4(\epsilon_f \epsilon_i)^2 - 2\right\} . \qquad (7.67)
$$

This is the Klein-Nishina formula for the cross section of Compton scattering. To continue, assume that the photons are unpolarised. To perform the summation over the polarisations, choose the photons momenta to be

$$k_i = \omega_i(1,\,0,\,0,\,1) \quad \text{and} \quad k_f = \omega_f(1,\,\sin\theta,\,0,\,\cos\theta) \tag{7.68}$$

and the real polarisation vectors

$$\begin{array}{rclrcl} \epsilon_i^{(1)} & = & (0,\,1,\,0,\,0)\,, & \epsilon_i^{(2)} & = & (0,\,0,\,1,\,0)\,, \\ \epsilon_f^{(1)} & = & (0,\,\cos\theta,\,0,\,-\sin\theta)\,, & \epsilon_f^{(2)} & = & (0,\,0,\,1,\,0) \end{array} \tag{7.69}$$

leading to

$$\begin{aligned} &\frac{\omega_f}{\omega_i} + \frac{\omega_i}{\omega_f} + 4(\epsilon_f\epsilon_i)^2 - 2 \longrightarrow \frac{1}{2}\sum_{\lambda,\,\lambda'=1}^{2}\left[\frac{\omega_f}{\omega_i} + \frac{\omega_i}{\omega_f} + 4\left(\epsilon_f^\lambda\epsilon_i^{\lambda'}\right)^2 - 2\right] \\ &= \; 2\left(\frac{\omega_f}{\omega_i} + \frac{\omega_i}{\omega_f}\right) + 2(1+\cos^2\theta) - 4 = 2\left(\frac{\omega_f}{\omega_i} + \frac{\omega_i}{\omega_f}\right) - 2\sin^2\theta \end{aligned} \tag{7.70}$$

and, thus, the unpolarised transition amplitude squared reads

$$\left|\tilde{\mathcal{S}}_{fi}\right|^2 \;=\; 2(4\pi\alpha)^2\left\{\left(\frac{\omega_f}{\omega_i} + \frac{\omega_i}{\omega_f}\right) - \sin^2\theta\right\}. \tag{7.71}$$

Then, the cross section becomes

$$\begin{aligned} \sigma \;=&\; \frac{2(4\pi\alpha)^2}{4m\omega_i}\int\frac{\mathrm{d}^3p_f}{(2\pi)^3 2E_f}\frac{\mathrm{d}^3k_f}{(2\pi)^3 2\omega_f}\left\{\left[\frac{\omega_f}{\omega_i} + \frac{\omega_i}{\omega_f} - \sin^2\theta\right]\right. \\ &\left. (2\pi)^4\delta^4(p_i + k_i - p_f - k_f)\right\} \\ =&\; \frac{8\pi^2\alpha^2}{m\omega_i}\int\frac{\mathrm{d}^3k_f}{(2\pi)^3 2\omega_f}\frac{\mathrm{d}^4p_f}{(2\pi)^4}\theta(E_f)\left\{\left[\frac{\omega_f}{\omega_i} + \frac{\omega_i}{\omega_f} - \sin^2\theta\right]\right. \\ &\left. (2\pi)^4\delta^4(p_i + k_i - p_f - k_f)(2\pi)\delta(p_f^2 - m^2)\right\} \\ =&\; \frac{8\pi^2\alpha^2}{m\omega_i}\int\frac{\mathrm{d}^3k_f}{(2\pi)^3 2\omega_f}\theta(m + \omega_i - \omega_f)\left\{\left[\frac{\omega_f}{\omega_i} + \frac{\omega_i}{\omega_f} - \sin^2\theta\right]\right. \\ &\left. (2\pi)\delta\left[(m + \omega_i - \omega_f)^2 - (\vec{k}_i - \vec{k}_f)^2 - m^2\right]\right\} \end{aligned}$$

$$
\begin{aligned}
&= \frac{8\pi^2\alpha^2}{m\omega_i}\int_0^{m+\omega_i}\frac{\omega_f^2\mathrm{d}\omega_f}{4\pi\omega_f}\int_{4\pi}\frac{\mathrm{d}\cos\theta\mathrm{d}\phi}{(2\pi)^2}\left\{\left[\frac{\omega_f}{\omega_i}+\frac{\omega_i}{\omega_f}-\sin^2\theta\right]\right. \\
&\qquad\qquad \left.(2\pi)\delta\left[2m(\omega_i-\omega_f)-2\omega_i\omega_f(1-\cos\theta)\right]\right\} \\
&= \frac{2\pi\alpha^2}{m\omega_i}\int_{-1}^{1}\mathrm{d}z\left\{\frac{\omega_i}{2m\left[1+\omega_i/m(1-z)\right]^2}\right. \\
&\qquad\qquad \left.\left[\frac{1}{1+\omega_i/m(1-z)}+1+\omega_i/m(1-z)-(1-z^2)\right]\right\} \\
&= \frac{\pi\alpha^2}{m^2}\int_{-1}^{1}\mathrm{d}z\left\{\frac{1}{[1+\epsilon(1-z)]^3}+\frac{1}{1+\epsilon(1-z)}-\frac{1-z^2}{[1+\epsilon(1-z)]^2}\right\},
\end{aligned}
\tag{7.72}
$$

where $\epsilon=\omega_i/m$. In the low-energy limit, i.e. for $\epsilon\to 0$ this yields the Thomson cross section

$$
\sigma_0 \approx \frac{\pi\alpha^2}{m^2}\int_{-1}^{1}\mathrm{d}z(1+z^2)=\frac{8\pi\alpha^2}{3m^2}\,. \tag{7.73}
$$

In the high-energy limit one finds

$$
\sigma \approx \frac{\pi\alpha^2}{m\omega_i}\int_{-1}^{1}\mathrm{d}z\frac{1}{\frac{m}{\omega_i}+(1-z)}=\frac{\pi\alpha^2}{m\omega_i}\log\frac{2\omega_i}{m}\,. \tag{7.74}
$$

High-energy limit

Consider once more the amplitudes for Compton scattering; this time, however, the calculation will be performed in the Lorentz gauge. Summing and squaring the amplitudes and thereby summing and averaging over outgoing and incoming polarisations one uses

$$
\sum\left[\epsilon_\nu(k_i)\epsilon^*_{\nu'}(k_i)\right]=-g_{\nu\nu'}\,. \tag{7.75}
$$

In the high-energy limit one may safely neglect the masses, and one finds

$$
\left|\mathcal{M}^{(a)}\right|^2 = \frac{e^4\mathrm{Tr}\left[\not p_f\gamma^\mu(\not p_i+\not k_i)\gamma^\nu\not p_i\gamma_\nu(\not p_i+\not k_i)\gamma_\mu\right]}{16(p_ik_i)^2}
$$

$$
\begin{aligned}
&= \frac{(-2)^2 e^4 \mathrm{Tr}\,[\not{p}_f(\not{p}_i+\not{k}_i)\not{p}_i(\not{p}_i+\not{k}_i)]}{16(p_ik_i)^2} \\
&= \frac{4e^4\,[8(p_fp_i+p_fk_i)(p_ip_i+p_ik_i)-4(p_ip_f)(p_i+k_i)^2]}{16(p_ik_i)^2} \\
&= \frac{2e^4(p_fk_i)}{(p_ik_i)} \\
\left|\mathcal{M}^{(b)}\right|^2 &= \frac{1}{4}\frac{e^4\mathrm{Tr}\,[\not{p}_f\gamma^\nu(\not{p}_i-\not{k}_f)\gamma^\mu\not{p}_i\gamma_\mu(\not{p}_i-\not{k}_f)\gamma_\nu]}{(2p_ik_f)^2} \\
&= \frac{(-2)^2 e^4 \mathrm{Tr}\,[\not{p}_f(\not{p}_i-\not{k}_f)\not{p}_i(\not{p}_i-\not{k}_f)]}{16(p_ik_f)^2} \\
&= \frac{4e^4\,[8(p_fp_i-p_fk_f)(p_ip_i-p_ik_f)-4(p_ip_f)(p_i-k_f)^2]}{16(p_ik_f)^2} \\
&= \frac{2e^4(p_fk_f)}{(p_ik_f)}\,. \\
\left|\mathcal{M}^{(a)}\overline{\mathcal{M}}^{(b)}\right| &= \frac{1}{4}\frac{e^4\mathrm{Tr}\,[\not{p}_f\gamma^\mu(\not{p}_i+\not{k}_i)\gamma^\nu\not{p}_i\gamma_\mu(\not{p}_i-\not{k}_f)\gamma_\nu]}{(2p_ik_f)(2p_ik_i)} \\
&= \frac{-8e^4(p_fp_i)\mathrm{Tr}\,[(\not{p}_i+\not{k}_i)(\not{p}_i-\not{k}_f)]}{16(p_ik_f)(p_ik_i)} \\
&= -\frac{2e^4(p_fp_i)(p_ik_i-p_ik_f-k_ik_f)}{(p_ik_f)(p_ik_i)}\,. \qquad (7.76)
\end{aligned}
$$

Taken together one finds

$$
\left|\tilde{\mathcal{S}}_{fi}\right|^2 = 2e^4\left[\frac{(p_fk_i)}{(p_ik_i)}+\frac{(p_fk_f)}{(p_ik_f)}-2\frac{(p_fp_i)(p_ik_i-p_ik_f-k_ik_f)}{(p_ik_f)(p_ik_i)}\right], \qquad (7.77)
$$

where $p_f = p_i + k_i - k_f$ has been used.

Mandelstam variables

Obviously, scalar products of four momenta involved in scattering processes are Lorentz invariants. For a $2 \to 2$ process, they are called Mandelstam variables and are given by

$$
\begin{aligned}
s &= (p_1+p_2)^2 = (p_3+p_4)^2 = E^2_{\mathrm{c.m.}} \\
t &= (p_1-p_3)^2 = (p_2-p_4)^2 \\
u &= (p_1-p_4)^2 = (p_2-p_3)^2\,, \qquad (7.78)
\end{aligned}
$$

where $p_{1,2}$ are incoming and $p_{3,4}$ are outgoing momenta and $E_{\text{c.m.}}$ is the total centre-of mass energy involved in the process. It is easy to show that

$$s + t + u = p_1^2 + p_2^2 + p_3^2 + p_4^2 \,. \tag{7.79}$$

For massless particles one finds

$$\begin{aligned} s &= 2p_1p_2 = 2p_3p_4 \\ t &= -2p_1p_3 = -2p_2p_4 = -\frac{s}{2}(1 - \cos\theta_{\text{c.m.}}) \\ u &= -2p_1p_4 = -2p_2p_3 = -\frac{s}{2}(1 + \cos\theta_{\text{c.m.}})\,, \end{aligned} \tag{7.80}$$

where $\theta_{\text{c.m.}}$ is the scattering angle between particles 1 and 3 in the centre-of mass frame. In this frame, the solid angle is given by

$$\mathrm{d}^2\Omega_{\text{c.m.}} = \mathrm{d}\cos\theta_{\text{c.m.}}\mathrm{d}\phi_{\text{c.m.}}\,. \tag{7.81}$$

Using the Mandelstam variables, the matrix element squared reads

$$\begin{aligned} \left|\tilde{\mathcal{S}}_{fi}\right|^2 &= 2e^4\left[\frac{(p_fk_i)}{(p_ik_i)} + \frac{(p_fk_f)}{(p_ik_f)} - 2\frac{(p_fp_i)(p_ik_i - p_ik_f - k_ik_f)}{(p_ik_f)(p_ik_i)}\right] \\ &= -32\pi^2\alpha^2\left[\frac{u}{s} + \frac{s}{u} + \frac{2t(s+u+t)}{su}\right] = -32\pi^2\alpha^2\left[\frac{s}{u} + \frac{u}{s}\right]\,. \end{aligned} \tag{7.82}$$

In Mandelstam variables, the cross section can be expressed as

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{16\pi\lambda(s,\, m_1^2,\, m_2^2)}\left|\tilde{\mathcal{S}}_{fi}(s,\, t)\right|^2\,, \tag{7.83}$$

where $m_{1,\,2}$ are the masses of the incoming particles and

$$\lambda(s,\, m_1^2,\, m_2^2) \;=\; [s - (m_1 + m_2)^2][s - (m_1 - m_2)^2]\,. \tag{7.84}$$

It should be also noted, that for total cross sections of massless particles, the range of t, the four-momentum transfer squared, equals $t \in [0,\, -s]$.

