Introduction

The following is an outline of von Neumann's argument against hidden variables, or "dispersion-free states" (states with zero uncertainty).

The thrust of the argument is that a theory with hidden variables cannot reproduce the expectation values of quantum mechanics. As we have confirmed QM's predictions for expectation values innumerable times, hidden variable theories are ruled out experimentally.

We consider a Hilbert space $\mathcal{H}$, a set of Hermitian linear operators, $\mathcal{O}$, which represent observables, and physical states, $|\psi\rangle \in \mathcal{H}$. For any given observable, $O \in \mathcal{O}$ there exists a set of eigenstates, $\{|\phi_n\rangle\} \in \mathcal{H}$ such that,

$$O|\phi_n\rangle = v_n(O)|\phi_n\rangle, \tag{1}$$

where $v_n(O)$ is the eigenvalue for the n^{th} eigenstate of the observable O..

We are interested in defining a function, $E(|\psi\rangle, O)$, the expectation value, which has the properties:

$$E : \mathcal{H} \otimes \mathcal{O} \quad \rightarrow \quad \mathbb{R}, \tag{2}$$

and

$$E(|\psi\rangle, O) \quad = \quad \langle\psi|O|\psi\rangle, \tag{3}$$

for all $\{|\psi\rangle, O\} \in \mathcal{H} \otimes \mathcal{O}$.

von Neumann's theorem

Theorem (von Nuemann 1932). *The general form of the function which fulfils these requirements is given by,*

$$E(|\psi\rangle, O) \quad = \quad \mathrm{Tr}\big(\rho_\psi O\big) \tag{4}$$

where ρ_ψ is a positive operator with the property

$$\mathrm{Tr}\big(\rho_\psi\big) \quad = \quad 1, \tag{5}$$

otherwise known as the density operator for the state $|\psi\rangle$.

We will not prove this theorem here but consider its relevance to the question of hidden variables.

Hidden variables

A hidden variable extension of quantum mechanics assumes that the state of the system is fully specified by the quantum state *and* some additional variable (or variables). Mathematically we describe this by extending the Hilbert space to include the space of Hidden variables, $\mathcal{L}$, such that,

$$\mathcal{H}_{HV} = \mathcal{H} \otimes \mathcal{L}. \tag{6}$$

A generic hidden variable theorem asserts that the state of the system is given by a point in this extended space, labelled by the quantum state and the hidden variable,

$$\psi_{HV} = (|\psi\rangle, \lambda), \tag{7}$$

where $|\psi\rangle \in \mathcal{H}$, $\lambda \in \mathcal{L}$, $\psi_{HV} \in \mathcal{H}_{HV}$. The whole point of a hidden variable description is that the addition of the extra variables specifies the exact state of the system beyond what can be achieved by a quantum state. In practice this means that the system has a "value" for each observable irrespective of any measurement being performed, for all observables simultaneously. Another way of stating this is that the state has no uncertainty associated with it. Recall that a statistical uncertainty on an observable is given by,

$$\Delta O = \sqrt{\langle O\rangle^2 - \langle O^2\rangle}, \tag{8}$$

so for a hidden variable state with no inherent uncertainty,

$$\langle O\rangle^2_{\psi_{HV}} = \langle O^2\rangle_{\psi_{HV}}. \tag{9}$$

This is equivalent to the fact that for a hidden variable description, the expectation value is assumed to be an eigenvalue.

Consider the subset of projection operators defined with respect to the quantum state $|\eta\rangle$, $P_\eta \in \mathcal{P} \subset \mathcal{O}$. Projection operators represent observables with a binary (yes or no) answer. They have the properties,

$$P_\eta = |\eta\rangle\langle\eta|, \tag{10}$$

$$P_\eta^2 = P_\eta, \tag{11}$$

$$E(\psi, P_\eta) \geqslant 0, \quad \forall\{\psi, \eta\} \in \mathcal{H}. \tag{12}$$

Note that the density operator, ρ_ψ is also a projection operator.

Von Neumann's argument against hidden variables

We now consider the expectation value for a hidden variable state, ψ_{HV}, which has no variance,

$$E(\psi_{HV}, P_\eta)^2 = E(\psi_{HV}, P_\eta^2) = E(\psi_{HV}, P_\eta), \tag{13}$$