7.1.5 Pair annihilation and creation

Result for pair annihilation

As the next process, consider pair annihilation of an electron positron pair into a pair of photons, $e^-(p_1)e^+(p_2) \to \gamma(k_1)\gamma(k_2)$. Assuming all particles to be massless, equivalent to the high-energy limit, the two amplitudes read

$$\begin{aligned}
\mathcal{M}^{(a)} &= \bar{v}(p_2)(-ie\gamma^\mu)\frac{i(\not{p}_1 - \not{k}_1)}{(p_1-k_1)^2}(-ie\gamma^\nu)u(p_1)\epsilon_\nu(k_1)\epsilon_\mu(k_2) \\
\mathcal{M}^{(a)} &= \bar{v}(p_2)(-ie\gamma^\nu)\frac{i(\not{p}_1 - \not{k}_2)}{(p_1-k_2)^2}(-ie\gamma^\mu)u(p_1)\epsilon_\nu(k_1)\epsilon_\mu(k_2)
\end{aligned} \tag{7.85}$$

Summing and squaring the amplitudes and summing and averaging over polarisations yields

$$\begin{aligned}
\left|\mathcal{M}^{(a)}\right|^2 &= \frac{1}{4}\frac{e^4\mathrm{Tr}\left[\not{p}_2\gamma^\mu(\not{p}_1-\not{k}_1)\gamma^\nu\not{p}_1\gamma_\nu(\not{p}_1-\not{k}_1)\gamma_\mu\right]}{(-2p_1k_1)^2} \\
&= \frac{2e^4(p_2k_1)}{(p_1k_1)} = 2e^4\frac{u}{t} \\
\left|\mathcal{M}^{(b)}\right|^2 &= \frac{1}{4}\frac{e^4\mathrm{Tr}\left[\not{p}_2\gamma^\nu(\not{p}_1-\not{k}_2)\gamma^\mu\not{p}_1\gamma_\mu(\not{p}_1-\not{k}_2)\gamma_\nu\right]}{(-2p_1k_2)^2} \\
&= \frac{2e^4(p_2k_2)}{(p_1k_2)} = 2e^4\frac{t}{u} \\
\left|\mathcal{M}^{(a)}\overline{\mathcal{M}}^{(b)}\right| &= \frac{1}{4}\frac{e^4\mathrm{Tr}\left[\not{p}_2\gamma^\mu(\not{p}_1-\not{k}_1)\gamma^\nu\not{p}_1\gamma_\mu(\not{p}_1-\not{k}_2)\gamma_\nu\right]}{(-2p_1k_1)(-2p_1k_2)} \\
&= \frac{-2e^4(p_1p_2)(k_1k_2-p_1k_1-p_1k_2)}{(p_1k_1)(p_1k_2)} = -2e^4\frac{s(s+t+u)}{ut} = 0\,.
\end{aligned} \tag{7.86}$$

Then, the total result for the transition amplitude squared reads

$$\left|\tilde{\mathcal{S}}_{fi}\right|^2 = 2e^4\left[\frac{u}{t}+\frac{t}{u}\right] = 32\pi^2\alpha^2\left[\frac{u}{t}+\frac{t}{u}\right]\,. \tag{7.87}$$

Pair creation

Finally, consider the reversed process, i.e. the creation of an electron-positron pair from two photons, $\gamma(k_1)\gamma(k_2) \to e^-(p_1)e^+(p_2)$. The amplitudes read

$$
\begin{aligned}
\mathcal{M}^{(a)} &= \bar{u}(p_1)(-ie\gamma^\mu)\frac{i(\not{p}_1 - \not{k}_1)}{(p_1-k_1)^2}(-ie\gamma^\nu)v(p_2)\epsilon^*_\nu(k_1)\epsilon^*_\mu(k_2)\\
\mathcal{M}^{(a)} &= \bar{u}(p_1)(-ie\gamma^\nu)\frac{i(\not{p}_1 - \not{k}_2)}{(p_1-k_2)^2}(-ie\gamma^\mu)v(p_2)\epsilon^*_\nu(k_1)\epsilon^*_\mu(k_2)
\end{aligned}
\tag{7.88}
$$

and emerge from the ones for pair annihilation by “barring” them completely. Therefore and because of the identical numbers of incoming polarisations to sum over, it does not come as a big surprise that the total amplitudes are identical as functions of the Mandelstam variables, i.e.

$$
\left|\tilde{\mathcal{S}}_{fi}\right|^2 = 2e^4\left[\frac{u}{t}+\frac{t}{u}\right] = 32\pi^2\alpha^2\left[\frac{u}{t}+\frac{t}{u}\right]. \tag{7.89}
$$

In fact, this is another manifestation of the property of crossing valid for the S-matrix. To put it more general: If there is a number of processes, which are related to each other by operations

$$
f_1f_2 \to f_3f_4 \quad \leftrightarrow \quad f_1\bar{f}_4 \to f_3\bar{f}_2 \tag{7.90}
$$

and the like, then the functional form of their total amplitudes squared as functions of the Mandelstam variables can be related, up to constant factors that emerge from different numbers of degrees of freedom to sum over.

7.2 QCD

7.2.1 Feynman rules

The Lagrangian of QCD can be decomposed in three parts as

$$
\mathcal{L}^{\rm QCD} = \mathcal{L}^{\rm classical} + \mathcal{L}^{\rm gaugefixing} + \mathcal{L}^{\rm ghosts}\,, \tag{7.91}
$$

where the classical piece reads

$$
\mathcal{L}^{\rm classical} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a + \sum_{\rm flavours}\bar{q}_i\left(i\not{\partial}\delta_{ab} + igT^a_{ij}\not{A}^a\right)q_j \tag{7.92}
$$

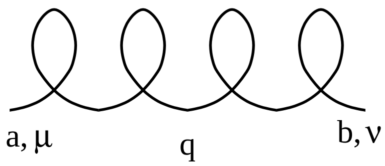
and the gauge-fixing and ghost term are given by

$$\mathcal{L}^{\text{gaugefixing}} = -\frac{1}{2\lambda}(\partial^\mu A^a_\mu)^2 \tag{7.93}$$

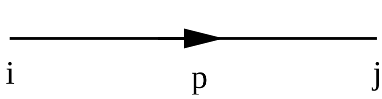
and

$$\mathcal{L}^{\text{ghosts}} = \partial_\mu \eta^\dagger_a \left(\partial^\mu \delta_{ab} - igf^{abc} A^\mu_b\right) \eta_c \,, \tag{7.94}$$

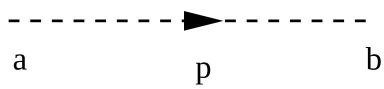
respectively. Here, and in the following, the Lorentz-indices are denoted by Greek letters μ, ν, …, whereas the colour indices are labelled by roman letters i, j and so on for colour indices in the fundamental representation and by a, b, and so on for colour indices in the adjoint representation. Then, in the gauge $\lambda = 1$, the propagators for the gluons, quarks and for the ghost fields are given by



$$= \frac{-i\delta^{ab} g^\mu \nu}{q^2} \tag{7.95}$$

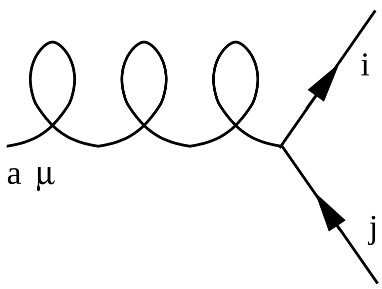


$$= \frac{i\delta^{ij}(\not{p} + m)}{p^2 - m^2} \tag{7.96}$$

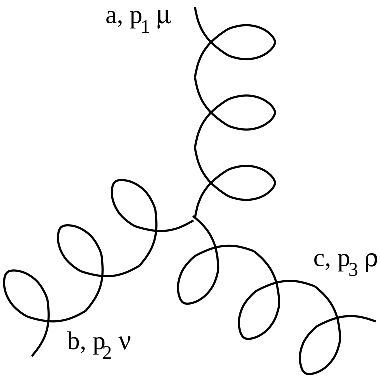


$$= \frac{i\delta^{ab}}{p^2} \tag{7.97}$$

The vertices are

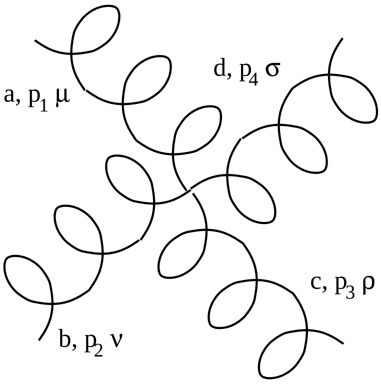


$$= -igT^a_{ij}\gamma^\mu \tag{7.98}$$

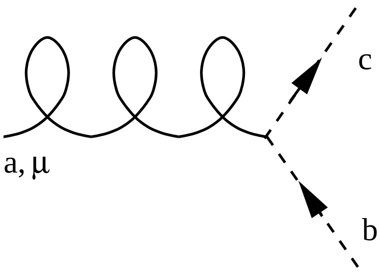


$$= \quad -gf^{abc}\begin{bmatrix} (p_1-p_2)^\rho g^{\mu\nu} \\ +(p_2-p_3)^\mu g^{\nu\rho} \\ +(p_3-p_1)^\nu g^{\rho\mu} \end{bmatrix} \tag{7.99}$$

(all momenta incoming)



$$= \quad -ig^2\begin{bmatrix} f^{xab}f^{xcd}\,(g^{\mu\rho}g^{\nu\sigma}-g^{\mu\sigma}g^{\nu\rho}) \\ +f^{xac}f^{xbd}\,(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\sigma}g^{\rho\nu}) \\ +f^{xad}f^{xbc}\,(g^{\mu\rho}g^{\nu\sigma}-g^{\mu\sigma}g^{\nu\rho}) \end{bmatrix} \tag{7.100}$$



$$= \quad gf^{abc}q^\mu\,. \tag{7.101}$$

7.2.2 Hadronic cross section in e^+e^- collisions

Neglecting for a moment the contribution of intermediate Z bosons, the transition amplitude for the production of a quark-antiquark pair in electron-positron annihilation, $e^-(p_1)e^+(p_2) \to q(p_3)\bar{q}(p_4)$, reads

$$\mathcal{S}_{fi} = \bar{u}(p_4)\,(-iee_q\gamma_\mu)\,v(p_3)\frac{-ig^{\mu\nu}}{(p_1+p_2)^2}\bar{v}(p_2)\,(-ie\gamma_\nu)\,u(p_1)\,, \tag{7.102}$$

where e_q denotes the fractional charge of the quark, i.e. either $2/3$ or $-1/3$ for up- or down-type quarks, respectively. Squaring and summing or averaging over polarizations and colours yields

$$\begin{aligned}
|\mathcal{S}_{fi}|^2 = & \frac{N_c e^4 e_q^2}{4s^2}\mathrm{Tr}\left[(\not{p}_4 + m_q)\gamma^\mu(\not{p}_3 - m_q)\gamma^\nu\right]\mathrm{Tr}\left[\not{p}_2\gamma_\mu\not{p}_1\gamma_\nu\right] \\
= & \frac{64N_c\pi^2\alpha^2 e_q^2}{s^2}\left[p_4^\mu p_3^\nu + p_3^\mu p_4^\nu - g^{\mu\nu}(p_3p_4 + m_q^2)\right]\left[p_2^\mu p_1^\nu + p_1^\mu p_2^\nu - g^{\mu\nu}p_1p_2\right] \\
= & \frac{64N_c\pi^2\alpha^2 e_q^2}{s^2}\left[2(p_3p_2)(p_4p_1) + 2(p_4p_2)(p_3p_1) + 2m_q^2(p_1p_2)\right] \\
= & \frac{64N_c\pi^2\alpha^2 e_q^2}{s^2}\left[\frac{s(s+2m_q^2)}{2} - 4(p_3p_2)(p_3p_1)\right], \qquad (7.103)
\end{aligned}$$

where the electron mass has been neglected and total four-momentum conservation has been employed. The number of colours for the quark-antiquark pair is given by $N_c = 3$. The cross section then reads

$$\begin{aligned}
\mathrm{d}\sigma = & \frac{(2\pi)^4\delta^4(p_1+p_2-p_3-p_4)\,|\mathcal{S}_{fi}|^2}{4p_1p_2}\frac{\mathrm{d}^3p_3}{(2\pi)^3 2E_3}\frac{\mathrm{d}^3p_4}{(2\pi)^3 2E_4} \\
= & \frac{64N_c\pi^2\alpha^2 e_q^2}{2s\cdot s^2}\left[\frac{s(s+2m_q^2)}{2} - 4(p_3p_2)(p_3p_1)\right]\frac{\mathrm{d}^3p_3}{(2\pi)^3 2E_3} \\
& \cdot(2\pi)\delta[(p_1+p_2-p_3)^2 - m_q^2]\theta(E_1+E_2-E_3)\,. \qquad (7.104)
\end{aligned}$$

Fixing the kinematics to be in the centre-of mass system of the incoming particles, one has

$$\begin{aligned}
p_{1,2} = & E(1,\,0,\,0,\,\pm 1)^T \quad \longrightarrow \quad s = 4E^2 \\
p_{3,4} = & (E, \pm\vec{p}) \quad \text{with } \vec{p} = \sqrt{E^2 - m_q^2}(\sin\theta,\,0,\,\cos\theta)^T \qquad (7.105)
\end{aligned}$$

leading to

$$\begin{aligned}
\mathrm{d}\sigma = & \frac{8N_c\pi\alpha^2 e_q^2}{s^2}\left[\frac{s+2m_q^2}{2} - (E_3^2 - (E_3^2 - m_q^2)\cos^2\theta)\right] \\
& \cdot\mathrm{d}\cos\theta\sqrt{E_3^2 - m_q^2}\,\mathrm{d}E_3\,\delta[(p_1+p_2-p_3)^2 - m_q^2]\theta\left(\frac{\sqrt{s}}{2} - E_3\right) \\
= & \frac{N_c\pi\alpha^2 e_q^2}{2s}\sqrt{1 - \frac{4m_q^2}{s}}\left[\left(1 + \frac{4m_q^2}{s}\right) + \left(1 - \frac{4m_q^2}{s}\right)\cos^2\theta\right]\mathrm{d}\cos\theta\,, \\
& \qquad (7.106)
\end{aligned}$$

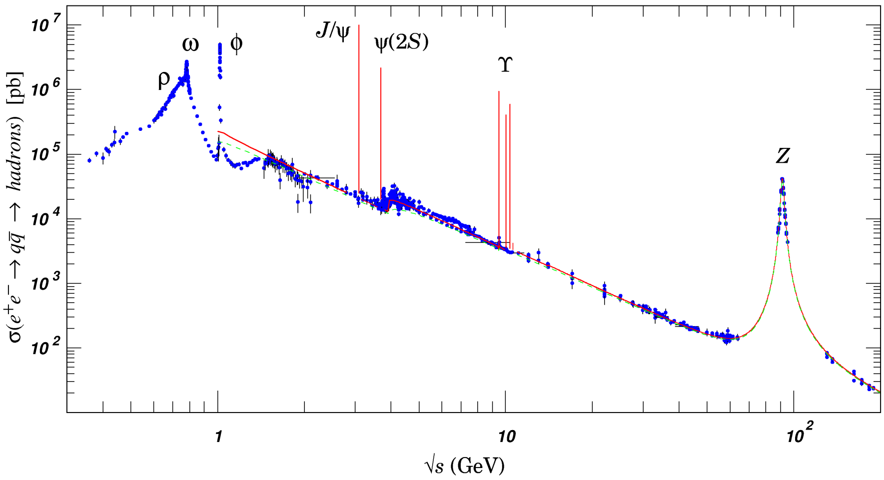


Figure 7.4: The total hadronic cross section, i.e. for e^+e^- →hadrons as a function of the centre-of mass energy. Note the step at roughly 3 GeV hinting at a new quark kicking in with mass of roughly 1.5 GeV. Cross sections are usually measured in barns; a barn is defined as $1\text{barn} = 10^{-28}\text{m}^2 \approx 0.000389379\text{GeV}^{-2}$.

where

$$E_3 \mathrm{d}E_3 = |\vec{p}_3|\,\mathrm{d}\,|\vec{p}_3| \tag{7.107}$$

and

$$\delta[(p_1+p_2-p_3)^2 - m_q^2] = \delta(4E^2 - 4EE_3) \to \frac{1}{2\sqrt{s}}\delta\left(E_3 - \frac{\sqrt{s}}{2}\right) \tag{7.108}$$

has been used. Integrating over the angle yields the total cross section for the production of a quark-antiquark pair in electron-positron annihilation, when only photons are present as intermediate particles,

$$\sigma^\gamma_{q\bar{q}} \;\; = \;\; \frac{N_c\pi\alpha^2 e_q^2}{2s}\cdot\beta\left[2\beta^2 + \frac{2}{3}\beta^2\right], \tag{7.109}$$

where $\beta = \sqrt{1-4m_q^2/s}$ is the quark velocity. A useful test concerning the number of colours and the masses of the quarks consists of taking the ratio

$$\mathcal{R}_{\text{had.}} = \frac{\sigma_{e^+e^-\to\text{hadrons}}(s)}{\sigma_{e^+e^-\to\mu^+\mu^-}(s)} \approx \frac{\sum\limits_q \sigma^\gamma_{q\bar{q}}(s)}{\sigma^\gamma_{e^+e^-\to\mu^+\mu^-}(s)}, \tag{7.110}$$

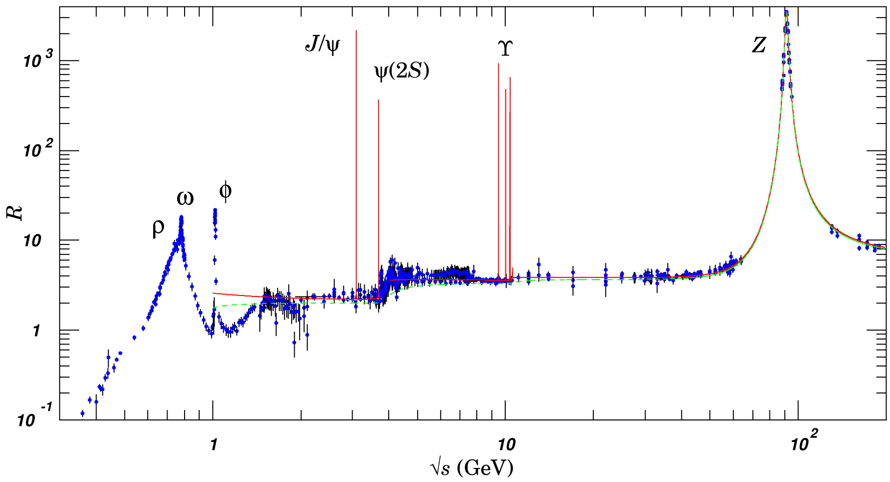


Figure 7.5: The ratio of the total hadronic cross section to the total cross section for muon production in electron-positron annihilation, $\mathcal{R}_{\text{had.}}$, as a function of the centre-of mass energy. Again, steps at different energies signal that new, heavier quarks enter the game.

where the sum is over all quakrs kinematically accessible. It should be noted that only in the region of relatively small centre-of mass energies (i.e. for $s \ll M_Z^2$), the cross sections above are valid approximations. Also, there are quark-antiquark bound states, mesons, with quantum numbers such that the photon can fluctuate into them directly. Examples include, for instance, the ρ- or the J/ψ-meson. Since such QCD bound states are beyond the reach of perturbation theory, for the purpose of analysis one should stay well away from these resonances. However, the total hadronic cross section in electron-positron annihilation and the ratio $\mathcal{R}_{\text{had.}}$ are depicted in Figs.7.4 and 7.5,respectively.

7.2.3 QCD production of heavy quarks

There are two basic processes which contribute to the purely QCD production of heavy quarks, such as the top-quark, namely

$$q\bar{q} \to Q\bar{Q} \quad \text{and} \quad gg \to Q\bar{Q}\,. \tag{7.111}$$

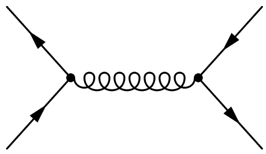


Figure 7.6: Production of a heavy quark pair through light quarks

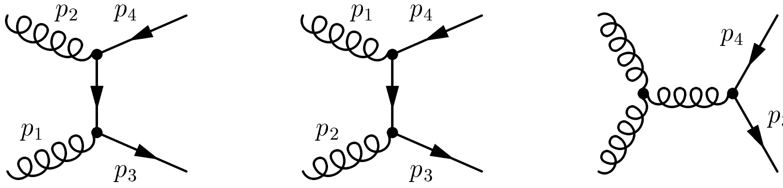


Figure 7.7: Production of a heavy quark pair through gluons. note the difference between the two "Abelian" graphs, the gluons are interchanged,

They are depicted in Figs.7.6 and gg2QQ,respectively.

Amplitudes for quark-induced heavy quark production

Starting with the first process of Eq.(7.111), the transition amplitude for $q(p_1)\bar{q}(p_2) \to Q(p_3)\bar{Q}(p_4)$, with the incoming quarks being massless, reads

$$\mathcal{S}_{fi} = \bar{u}_i(p_4)\left(-ig_s T^a_{ij}\gamma_\mu\right) v_j(p_3)\frac{-i\delta_{ab}g^{\mu\nu}}{(p_1+p_2)^2}\bar{v}_k(p_2)\left(-ig_s T^b_{kl}\gamma_\nu\right) u_l(p_1)\,, \tag{7.112}$$

where the i, j, k, and l denote the colour indices of the fundamental and a and b the colour indices in the adjoint representation, respectively. Simple inspection shows that one of the new features in the computation of pure QCD processes lies in the evaluation of expressions involving the generators of QCD, the T^a. Important properties of the generators of the fundamental and adjoint representation can be found in the appendix.
From them, one has

$$T^a_{ij}T^b_{kl}\delta_{ab} \;=\; T^a_{ij}T^a_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{jk} - \frac{1}{3}\delta_{ij}\delta_{kl}\right)\,. \tag{7.113}$$

When squaring the transition amplitude, the colour factor, taking care of proper averages becomes

$$
\begin{aligned}
\mathcal{C} &:= \frac{1}{3^2}\sum_{i,j,k,l=1}^{3}\left[\frac{1}{2}\left(\delta_{il}\delta_{jk}-\frac{1}{3}\delta_{ij}\delta_{kl}\right)\right]^2 = \frac{1}{36}\left[\delta_{il}^2\delta_{jk}^2-\frac{2}{3}\delta_{il}\delta_{jk}\delta_{ij}\delta_{kl}+\frac{1}{9}\delta_{ij}^2\delta_{kl}^2\right] \\
&= \frac{1}{36}\left[\delta_{ii}\delta_{jj}-\frac{2}{3}\delta_{ii}+\frac{1}{9}\delta_{ii}\delta_{kk}\right] = \frac{1}{36}\left[9-\frac{2}{3}\cdot 3+\frac{1}{9}\cdot 9\right] = \frac{2}{9}\,. \qquad (7.114)
\end{aligned}
$$

Then, using the result of Eq.(7.103) and replacing the factor N_c by the colour factor above, the transition amplitude squared reads

$$
\begin{aligned}
\left|\mathcal{S}_{fi}^{q\bar{q}\to Q\bar{Q}}\right|^2 &= \frac{2}{9}\,\frac{64\pi^2\alpha_s^2}{s^2}\left[\frac{s(s+2m_q^2)}{2}-4(p_3p_2)(p_3p_1)\right] \\
&= \frac{64\pi^2\alpha_s^2}{9}\,\frac{\left[(t-m^2)^2+(u-m^2)^2+2m^2s\right]}{s^2}\,, \qquad (7.115)
\end{aligned}
$$

where, in full analogy to the electromagnetic fine-structure constant,

$$
\alpha_s = \frac{g_s^2}{4\pi}\,. \qquad (7.116)
$$

For massless quarks in the final state, which are distinct from the initial ones, the transition amplitude squared reduces to the well-known result

$$
\left|\mathcal{S}_{fi}^{q\bar{q}\to q'\bar{q}'}\right|^2 = \frac{64\pi^2\alpha_s^2}{9s^2}\left[s^2-2tu\right] = \frac{64\pi^2\alpha_s^2}{9}\left[\frac{t^2+u^2}{s^2}\right]\,. \qquad (7.117)
$$

Amplitudes for gluon-induced heavy quark production

Due to the non-Abelian character of QCD, there are three instead of two diagrams for the pair production of heavy quarks triggered by gluons. The first two ones are, up to colour factors, identical to the ones already present for, e.g. QED, whereas the third one stems from the two incoming gluons fusing into a third, virtual one, which in turn decays into the two quarks. For the process $g(p_1)+g(p_2)\to Q(p_3)\bar{Q}(p_4)$, the three emerging amplitudes therefore read

$$
\begin{aligned}
\mathcal{M}^{(a)} &= \bar{u}_i(p_3)(-ig_s\gamma^\mu T^a_{ij})\frac{i\delta_{jk}(\not{p}_3-\not{p}_1+m)}{(p_3-p_1)^2-m^2}(-ig_s\gamma^\nu T^b_{kl})v_l(p_4)\epsilon_\mu(p_1)\epsilon_\nu(p_2) \\
&= -\frac{ig_s^2T^a_{ij}T^b_{jl}}{(p_3-p_1)^2-m^2}\bar{u}_i(p_3)\left[2p_3^\mu\gamma^\nu-\gamma^\mu\not{p}_1\gamma^\nu\right]v_l(p_4)\epsilon_\mu(p_1)\epsilon_\nu(p_2)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{M}^{(b)} &= \bar{u}_i(p_3)(-ig_s\gamma^\nu T^b_{ij})\frac{i\delta_{jk}(\not{p}_3-\not{p}_2+m)}{(p_3-p_2)^2-m^2}(-ig_s\gamma^\mu T^a_{kl})v_l(p_4)\epsilon_\mu(p_1)\epsilon_\nu(p_2)\\
&= -\frac{ig_s^2 T^b_{ij}T^a_{jl}}{(p_3-p_2)^2-m^2}\bar{u}_i(p_3)\left[2p_3^\nu\gamma^\mu-\gamma^\nu\not{p}_2\gamma^\mu\right]v_l(p_4)\epsilon_\mu(p_1)\epsilon_\nu(p_2)\\
\mathcal{M}^{(c)} &= \bar{u}_i(p_3)(-ig_s\gamma^\rho T^c_{il})v_l(p_4)\frac{-ig_{\rho\sigma}}{(p_1+p_2)^2}\left[-g_s f^{abc}T^{\sigma\mu\nu}(-p_1-p_2,\,p_1,\,p_2)\right]\\
&\quad \epsilon_\mu(p_1)\epsilon_\nu(p_2)\\
&= -\frac{ig_s^2\left[T^a,\,T^b\right]_{ij}}{(p_1+p_2)^2}\bar{u}_i(p_3)\left[-2p_1^\nu\gamma^\mu+2p_2^\mu\gamma^\nu+(\not{p}_1-\not{p}_2)g^{\mu\nu}\right]v_l(p_4)\\
&\quad \epsilon_\mu(p_1)\epsilon_\nu(p_2)\,,
\end{aligned}
\tag{7.118}
$$

when the equations of motion are employed to yield

$$
\bar{u}_i(p_3)\gamma^\mu(\not{p}_3+m) \;=\; \bar{u}_i(p_3)\left[(-\not{p}_3+m)\gamma^\mu+2p_3^\mu\right]\,. \tag{7.119}
$$

For the amplitude containing the non-Abelian vertex one uses that

$$
\begin{aligned}
&T^{\sigma\mu\nu}(-p_1-p_2,\,p_1,\,p_2)\\
&\quad = \quad (-2p_1-p_2)^\nu g^{\sigma\mu}+(p_1-p_2)^\sigma g^{\mu\nu}+(2p_2-p_1)^\mu g^{\nu\sigma}\\
&\quad \longrightarrow \quad -2p_1^\nu g^{\sigma\mu}+(p_1-p_2)^\sigma g^{\mu\nu}+2p_2^\mu g^{\nu\sigma}
\end{aligned}
\tag{7.120}
$$

because of $p_1\cdot\epsilon(p_1)=p_2\cdot\epsilon(p_2)=0$. Also, one of the colour identities, namely

$$
if^{abc}T^c_{ij}=\left[T^a,\,T^b\right]_{ij}=T^a_{ik}T^b_{kj}-T^b_{ik}T^a_{kj} \tag{7.121}
$$

has been employed. Before squaring the amplitudes one may extract the colour factors related to each of the products $|\mathcal{M}^{(a)}\mathcal{M}^{*(b)}|=\mathcal{C}^{ab}$. They read

$$
\begin{aligned}
\mathcal{C}^{aa} &:= \frac{1}{8^2}\left|T^a_{ij}\delta_{jk}T^b_{kl}\right|^2=\frac{1}{64}T^a_{ij}T^b_{jl}T^b_{lk}T^a_{ki}\\
&= \frac{1}{256}\left(3\delta_{jk}-\frac{1}{3}\delta_{ij}\delta_{ki}\right)\left(3\delta_{jk}-\frac{1}{3}\delta_{jl}\delta_{lk}\right)=\frac{1}{12}\\
\mathcal{C}^{bb} &:= \frac{1}{64}\left|T^b_{ij}\delta_{jk}T^a_{kl}\right|^2=\mathcal{C}^{aa}=\frac{1}{12}\\
\mathcal{C}^{ab} &:= \frac{1}{64}T^a_{ij}T^b_{jl}T^a_{lk}T^b_{ki}=\frac{1}{256}\left(\delta_{ik}\delta_{jl}-\frac{1}{3}\delta_{ij}\delta_{lk}\right)\left(\delta_{ji}\delta_{lk}-\frac{1}{3}\delta_{jl}\delta_{ki}\right)=-\frac{1}{96}\\
\mathcal{C}^{cc} &:= \frac{1}{64}\left|if^{abc}T^c_{ij}\right|^2=2\left(\mathcal{C}^{aa}-\mathcal{C}^{ab}\right)=\frac{3}{16}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{C}^{ac} &:= \frac{1}{64} T^a_{ij} T^b_{jl} \left(T^a_{lk} T^b_{ki} - T^b_{lk} T^a_{ki} \right) = \mathcal{C}^{ab} - \mathcal{C}^{aa} = -\frac{3}{32} \\
\mathcal{C}^{bc} &:= \frac{1}{64} T^b_{ij} T^a_{jl} \left(T^a_{lk} T^b_{ki} - T^b_{lk} T^a_{ki} \right) = \mathcal{C}^{ab} - \mathcal{C}^{aa} = -\frac{3}{32} \,. \qquad (7.122)
\end{aligned}
$$

In a next step the amplitudes squared have to be evaluated. This will be done in the Lorentz gauge, as already hindsighted by the gluon propagator in $\mathcal{M}^{(c)}$. Making use of

$$
\begin{aligned}
s &= (p_1 + p_2)^2 = (p_3 + p_4)^2 = 2p_3 p_4 + 2m^2 \\
t &= (p_1 - p_3)^2 = m^2 - 2p_1 p_3 \\
u &= (p_1 - p_4)^2 = m^2 - 2p_1 p_4 \qquad (7.123)
\end{aligned}
$$

Note that by employing the identiy above, cf. Eq.(7.121), one can check that the total amplitude is manifestly gauge invariant, i.e. it vanishes identically to zero when contracted with either $p_{1\mu}$ or $p_{2\nu}$ instead of their respective polarization vectors.

$$
\begin{aligned}
\mathcal{S}_{fi} \sim\; & -ig_s^2 \bar{u}_i(p_3) \left\{ T^a_{ij} T^b_{jl} \frac{2p_3^\mu \gamma^\nu - \gamma^\mu \not{p}_1 \gamma^\nu}{t - m^2} + T^b_{ij} T^a_{jl} \frac{2p_3^\nu \gamma^\mu - \gamma^\nu \not{p}_2 \gamma^\mu}{u - m^2} \right. \\
& \left. + \left[T^a,\, T^b \right]_{ij} \frac{-2p_1^\nu \gamma^\mu + (\not{p}_1 - \not{p}_2) g^{\mu\nu} + 2p_2^\mu \gamma^\nu}{s} \right\} v_l(p_4) \,. \qquad (7.124)
\end{aligned}
$$

Plugging in both vectors, to keep things easy, and suppressing colour indices, one finds

$$
\begin{aligned}
\mathcal{S}_{fi} \to \bar{u} & \left\{ T^a T^b \frac{2p_1 p_3 \not{p}_2}{-2p_1 p_3} + T^a T^b \frac{2p_2 p_3 \not{p}_1}{-2p_2 p_3} - \left[T^a,\, T^b \right] \frac{p_1 p_2 (\not{p}_1 - \not{p}_2)}{2p_1 p_2} \right\} v \\
= & -\bar{u} \left\{ T^a T^b \frac{\not{p}_2 + \not{p}_1}{2} + T^a T^b \frac{\not{p}_2 + \not{p}_1}{2} \right\} v = -\frac{\left[T^a,\, T^b \right]}{2} \bar{u}(\not{p}_3 + \not{p}_4) v \\
= & -\frac{\left[T^a,\, T^b \right]}{2} \bar{u}(p_3)(-m + m) v(p_4) = 0 \qquad (7.125)
\end{aligned}
$$

as advertised. This allows to replace the sum over the gluon polarizations with the metric tensor, i.e. to replace

$$
\sum \epsilon_\mu(p_1) \epsilon^*_{\mu'}(p_1) = -g_{\mu\mu'} \,. \qquad (7.126)
$$

This allows to find compact expressions for the individual contributions,

$$
\begin{aligned}
&\left|\mathcal{M}^{(a)}\right|^2 \\
&= \frac{\mathcal{C}^{aa} g_s^4 \mathrm{Tr}\left[(\not{p}_3+m)\gamma^\mu(\not{p}_3-\not{p}_1+m)\gamma^\nu(-\not{p}_4-m)\gamma_\nu(\not{p}_3-\not{p}_1+m)\gamma_\mu\right]}{4\left[(p_1-p_3)^2-m^2\right]^2} \\
&= \frac{-4 g_s^4 \mathrm{Tr}\left[(\not{p}_3-2m)(\not{p}_3-\not{p}_1+m)(\not{p}_4-2m)(\not{p}_3-\not{p}_1+m)\right]}{48\,(t-m^2)^2} \\
&= \frac{2 g_s^4\left[(p_1p_3)(p_1p_2-p_1p_3)+m^2(p_1p_2-p_2p_3)-m^4\right]}{3\,(t-m^2)^2} \\
&= \frac{g_s^4\left[(m^2-t)(m^2-u)-2m^2(t+m^2)\right]}{6\,(t-m^2)^2}
\end{aligned}
\tag{7.127}
$$

$$
\begin{aligned}
&\left|\mathcal{M}^{(b)}\right|^2 = \left|\mathcal{M}^{(b)}\right|^2_{p_1\leftrightarrow p_2,\, t\leftrightarrow u} \\
&= \frac{g_s^4\left[(m^2-t)(m^2-u)-2m^2(u+m^2)\right]}{6\,(u-m^2)^2}
\end{aligned}
\tag{7.128}
$$

$$
\begin{aligned}
&\left|\mathcal{M}^{(a)}\mathcal{M}^{*(b)}\right| \\
&= \frac{\mathcal{C}^{ab} g_s^4 \mathrm{Tr}\left[(\not{p}_3+m)\gamma^\mu(\not{p}_3-\not{p}_1+m)\gamma^\nu(-\not{p}_4-m)\gamma_\mu(\not{p}_3-\not{p}_2+m)\gamma_\nu\right]}{4\left[(p_1-p_3)^2-m^2\right]\left[(p_2-p_3)^2-m^2\right]} \\
&= \frac{2 g_s^4 \mathrm{Tr}\left[(\not{p}_3+m)\gamma^\mu(\not{p}_3-\not{p}_2+m)(\not{p}_3-\not{p}_2+m)\gamma_\mu(\not{p}_4+m)\right]}{384\left[(p_1-p_3)^2-m^2\right]\left[(p_2-p_3)^2-m^2\right]} \\
&= \frac{g_s^4 \mathrm{Tr}\left[(4p_4p_3-4m^2-2m\not{p}_4+2m\not{p}_3)(\not{p}_3-\not{p}_1+m)(\not{p}_3-\not{p}_2+m)\right]}{192\left[(p_1-p_3)^2-m^2\right]\left[(p_2-p_3)^2-m^2\right]} \\
&= \frac{g_s^4 m^2[2m^2-(p1p2)]}{24\left[(p_1-p_3)^2-m^2\right]\left[(p_2-p_3)^2-m^2\right]} \\
&= -\frac{g_s^4 m^2(s-4m^2)}{24(t-m^2)(u-m^2)}
\end{aligned}
\tag{7.129}
$$

$$
\begin{aligned}
&\left|\mathcal{M}^{(c)}\right|^2 \\
&= \frac{\mathcal{C}^{cc} g_s^4}{4s^2}\mathrm{Tr}\left[(\not{p}_3+m)(-2p_1^\nu\gamma^\mu+2p_2^\mu\gamma^\nu+(\not{p}_1-\not{p}_2)g^{\mu\nu})\right. \\
&\qquad\left.(-\not{p}_4-m)(-2p_1^\nu\gamma^\mu+2p_2^\mu\gamma^\nu+(\not{p}_1-\not{p}_2)g^{\mu\nu})\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{3g_s^4}{64s^2}\left\{4\mathrm{Tr}\left[(\not{p}_3+m)\not{p}_1(\not{p}_4+m)\not{p}_2\right]+4\mathrm{Tr}\left[(\not{p}_3+m)\not{p}_2(\not{p}_4+m)\not{p}_1\right]\right\} \\
&= \frac{-3g_s^4\left[(p_1p_3)(p_2p_4)+(p_1p_4)(p_2p_3)-(p_1p_2)(p_3p_4-m^2)\right]}{2s^2} \\
&= \frac{-3g_s^4\left[(p_1p_3)^2+(p_1p_4)^2-(p_1p_2)(p_1p_2-2m^2)\right]}{2s^2} \\
&= \frac{3g_s^4(m^2-t)(m^2-u)}{4s^2}
\end{aligned}
\tag{7.130}
$$

$$
\begin{aligned}
&\left|\mathcal{M}^{(a)}\mathcal{M}^{*(c)}\right| \\
&= \frac{\mathcal{C}^{ac}g_s^4\mathrm{Tr}\left[(\not{p}_3+m)\gamma^\mu(\not{p}_1-\not{p}_3+m)\gamma^\nu(-\not{p}_4-m)\gamma^\rho\right]T_{\rho\mu\nu}(-p_1-p_2,\,p_1,\,p_2)}{4s(t-m^2)} \\
&= \frac{-3g_s^4}{32s(t-m^2)}\Bigg\{\mathrm{Tr}\left[(\not{p}_3-2m)(\not{p}_3-\not{p}_1+m)\not{p}_1(\not{p}_4+m)\right] \\
&\qquad -\mathrm{Tr}\left[(\not{p}_3+m)\not{p}_2(\not{p}_3-\not{p}_1+m)p(\not{p}_4-2m)\right] \\
&\qquad -\mathrm{Tr}\left[(\not{p}_3+m)(\not{p}_3-\not{p}_1-2m)(\not{p}_4+m)(\not{p}_1-\not{p}_2)\right]\Bigg\} \\
&= \Big[(p_3p_4)((p_3p_4)-(p_1p_4))-4(p_1p_3)(p_3p_4) \\
&\qquad +4m^2(2p_1p_4-4p_1p_3-3p_3p_4)-5m^4\Big] \\
&= \frac{-3g_s^4\left[m^2(p_1p_3-p_1p_4)-(p_1p_3)(p_1p_4)\right]}{16s(t-m^2)} \\
&= \frac{3g_s^4[(m^2-t)(m^2-u)+m^2(u-t)]}{16s(t-m^2)}
\end{aligned}
\tag{7.131}
$$

$$
\begin{aligned}
\left|\mathcal{M}^{(b)}\mathcal{M}^{*(c)}\right| &= \left|\mathcal{M}^{(a)}\mathcal{M}^{*(c)}\right|_{p_1\leftrightarrow p_2,t\leftrightarrow u} \\
&= \frac{3g_s^4[(m^2-t)(m^2-u)+m^2(t-u)]}{16s(u-m^2)}\,.
\end{aligned}
\tag{7.132}
$$

The total amplitude squared then reads

$$
\begin{aligned}
|\mathcal{S}_{fi}|^2 &= \pi^2\alpha_s^2\Bigg\{\frac{8[(t-m^2)(u-m^2)-2m^2(m^2+t)]}{3(t-m^2)^2} \\
&\qquad +\frac{8[(u-m^2)(t-m^2)-2m^2(m^2+u)]}{3(u-m^2)^2}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{12(t-m^2)(u-m^2)}{s^2} \\
&-\frac{2m^2(s-4m^2)}{3(t-m^2)(u-m^2)} \\
&+\frac{6[(t-m^2)(u-m^2)+m^2(u-t)]}{s(t-m^2)} \\
&\left.+\frac{6[(u-m^2)(t-m^2)+m^2(t-u)]}{s(u-m^2)}\right\} . \qquad (7.133)
\end{aligned}
$$

In the massless limit this yields the well-known result

$$
\begin{aligned}
|\mathcal{S}_{fi}|^2 &= \frac{g_s^4}{6}\left[\frac{u}{t}+\frac{t}{u}\right]-\frac{3g_s^4}{8}\frac{t^2+u^2}{s^2}+\frac{3g_s^4}{8}\left[1+\frac{u}{s}+\frac{t}{s}\right] \\
&= \frac{g_s^4}{6}\left[\frac{u}{t}+\frac{t}{u}\right]-\frac{3g_s^4}{8}\frac{t^2+u^2}{s^2} \qquad (7.134)
\end{aligned}
$$

for the transition amplitude squared.

Chapter 8

Renormalisation of gauge theories

8.1 QED at one-loop

The QED Lagrangian in Feynman gauge ($\alpha = 1$ in front of the gauge fixing term) is given by

$$\mathcal{L}^{\text{cl.}} \;=\; -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\alpha}(\partial_\mu A^\mu)^2 + i\bar{\psi}\not{\partial}\psi - m\bar{\psi}\psi - e\bar{\psi}\not{A}\psi\,, \tag{8.1}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,. \tag{8.2}$$

8.1.1 Regularisation

Catalogue of diagrams

Before starting out with the program of regularisation, the superficial degree of divergence for each graph in QED should be counted. This is to list all relevant diagrams that have to be regularised. Defining, as before L as the number of loops, V as the number of vertices and E_ψ, I_ψ and E_A, I_A as the numbers of external (E) or internal (I) fermion (ψ) or photon fields (A), the superficial degree of divergence for a QED graph is

$$D = 4L - 2I_A - I_\psi\,. \tag{8.3}$$

The goal now is to cast this into a form such that only the external legs show up, allowing to catalogise the generic types of diagrams or n-point functions that need to be cured. Since there is only one kind of vertex in QED with two fermion and one photon line, it is easy to convince oneself that

$$V = I_\psi + \frac{1}{2}E_\psi \tag{8.4}$$

and that

$$V = 2I_A + E_A\,. \tag{8.5}$$

Also, the number of independent momenta is equal to L, which in turn equals the number of internal lines minus the number of vertices (momentum conservation at each vertex) plus one (total four momentum conservation),

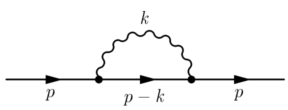
$$L = I_\psi + I_A - V + 1\,. \tag{8.6}$$

Putting everything together, one finds

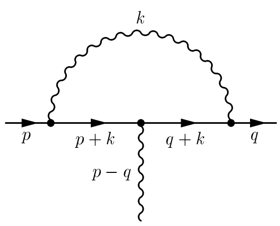
$$D = 4 - \frac{3}{2}E_\psi - E_A\,. \tag{8.7}$$

This is quite fortunate, since once again (after the example of ϕ^4 theory) it shows that the total divergence of any Feynman diagram, no matter how complicated, just depends on the number and type of external particles. In fact, the only superficially divergent graphs are

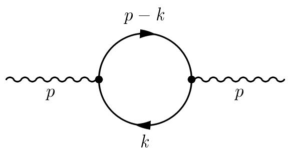
1. the electron self-energy ($D = 1$),



2. the electron-photon vertex ($D = 0$),



3. the photon vacuum polarisation (nominally $D = 2$, but gauge invariance reduces the degree of divergence by 2),



4. the three-photon graphs ($D = 1$); they vanish because there are two graphs, one with the electron clockwise and one with the electron counterclockwise leading to exactly the same results with opposite sign - this is a minifestation of Furry's theorem, and, finally,

5. the light-by-light scattering ($D = 0$). This graph becomes convergent by employing gauge invariance.

Self-energy of the electron

Working in the Feynman gauge and suppressing the spinors the one-loop correction on the fermion line, cf. Fig.8.1, also known as self-energy, reads

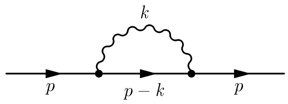


Figure 8.1: Electron self energy graph at order e^2.

$$
\begin{aligned}
\Sigma(p) &= \\
&= i(-i)(ie)^2\mu^{4-D}\int\frac{\mathrm{d}^Dk}{(2\pi)^D}\frac{\gamma^\mu\left(\not{p}-\not{k}+m\right)\gamma^\nu\left(g_{\mu\nu}\right)}{\left[(p-k)^2-m^2\right]k^2} \\
&= -e^2\mu^{4-D}\int\frac{\mathrm{d}^Dk}{(2\pi)^D}\frac{(2-D)\left(\not{p}-\not{k}\right)+Dm}{\left[(p-k)^2-m^2\right]k^2} \\
&= -e^2\mu^{4-D}\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^Dk}{(2\pi)^D}\frac{(2-D)\left(\not{p}-\not{k}\right)+Dm}{\left[k^2-2xp\cdot k+xp^2-xm^2\right]^2}
\end{aligned}
$$

$$
\begin{aligned}
&= -e^2\mu^{4-D}\int\limits_0^1 \mathrm{d}x \int \frac{\mathrm{d}^D K}{(2\pi)^D}\frac{(2-D)\left((1-x)\not{p}-\not{K}\right)+Dm}{\left[K^2+x(1-x)p^2-xm^2\right]^2} \\
&= -\frac{ie^2\mu^{4-D}\Gamma\left(2-\frac{D}{2}\right)}{(4\pi)^{D/2}\Gamma(2)}\int\limits_0^1 \mathrm{d}x \frac{(2-D)(1-x)\not{p}+Dm}{\left[x(x-1)p^2+xm^2\right]^{2-D/2}}\,,
\end{aligned} \tag{8.8}
$$

where, in intermediate steps, a Feynamn trick of Eq.(5.29) and the substitution $k^\mu \to K^\mu = k^\mu - xp^\mu$ has been applied. Also, Eq.(5.15) has been used for the evaluation of the D-dimensional integral. To continue, one uses $D = 4-2\eta$ and expands around small η. Using

$$
\hat{\mu}^2 = 4\pi e^{-\gamma_E}\mu^2 \quad \text{and} \quad M^2(x) = (x^2-x)p^2 + xm^2 \tag{8.9}
$$

this yields

$$
\begin{aligned}
\Sigma(p) &= -\frac{ie^2}{(4\pi)^2}\frac{1}{\eta}\left[1+\eta\log\left(\hat{\mu}^2\right)\right] \\
&\quad \int\limits_0^1 \mathrm{d}x\left[(2\eta-2)(1-x)\not{p}+(4-2\eta)m\right]\left[1-\eta\log M^2(x)\right] \\
&= -\frac{ie^2}{(4\pi)^2}\left\{\frac{-\not{p}+4m}{\eta}+\not{p}-2m+\right. \\
&\qquad\qquad \left.\int\limits_0^1 \mathrm{d}x\left[2(1-x)\not{p}+4m\right]\log\frac{M^2(x)}{\hat{\mu}^2}\right\}\,.
\end{aligned} \tag{8.10}
$$

The finite integral can be done analytically.

Vacuum polarisation

The one-loop correction to the photon line is mediated through a fermion loop, also known as vacuum polarisation, cf.Fig.8.2. It reads

$$
\begin{aligned}
&\Pi_{\mu\nu}(p) = \\
&= -i^2(ie)^2\mu^{4-D}\int\frac{\mathrm{d}^4k}{(2\pi)^D}\mathrm{Tr}\left[\frac{\gamma_\mu(\not{p}+\not{k}+m)\gamma_\nu(\not{k}+m)}{[(p+k)^2-m^2][k^2-m^2]}\right] \\
&= -4e^2\mu^{4-D}\int\frac{\mathrm{d}^4k}{(2\pi)^D}\frac{(p+k)_\mu k_\nu+(p+k)_\nu k_\mu-[(p+k)\cdot k-m^2]g_{\mu\nu}}{[(p+k)^2-m^2][k^2-m^2]}
\end{aligned}
$$

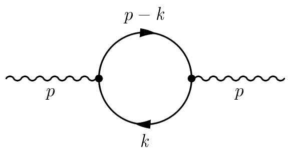


Figure 8.2: Vacuum polarisation graph at order e^2.

$$
\begin{aligned}
&= -4e^2\mu^{4-D}\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^4k}{(2\pi)^D} \\
&\qquad \times \frac{(p+k)_\mu k_\nu + (p+k)_\nu k_\mu - [(p+k)\cdot k - m^2]g_{\mu\nu}}{[k^2 + 2xp\cdot k + x(p^2-m^2)]^2} \\
&= -4e^2\mu^{4-D}\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^4K}{(2\pi)^D} \\
&\qquad \times \frac{2K_\mu K_\nu - 2x(1-x)p_\mu p_\nu - [K^2 - x(1-x)p^2 - m^2]g_{\mu\nu}}{[K^2 + x(1-x)p^2 - m^2]^2} \\
&= -4e^2\mu^{4-D}\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^4K}{(2\pi)^D} \left\{ \frac{2g_{\mu\nu}}{D}\frac{K^2}{[K^2 + x(1-x)p^2 - m^2]^2} \right. \\
&\qquad \left. - \frac{g_{\mu\nu}}{K^2 + x(1-x)p^2 - m^2} - \frac{2x(1-x)(p_\mu p_\nu - p^2 g_{\mu\nu})}{[K^2 + x(1-x)p^2 - m^2]^2} \right\} \\
&= -\frac{4ie^2\mu^{4-D}}{(4\pi)^D/2}\int_0^1 \mathrm{d}x \left\{ -\frac{2g_{\mu\nu}}{D}\frac{\Gamma\left(1+\frac{D}{2}\right)\Gamma\left(1-\frac{D}{2}\right)}{\Gamma\left(\frac{D}{2}\right)\Gamma\left(2\right)}\frac{1}{[M^2(x)]^{1-D/2}} \right. \\
&\qquad \left. + g_{\mu\nu}\frac{\Gamma\left(1-\frac{D}{2}\right)}{[M^2(x)]^{1-D/2}} - \frac{2x(1-x)(p_\mu p_\nu - p^2 g_{\mu\nu})}{[M^2(x)]^{2-D/2}}\frac{\Gamma\left(2-\frac{D}{2}\right)}{\Gamma\left(2\right)} \right\} \\
&= \frac{4ie^2\mu^{4-D}\Gamma\left(2-\frac{D}{2}\right)}{(4\pi)^D/2}\left(p_\mu p_\nu - p^2 g_{\mu\nu}\right)\int_0^1 \mathrm{d}x \frac{2x(1-x)}{[M^2(x)]^{2-D/2}}\,, \qquad (8.11)
\end{aligned}
$$

where the minus sign and the trace stem from the fermion loop. In the computation above the substitution $k^\mu \to K^\mu = k^\mu + xp^\mu$ and the usual integral identities already employed for the evaluation of the self-energy diagram have been applied. Also, the function $M^2(x)$ equals the one introduced before. Setting $D = 4 - 2\eta$ the result above is expanded around small η yielding

$$\begin{aligned}\Pi_{\mu\nu}(p) &= \frac{4ie^2\left(p_\mu p_\nu - p^2 g_{\mu\nu}\right)}{(4\pi)^2}\frac{1}{\eta}\left[1+\eta\log\left(\hat{\mu}^2\right)\right] \\ &\qquad \int\limits_0^1 \mathrm{d}x 2x(1-x)\left[1-\eta\log M^2(x)\right] \\ &= \frac{4ie^2\left(p_\mu p_\nu - p^2 g_{\mu\nu}\right)}{(4\pi)^2}\left[\frac{2}{3\eta}+\int\limits_0^1 \mathrm{d}x 2x(1-x)\log\frac{M^2(x)}{\hat{\mu}^2}\right] .\end{aligned} \tag{8.12}$$

Again, the remaining integral is finite and can be computed analytically.

Vertex correction

The vertex correction at one-loop, cf.Fig.8.3 is given by

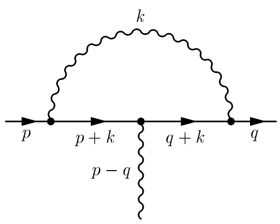


Figure 8.3: Electron-photon vertex correction graph at order e^3.

$$\begin{aligned}&\Gamma_\rho(p,\, q) = \\ &= i^2(-i)(ie)^3\mu^{4-D}\int\frac{\mathrm{d}^4k}{(2\pi)^D}\frac{g^{\mu\nu}\gamma_\mu(\not{p}+\not{k}+m)\gamma_\rho(\not{q}+\not{k}+m)\gamma_\nu}{[(p+k)^2-m^2][(q+k)^2-m^2]k^2} \\ &= 2e^3\mu^{4-D}\int\limits_0^1 \mathrm{d}x\int\limits_0^{1-x}\mathrm{d}y\int\frac{\mathrm{d}^4k}{(2\pi)^D} \\ &\qquad \frac{\gamma_\mu(\not{p}+\not{k}+m)\gamma_\rho(\not{q}+\not{k}+m)\gamma^\mu}{[k^2+(2xp+2yq)\cdot k+xp^2+yq^2-(x+y)m^2]^3}\end{aligned}$$

$$= \quad 2e^3\mu^{4-D}\int\limits_0^1 \mathrm{d}x \int\limits_0^{1-x}\mathrm{d}y \int \frac{\mathrm{d}^4K}{(2\pi)^D}$$
$$\frac{\gamma_\mu(\not{K} - y\not{q} + (1-x)\not{p} + m)\gamma_\rho(\not{K} + (1-y)\not{q} - x\not{p} + m)\gamma^\mu}{[K^2 + x(1-x)p^2 + y(1-y)q^2 - 2xyp\cdot q - (x+y)m^2]^3} \tag{8.13}$$

To continue, the numerator has to be inspected in some detail. To separate out those pieces leading to ultraviolet divergences, one has to separate the terms quadratic in K from the others. Then, with

$$-V^2(x,\,y) = x(1-x)p^2 + y(1-y)q^2 - 2xyp\cdot q - (x+y)m^2 \tag{8.14}$$

the divergent part of the vertex correction reads

$$\begin{aligned}
\Gamma_\rho^{\mathrm{U.V.}}(p,\,q) &= -2(D-2)e^3\mu^{4-D}\int\limits_0^1 \mathrm{d}x \int\limits_0^{1-x}\mathrm{d}y \int \frac{\mathrm{d}^4K}{(2\pi)^D}\frac{\not{K}\gamma_\rho\not{K}}{[K^2 - V^2(x,\,y)]^3}\\
&= \frac{2(2-D)^2\gamma_\rho}{D}e^3\mu^{4-D}\int\limits_0^1 \mathrm{d}x \int\limits_0^{1-x}\mathrm{d}y \int \frac{\mathrm{d}^4K}{(2\pi)^D}\frac{K^2}{[K^2 - V^2(x,\,y)]^3}\\
&= \frac{2ie^3\mu^{4-D}}{(4\pi)^{D/2}}\gamma_\rho\frac{(2-D)^2\Gamma\left(1+\frac{D}{2}\right)\Gamma\left(2-\frac{D}{2}\right)}{\Gamma\left(\frac{D}{2}\right)\Gamma\left(3\right)D}\int\limits_0^1 \mathrm{d}x \int\limits_0^{1-x}\mathrm{d}y\frac{1}{[V^2(x,\,y)]^{2-D/2}}\\
&= \frac{2ie^3}{(4\pi)^2}\gamma_\rho\frac{1-2\eta}{\eta}\left[1+\eta\log\hat{\mu}^2\right]\int\limits_0^1 \mathrm{d}x \int\limits_0^{1-x}\mathrm{d}y\left[1-\eta\log V^2(x,\,y)\right]\\
&= \frac{ie^3}{(4\pi)^2}\gamma_\rho\left[\frac{1}{\eta} - 2 - 2\int\limits_0^1 \mathrm{d}x \int\limits_0^{1-x}\mathrm{d}y\log\frac{V^2(x,\,y)}{\hat{\mu}^2}\right]\,.
\end{aligned} \tag{8.15}$$

Again the last integral yields a final result, as well as those parts that have been neglected before.

8.1.2 Renormalisation

Counterterms

Starting from the QED Lagrangian, Eq.(8.1), one is tempted to try a counterterm Lagrangian of the form

$$\mathcal{L}^{\mathrm{c.t.}} \quad = \quad -\frac{K_3}{4}F_{\mu\nu}F^{\mu\nu} - \frac{K_\alpha}{2\alpha}(\partial_\mu A^\mu)^2 + iK_2\bar{\psi}\not{\partial}\psi - K_m m\bar{\psi}\psi - K_1 e\bar{\psi}\not{A}\psi\,.$$

$$(8.16)$$

Then the renormalised Lagrangian

$$\mathcal{L}^{\text{ren.}} = \mathcal{L}^{\text{cl.}} + \mathcal{L}^{\text{c.t.}} \tag{8.17}$$

can be rewritten in terms of bare quantities

$$\begin{aligned}
\psi_0 &= \sqrt{1+K_2}\,\psi := \sqrt{Z_2}\,\psi \\
A_0^\mu &= \sqrt{1+K_3}\,A^\mu := \sqrt{Z_3}\,A^\mu \\
e_0 &= \frac{1+K_1}{(1+K_2)\sqrt{1+K_3}}\,e := \frac{Z_1}{Z_2\sqrt{Z_3}}\,e \\
m_0 &= \frac{1+K_m}{1+K_2}\,m := \frac{Z_m}{Z_2}\,m \\
\alpha_0^{-1} &= \frac{1+K_\alpha}{1+K_3}\alpha^{-1} := \frac{Z_\alpha}{Z_3}\alpha^{-1}
\end{aligned} \tag{8.18}$$

and reads

$$\begin{aligned}
\mathcal{L}^{\text{ren.}} &= \frac{1}{4}F_{\mu\nu 0}F_0^{\mu\nu} + \frac{1}{2\alpha_0}(\partial_\mu A_0^\mu)^2 + i\bar{\psi}_0\partial\!\!\!/\psi_0 + im_0\bar{\psi}_0\psi_0 + ie\bar{\psi}_0 A\!\!\!/_0\psi_0 \\
&= \frac{Z_3}{4}F_{\mu\nu}F^{\mu\nu} + \frac{Z_\alpha}{2\alpha}(\partial_\mu A^\mu)^2 + iZ_2\bar{\psi}\partial\!\!\!/\psi + iZ_m m\bar{\psi}\psi + eZ_1\bar{\psi}A\!\!\!/\psi\,.
\end{aligned} \tag{8.19}$$

Written in this form it is rather straightforward to see that gauge invariance will be preserved under loop corrections if $Z_1 = Z_2$. In fact this is exactly the case as will be discussed in the following. However, this is not so obvious, since by putting in a gauge fixing term the Z_i, or, basically, the counter terms, became gauge dependent.
Now, from the various contributions calculated in the previous section, the counter terms can be read off. From the self energy

$$\Sigma(p) = \frac{ie^2}{(4\pi)^2}\frac{p\!\!\!/ - 4m}{\eta} + \text{finite terms} \tag{8.20}$$

one finds

$$\begin{aligned}
K_2 &= \frac{e^2}{(4\pi)^2}\left[\frac{1}{\eta} + F_2\left(\eta,\,\frac{m}{\hat{\mu}}\right)\right] \\
K_m &= -\frac{4e^2}{(4\pi)^2}\left[\frac{1}{\eta} + F_m\left(\eta,\,\frac{m}{\hat{\mu}}\right)\right],
\end{aligned} \tag{8.21}$$

where the finite pieces are encapsulated in the finite functions F, which are analytic for $\eta \to 0$. The vertex correction, given by

$$\Pi_{\mu\nu}(p) \quad = \quad \frac{ie^2\,(p_\mu p_\nu - p^2 g_{\mu\nu})}{(4\pi)^2}\left[\frac{4}{3\eta} + \text{finite terms}\right] \tag{8.22}$$

yields

$$\begin{aligned}
K_3 &= -\frac{e^2}{(4\pi)^2}\left[\frac{4}{3\eta} + F_3\left(\eta,\, \frac{m}{\hat{\mu}}\right)\right] \\
K_\alpha &= \frac{e^2}{(4\pi)^2}\left[\frac{4}{3\eta} + F_\alpha\left(\eta,\, \frac{m}{\hat{\mu}}\right)\right],
\end{aligned} \tag{8.23}$$

where the F have the same properties as the ones above. The renormalisation of α corresponds to the correction of the longitudinal part of the photon propagator, as given by the term $\sim p_\mu p_\nu$. Lastly, the vertex corrections

$$\Gamma_\rho^{\mathrm{U.V.}}(p,\, q) = \frac{ie^3}{(4\pi)^2}\gamma_\rho\left[\frac{1}{\eta} + \text{finite terms}\right] \tag{8.24}$$

give

$$K_1 \quad = \quad -\frac{e^2}{(4\pi)^2}\left[\frac{1}{\eta} + F_1\left(\eta,\, \frac{m}{\hat{\mu}}\right)\right], \tag{8.25}$$

where again F_1 is the finite piece of the counter term.
To summarize

$$\begin{aligned}
Z_1 &= 1 - \frac{e^2}{(4\pi)^2}\left(\frac{1}{\eta} + F_1\right) + \mathcal{O}(e^4) \\
Z_2 &= 1 - \frac{e^2}{(4\pi)^2}\left(\frac{1}{\eta} + F_2\right) + \mathcal{O}(e^4) \\
Z_3 &= 1 - \frac{e^2}{(4\pi)^2}\left(\frac{4}{3\eta} + F_3\right) + \mathcal{O}(e^4) \\
Z_m &= 1 - \frac{e^2}{(4\pi)^2}\left(\frac{4}{\eta} + F_m\right) + \mathcal{O}(e^4) \\
Z_\alpha &= 1 - \frac{e^2}{(4\pi)^2}\left(\frac{4}{3\eta} + F_\alpha\right) + \mathcal{O}(e^4).
\end{aligned} \tag{8.26}$$

In fact, to this order in perturbation theory, the divergent pieces of $Z_{1,2}$ satisfy the relation $Z_1 = Z_2$. The bare charge therefore can be expressed as

$$e_0 = \mu^\eta \left[1 + \frac{e^2}{(4\pi)^2} \frac{2}{3\eta} + \text{finite terms} + \mathcal{O}(e^4) \right] . \tag{8.27}$$

Thus, by using a mass independent renormalisation prescription, one can read off the scale variation of the coupling constant,

$$\hat{\mu} \frac{\partial e}{\partial \hat{\mu}} = \frac{e^3}{12\pi^2} , \tag{8.28}$$

in fact the same sign as for the scalar ϕ^4 theory. The solution to the equation above yields a running coupling constant given by

$$e^2(\mu) = \frac{e^2(\mu_0)}{1 - \frac{e^2(\mu_0)}{6\pi^2} \log \frac{\mu}{\mu_0}} , \tag{8.29}$$

leading to the Landau singularity at

$$\mu_L = \mu_0 \exp \left[\frac{6\pi^2}{e^2(\mu_0)} \right] . \tag{8.30}$$

The fact that the coupling constant decreases for smaller scales i.e. longer distances is good news: it proves that QED as a theory is perfectly suited to define asymptotic states and according scattering cross sections.

Ward identities: simple version

When moving to an all-orders discussion of QED and its renormalisation, the work will be vastly simplified by a set of identities known as Ward-Takahashi identities. Their benefit is that they reduce the number of independent renormalisation constants Z_i, and they are intimately related to the gauge structure of the theory. One such identity is already present in the one-loop renomrlisation constants listed in Eq.(8.26), namely $Z_1 = Z_2$. This identity could have been seen already before explicit calculation, to see how this works recall the expression for the self energy correction,

$$\Sigma(p) = -e^2 \mu^{4-D} \int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{\gamma^\mu (\not{p} - \not{k} + m) \gamma_\mu}{[(p-k)^2 - m^2] k^2} \tag{8.31}$$

and compute its derivative

$$
\begin{aligned}
-\frac{\partial\Sigma(p)}{\partial p^\rho} &= -e^2\mu^{4-D}\int\frac{\mathrm{d}^Dk}{(2\pi)^D}\frac{1}{k^2}\gamma^\mu\left[-\frac{\partial}{\partial p^\rho}\frac{1}{\not{p}-\not{k}-m}\right]\gamma_\mu \\
&= e^2\mu^{4-D}\int\frac{\mathrm{d}^Dk}{(2\pi)^D}\frac{1}{k^2}\gamma^\mu\left[\frac{1}{\not{p}-\not{k}-m}\gamma_\rho\frac{1}{\not{p}-\not{k}-m}\right]\gamma_\mu \\
&= e^2\mu^{4-D}\int\frac{\mathrm{d}^Dk}{(2\pi)^D}\frac{\gamma^\mu(\not{p}-\not{k}+m)\gamma_\rho(\not{p}-\not{k}+m)\gamma_\mu}{k^2[(p-k)^2-m^2]^2}\,, \qquad (8.32)
\end{aligned}
$$

which is, up to a constant factor (the coupling constant e) identical with the vertex correction, if the two fermion momenta are identical,

$$
\begin{aligned}
\Gamma_\rho(p,\,p) &= \lim_{q\to p}\Gamma_\rho(p,\,q) \\
&= \lim_{q\to p}e^3\mu^{4-D}\int\frac{\mathrm{d}^4k}{(2\pi)^D}\frac{\gamma_\mu(\not{p}-\not{k}+m)\gamma_\rho(\not{q}-\not{k}+m)\gamma_\mu}{[(p-k)^2-m^2][(q-k)^2-m^2]k^2} \\
&= e^3\mu^{4-D}\int\frac{\mathrm{d}^4k}{(2\pi)^D}\frac{\gamma_\mu(\not{p}-\not{k}+m)\gamma_\rho(\not{p}-\not{k}+m)\gamma_\mu}{[(p-k)^2-m^2]^2k^2}\,. \qquad (8.33)
\end{aligned}
$$

This explains why the two ultraviolet structures of the respective diagrams can be related. To generalize the procedure above to obtain a more useful form, consider

$$
\frac{1}{\not{p}-m}-\frac{1}{\not{p}+\not{q}-m} = \frac{\not{p}+\not{q}-m-\not{p}+m}{[\not{p}-m][\not{p}+\not{q}-m]} = \frac{\not{q}}{[\not{p}-m][\not{p}+\not{q}-m]}\,, \qquad (8.34)
$$

which allows to identify

$$
(p-q)^\rho\,\Gamma_\rho(p,\,q) = -\left[\Sigma(p)-\Sigma(q)\right]. \qquad (8.35)
$$

To show that this relationship holds true to all orders, consider a specific member contributing to the electron self-energy at arbitrary order. Then, schematically, one has, for an external electron with momentum p_0

$$
\Sigma(p_0) \sim \sum_{\mathrm{N=numberofloops}}\prod_{\mathrm{N}}\int\left(\prod_{i=0}^{N}\not{a}_i\frac{1}{\not{p}_i-m}\right)\not{a}_n\,, \qquad (8.36)
$$

where the $p_i = p_0 + q_i$ with the q some loop momenta and $p_n = p_0$. The a_i denote the momentum-dependent photon lines that are connected to other fermion loops, which are not shown explicitly. As an example consider a graph, where the photon of the self-energy correction is itself corrected by some vacuum polarisation, cf.Fig.8.4. Then, after performing the traces of the vacuum polarisation piece, the γ matrices of the photon vertices on the electron line are contracted with loop momenta from the vacuum bubble. is

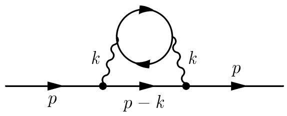


Figure 8.4: Electron self energy graph at higher order.

given by In the scheme above, these guys are denoted by the a_i. From the considerations just made it is clear that the total momenta inside the a_i must add up to zero - after all there is only one incoming and one outgoing particle, both with momentum p_0. Similarly, a corresponding diagram contributing to the vertex correction can be represented as

$$\begin{aligned}\Gamma_\rho(p_0, p_N) \quad &\sim \sum_{\mathrm{N=numberofloops}} \prod_{\mathrm{N}} \int \sum_{r=1}^{N-1} \left(\prod_{i=0}^{r} \not{a}_i \frac{1}{\not{p}_i - m} \right) \gamma_\rho \frac{1}{\not{p}_r + \not{q} - m} \\ &\qquad \left(\prod_{i=r+1}^{N} \not{a}_i \frac{1}{\not{p}_i + \not{q} - m} \right) \not{a}_n \,. \end{aligned} \tag{8.37}$$

Here, q denotes the momentum of the external photon line and the external fermion momenta are p_0 and p_N. Obviously, the photon is inserted at point r, and the sum is over all positions r. The essential point of the proof is that there is a similarity of the self-energy and the vertex graphs. Graphocally speaking, taking any self-energy graph and making all possible insertions of an extra photon along all fermion propgators leads to the sum over all vertex correction diagrams. To show this, contract $\Gamma_\rho(p_0, p_N)$ with q^ρ and use the previous identity. Then each term of the sum over r splits into two pieces with a relative minus sign. Performing the sum it becomes apparent that only the first and the last term survive, leading to

$$q^\rho \Gamma_\rho(p_0, p_N) \quad \sim \sum_{\mathrm{N=numberofloops}} \prod_{\mathrm{N}} \int \sum_{r=1}^{N-1} \left(\prod_{i=0}^{r-1} \not{a}_i \frac{1}{\not{p}_i - m} \right) \not{a}_r$$

$$\left(\frac{1}{\not{p}_r - m} - \frac{1}{\not{p}_r + \not{q} - m}\right)\left(\prod_{i=r+1}^{N} \not{a}_i \frac{1}{\not{p}_i + \not{q} - m}\right)\not{a}_n$$
$$= \Sigma(p_0) - \Sigma(p_N = p_0 + q)\,. \tag{8.38}$$

The proof is complete by realizing that attaching the photon along closed fermion lines (i.e. vacuum polarisation insertions) leads to exact pairwise cancellations yielding a contribution of exaclty zero from such closed loops. These considerations prove the Ward-Takahashi identity to all orders.

BRS transformations

The Ward-Takahashi identities discussed above play a crucial role for the proof of renormalizability of QED. To cast them into a more general form, however, it is useful to make something that looks like a little detour. The main point of this detour is to define a symmetry valid for gauge theories which covers up for the gauge-invariance breaking effects of the gauge-fixing term.

Because of local gauge invariance not all of the connected Greens functions generated by

$$\begin{aligned} Z[J_\mu,\, \bar{\chi},\, \chi] &= Z[0,\, 0,\, 0]\exp\left\{i\mathcal{W}[J]\right\} \\ &= \mathcal{N}\int \mathcal{D}A_\mu \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left\{i\mathcal{S} + \int \mathrm{d}^4x \left[J_\mu A^\mu + i\bar{\chi}\psi + i\bar{\psi}\chi\right]\right\} \end{aligned} \tag{8.39}$$

where

$$\mathcal{S} = \int \mathrm{d}^4x \left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\alpha}(\partial_\mu A^\mu)^2 + i\bar{\psi}\not{\partial}\psi - m\bar{\psi}\psi - e\bar{\psi}\not{A}\psi\right] \tag{8.40}$$

are independent. In fact, due to the gauge fixing term and the various sources, the generating functional above is not invariant under the local gauge transformations

$$\begin{aligned} \delta A_\mu(x) &= \frac{1}{e}\partial_\mu \Lambda(x) \\ \delta\psi(x) &= = -i\Lambda(x)\psi(x) \\ \delta\bar{\psi}(x) &= = i\bar{\psi}(x)\Lambda(x)\,. \end{aligned} \tag{8.41}$$

In this and the next section a set of functional constraints on $\mathcal{W}$ will be derived which will result in relations between various Greens functions known as Ward identities. To derive these constraints, the approach due to Becchi, Rouet, and Stora will be used: it generlizes neatly to the case of Yang-Mills theories.

The first step of their approach consists in restoring a weaker form of invariance even in the presence of the gauge fixing term. In doing so, the sources will be negrlected for a moment. This restoration is achieved by switching to a ghost representation of the constant determinant stemming from the gauge fixing term, resulting also in a different normalization factor. Then the new action, including the ghost fields η, is

$$\mathcal{S}' = \int \mathrm{d}^4x \left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\alpha}(\partial_\mu A^\mu)^2 + \bar{\psi}\,(i\partial\!\!\!/ - m - eA\!\!\!/)\,\bar{\psi} + i\partial_\mu\eta^*\partial^\mu\eta\right] \tag{8.42}$$

This new action is invariant under the special gauge transformations

$$\begin{aligned}
\delta A_\mu(x) &= \frac{1}{e}\partial_\mu\left[\xi^*\eta(x) + \xi\eta^*(x)\right] \\
\delta\psi(x) &= = -i\left[\xi^*\eta(x) + \xi\eta^*(x)\right]\psi(x) \\
\delta\bar{\psi}(x) &= = i\bar{\psi}(x)\left[\xi^*\eta(x) + \xi\eta^*(x)\right] \\
\delta\eta(x) &= -\frac{i}{\alpha e}(\partial\cdot A)\xi \\
\delta\eta^*(x) &= \frac{i}{\alpha e}(\partial\cdot A)\xi^*\,,
\end{aligned} \tag{8.43}$$

where ξ are arbitrary complex Grassmann numbers independent of x. The transformations of Eq.(8.43) are also known also as BRS-transformations. Applying them, it is quite simple to check that the kinetic term of the gauge fields and the fermion piece of the action are invariant; hence the action is shifted by

$$\begin{aligned}
\delta\mathcal{S}' = \frac{1}{e}\int \mathrm{d}^4x &\left\{\frac{1}{\alpha}(\partial\cdot A)\partial^2\left[\xi^*\eta(x) + \xi\eta^*(x)\right]\right. \\
&\left. -\frac{1}{\alpha}(\partial\cdot A)\partial^2\left[\xi^*\eta(x)\right] - \frac{1}{\alpha}(\partial\cdot A)\partial^2\left[\xi\eta^*(x)\right]\right\} = 0
\end{aligned} \tag{8.44}$$

as advertised. To obtain this result the variation of the ghost term of the new action has been integrated by parts and the identity

$$(\omega\chi)^* = \chi^*\omega^* \tag{8.45}$$

for Grassmann numbers has been used. Inclusing sources one has to add source terms σ for the ghost fields; again, they are scalars obeying Fermi statistics. The generating functional then reads

$$\begin{aligned}\exp\{i\mathcal{W}[J]\} &= \mathcal{N}'\int \mathcal{D}A_\mu\mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\eta^*\mathcal{D}\eta \\ &\qquad \exp\left\{i\mathcal{S}' + \int \mathrm{d}^4x\left[J_\mu A^\mu + i\bar{\chi}\psi + i\bar{\psi}\chi + i\eta^*\sigma + i\eta\sigma^*\right]\right\}\end{aligned} \tag{8.46}$$

Applying now a BRS transformation, two things can be noted: first, the Jacobian is unity; second, since the action is invariant under the transformations only the source terms are affected. Then, the generating functional, after having applied the BRS transformation, yields

$$\begin{aligned}\exp\{i\mathcal{W}[J]\} &= \mathcal{N}'\int \mathcal{D}A_\mu\mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\eta^*\mathcal{D}\eta \\ &\qquad \exp\left\{i\mathcal{S}' + \int \mathrm{d}^4x\left[J_\mu A^\mu + i\bar{\chi}\psi + i\bar{\psi}\chi + i\eta^*\sigma + i\eta\sigma^*\right]\right\} \\ &\qquad \exp\left\{\int \mathrm{d}^4x\left[J_\mu \delta A^\mu + i\bar{\chi}\delta\psi + i\delta\bar{\psi}\chi + i\delta\eta^*\sigma + i\delta\eta\sigma^*\right]\right\}.\end{aligned} \tag{8.47}$$

Specialising to those BRS transformations for which ξ is real, i.e. $\xi^* = \xi$, and demanding that the Greens functions remain invariant means that the difference of both corresponding generating functionals vanishes. By expanding the exponential in ξ and noting that the second term $\sim \xi^2$ is zero because of $\xi^2 = 0$, one has

$$\begin{aligned}0 = &\int \mathcal{D}A_\mu\mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\eta^*\mathcal{D}\eta \\ &\qquad \exp\left\{i\mathcal{S}' + \int \mathrm{d}^4x\left[J_\mu A^\mu + i\bar{\chi}\psi + i\bar{\psi}\chi + i\eta^*\sigma + i\eta\sigma^*\right]\right\} \\ &\qquad \left\{\int \mathrm{d}^4x\left[J_\mu\partial^\mu\frac{\eta+\eta^*}{e} - i\bar{\chi}(\eta+\eta^*)\psi + i\bar{\psi}(\eta+\eta^*)\chi \right.\right. \\ &\qquad\qquad \left.\left. + \frac{i}{\alpha e}(\partial\cdot A)(\sigma-\sigma^*)\right]\right\}.\end{aligned} \tag{8.48}$$

Effective action and Ward-Takahashi identities

To continue, one may introduce the effective action through a Legendre transformation from the path integral.

$$\Gamma[\phi_{\rm cl.}] = \mathcal{W}[J] - \int {\rm d}^4x \left(J\cdot A_{\rm cl.} + i\bar\chi\psi_{\rm cl.} + i\bar\psi_{\rm cl.}\chi + \eta^*_{\rm cl.}\sigma + \eta_{\rm cl.}\sigma^*\right) . \quad (8.49)$$

this effective action serves as the generating functional of all n point functions, i.e. proper vertices, and it is formulated in terms of the classical fields. There,m the J_μ, etc., are the classical sources, defined through equation of the form

$$J_\mu = -\frac{\delta\Gamma}{\delta A^\mu_{\rm cl.}} . \quad (8.50)$$

In terms of Γ the equation above, Eq.8.48, can be written in manageable form as

$$\begin{aligned} 0 &= \int {\rm d}^4x \left[-\frac{1}{e}\frac{\partial\Gamma}{\partial A^\mu_{\rm cl.}}\partial^\mu(\eta_{\rm cl.}+\eta^*_{\rm cl.}) \right. \\ &\quad + i\frac{\partial\Gamma}{\partial\psi_{\rm cl.}}(\eta_{\rm cl.}+\eta^*_{\rm cl.})\psi_{\rm cl.} - i\bar\psi_{\rm cl.}(\eta_{\rm cl.}+\eta^*_{\rm cl.})\frac{\partial\Gamma}{\partial\bar\psi_{\rm cl.}} \\ &\quad \left. + \frac{i}{\alpha e}\partial\cdot A_{\rm cl.}\frac{\partial\Gamma}{\partial\eta_{\rm cl.}} - \frac{i}{\alpha e}\partial\cdot A_{\rm cl.}\frac{\partial\Gamma}{\partial\eta^*_{\rm cl.}}\right] . \end{aligned} \quad (8.51)$$

This is the most transparent form of the Ward-Takhashi identities for QED. In the following it will be applied to some of the simplest cases.

1. The dependence of Γ on the gost fields is simple, since they do not interact. Therefore one can write

   $$\Gamma = i\int {\rm d}^4x{\rm d}^4y\eta^*_{\rm cl.}(x)\Delta^{-1}(x-y)\eta_{\rm cl.}(y) + \Gamma'[A^\mu_{\rm cl}, \psi_{\rm cl}, \bar\psi_{\rm cl}] \quad (8.52)$$

   with the free propagator

   $$\Delta(x-y) = -\partial^2\delta^4(x-y) . \quad (8.53)$$

   In contrast the expression for Γ' is not so simple. It starts out with

   $$\begin{aligned} \Gamma' &= \int {\rm d}^4x{\rm d}^4y \left[\bar\psi_{\rm cl.}(x)S_F^{-1}(x-y)\psi_{\rm cl.}(y) + \frac{1}{2}A^\mu_{\rm cl}(x)\Delta^{-1}_{\mu\nu}(x-y)A^\nu_{\rm cl}(y)\right] \\ &\quad + \int {\rm d}^4x{\rm d}^4y{\rm d}^4z \left[\bar\psi_{\rm cl.}(x)A^\mu_{\rm cl}(y)\Gamma_\mu(x,y,z)\psi_{\rm cl.}(z) + \ldots\right] , \end{aligned}$$
   (8.54)

where Γ_μ is the three point function and S_F and $\Delta_{\mu\nu}$ are the free field propagators for the fermion and photon field, respectively. Of course, Γ' contains many more interaction terms corresponding to various n point functions.

applying now Eq.(8.51) to Eqs.(8.52) and (8.54) and keeping only those terms that contain the photon or the ghost fields, i.e. A^μ or $\eta + \eta^*$, one obtains in momentum space

$$k^\mu \Delta^{-1}_{\mu\nu}(k) + \frac{1}{\alpha} k_\nu k^2 = 0 \,. \tag{8.55}$$

Writing the inverse propagator in a general form as

$$\Delta^{-1}_{\mu\nu}(k) = A(k^2) g_{\mu\nu} + B(k^2) k_\mu k_\nu \tag{8.56}$$

and plugging it into the equation above one finds the Ward identity

$$A(k^2) + B(k^2) k^2 + \frac{1}{\alpha} k^2 = 0 \tag{8.57}$$

and in Feynman gauge ($\alpha = 1$) the general form of the inverse propgator is

$$\Delta^{-1}_{\mu\nu}(k) = -g_{\mu\nu} k^2 + \left(g_{\mu\nu} k^2 - k_\mu k_\nu \right) F(k^2) \,. \tag{8.58}$$

Here $F(k^2)$ is at least of order e^2.

2. As another application, consider terms which contain ψ, $\bar{\psi}$ and the η's. One finds

$$\frac{\partial}{\partial y_\mu} \Gamma_\mu(x, y, z) - i S_F^{-1}(x - z) \delta(z - y) + i S_F^{-1}(x - z) \delta(x - y) = 0 \tag{8.59}$$

In momentum space this yields

$$(p - p')^\mu \Gamma_\mu(p, p - p', p') = S_F^{-1}(p) - S_F^{-1}(p') \,, \tag{8.60}$$

an identity that has been encountered and discussed before.

In the respective perturbative expansion of both terms this is easy to test at leading order.

$$\begin{aligned} \Gamma_\mu &= i\gamma_\mu + \ldots \\ S_F^{-1}(p) &= i(\not{p} - m) + \ldots \end{aligned} \tag{8.61}$$

and since S_F^{-1} is multiplicatively renormlised by Z_2 and Γ_μ by Z_1 the identity above enforces

$$Z_1 = Z_2\,. \tag{8.62}$$

Moreover, it is clear that it would be not a smart move to adopt any subtraction procedure (i.e. any prescription for the construction of counterterms) that violates the Ward identities because then the BSR symmetry would be violated.

Renormalisation prescriptions

Recalling the result of the vacuum polarisation diagram and the correspnoding counterterm, cf. Eqs.(8.12) and (8.23), yields for the photon propagator uo to order e^4

$$\begin{aligned}\Delta_{\mu\nu}(p) \;=\;& \frac{ig_{\mu\nu}}{p^2}\left[1+\frac{e^2}{2\pi^2}\left(-\int\limits_0^1 \mathrm{d}x x(1-x)\log\frac{m^2+p^2x(1-x)}{\hat{\mu}^2}-\frac{1}{6}F_3\right)\right]\\ &-\frac{ip_\mu p_\nu}{p^4}\left[\frac{e^2}{2\pi^2}\left(-\int\limits_0^1 \mathrm{d}x x(1-x)\log\frac{m^2+p^2x(1-x)}{\hat{\mu}^2}-\frac{1}{6}F_\alpha\right)\right]\,,\end{aligned} \tag{8.63}$$

where the $F_{3,\alpha}$ were a priori unknown terms in the counter terms. Requiring that for $p^2\to 0$ he photon propagator remains in the same form fixes these terms. In other words from

$$\left.\left\{\Delta_{\mu\nu}(p) = \frac{ig_{\mu\nu}}{p^2}\right\}\right|_{p^2=0} \tag{8.64}$$

it follows that

$$F_3 = F_\alpha = -\log\frac{m^2}{\hat{\mu}^2}\,. \tag{8.65}$$

Appendix A

Some useful formulae

A.1 Identities for γ matrices

All of them stem from two simple properties of the γ-matrices:

1. Independent of the number of space-time dimensions, the γ matrices act on a four-dimensional space of spinor indices.

2. Independent of the number of space-time dimensions, the γ matrices satisfy

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\,. \tag{A.1}$$

From these properties, the following identities for traces can be derived:

$$\begin{aligned}
\mathrm{Tr}[1] &= \sum_{\alpha=1}^{4}[\delta_{\alpha\alpha}] = 4\\
\mathrm{Tr}[\gamma^\mu\gamma^\nu] &= \mathrm{Tr}[\gamma^\mu\gamma^\nu] = \frac{1}{2}\mathrm{Tr}[\gamma^\mu\gamma^\nu + \gamma^\mu\gamma^\nu] = \mathrm{Tr}[g^{\mu\nu}] = 4g^{\mu\nu}\\
\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] &= 4\left[g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right]\\
\mathrm{Tr}[\not{a}_1\not{a}_2\not{a}_3\dots\not{a}_n] &= \begin{cases} 0 & n = \mathrm{odd}\\ a_1\cdot a_2\mathrm{Tr}[\not{a}_3\dots\not{a}_n] - a_1\cdot a_3\mathrm{Tr}[\not{a}_2\not{a}_4\dots\not{a}_n] & \\ \quad +\dots + a_1\cdot a_n\mathrm{Tr}[\not{a}_2\not{a}_3\dots\not{a}_{n-1}] & n = \mathrm{even}\end{cases}\\
\mathrm{Tr}[\gamma_5] &= \mathrm{Tr}[\not{a}\not{b}\gamma_5] = 0\\
\mathrm{Tr}[\not{a}\not{b}\not{c}\not{d}\gamma_5] &= 4i\epsilon^{\mu\nu\rho\sigma}a_\mu b_\nu c_\rho d_\sigma
\end{aligned} \tag{A.2}$$

Furthermore, one may derive identities for products of γ matrices in D dimensions. Starting from

$$\gamma^\mu \gamma_\mu = \frac{1}{2} \left(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu\right) g_{\mu\nu} = g_{\mu\nu} g^{\mu\nu} = g^\mu_\mu = D \tag{A.3}$$

one has

$$\begin{aligned}
\gamma^\mu \gamma^\nu \gamma_\mu &= \left(-\gamma^\nu \gamma^\mu + 2g^{\mu\nu}\right) \gamma_\mu = (-D+2)\gamma^\nu \\
\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu &= \left(-\gamma^\nu \gamma^\mu + 2g^{\mu\nu}\right) \gamma^\rho \gamma_\mu = -(-D+2)\gamma^\nu \gamma^\rho + 2\gamma^\rho \gamma^\nu \\
&= (D-4)\gamma^\nu \gamma^\rho + 4g^{\nu\rho} \\
\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\mu &= \left(-\gamma^\nu \gamma^\mu + 2g^{\mu\nu}\right) \gamma^\rho \gamma^\sigma \gamma_\mu \\
&= 2\gamma^\rho \gamma^\sigma \gamma^\nu - \gamma^\nu \left(-\gamma^\rho \gamma^\mu + 2g^{\rho\mu}\right) \gamma^\sigma \gamma_\mu \\
&= 2\gamma^\rho \gamma^\sigma \gamma^\nu - 2\gamma^\nu \gamma^\sigma \gamma^\rho + \gamma^\nu \gamma^\rho \gamma^\mu \gamma^\sigma \gamma_\mu \\
&= 2\gamma^\rho \gamma^\sigma \gamma^\nu - 2\gamma^\nu \gamma^\sigma \gamma^\rho + (2-D)\gamma^\nu \gamma^\rho \gamma^\sigma \\
&= -2\gamma^\rho \gamma^\sigma \gamma^\nu - (D-2)\gamma^\nu \gamma^\rho \gamma^\sigma \,.
\end{aligned} \tag{A.4}$$

A.2 Colour algebra

The defining equations are

$$\begin{aligned}
\left[T^a, T^b\right] &= i f^{abc} T^c \\
\left\{T^a, T^b\right\} &= \frac{1}{3}\delta^{ab} + d^{abc} T^c \,,
\end{aligned} \tag{A.5}$$

where the f_{abc} are antisymmetric structure constants. The only non-zero values are for

a	1	1	1	2	2	3	3	4	6
b	2	4	5	4	5	4	6	5	7
c	3	7	6	6	7	5	7	8	8
f_{abc}	1	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$

A convenient representation of the generators T^a is

$$
\begin{aligned}
T_1 &= \tfrac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad & T_2 &= \tfrac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
T_3 &= \tfrac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad & T_4 &= \tfrac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\
T_5 &= \tfrac{1}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad & T_6 &= \tfrac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\
T_7 &= \tfrac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad & T_8 &= \tfrac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
\end{aligned}
\tag{A.6}
$$

They satisfy, independent of representation,

$$
\begin{aligned}
T^a_{ij}T^a_{kl} &= \frac{1}{2}\left(\delta_{il}\delta_{jk} - \frac{1}{3}\delta_{ij}\delta_{kl}\right) \\
\mathrm{Tr}\left[T^a\right] &= 0 \\
\mathrm{Tr}\left[T^aT^b\right] &= \frac{1}{2}\delta^{ab} \\
\mathrm{Tr}\left[T^aT^bT^c\right] &= \frac{1}{4}\left(d^{abc} + if^{abc}\right) \\
\mathrm{Tr}\left[T^aT^bT^aT^d\right] &= -\frac{1}{12}\delta^{bd}\,.
\end{aligned}
\tag{A.7}
$$

The structure constants satisfy in turn Jacobi identities,

$$
\begin{aligned}
f_{abe}f_{ecd} + f_{cbe}f_{aed} + f_{dbe}f_{ace} &= 0 \\
f_{abe}d_{ecd} + f_{cbe}d_{aed} + f_{dbe}d_{ace} &= 0\,,
\end{aligned}
\tag{A.8}
$$

and also

$$
f_{abc}f_{cde} = \frac{2}{3}\left(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}\right) + \left(d_{ace}d_{bde} - d_{bce}d_{ade}\right)\,. \tag{A.9}
$$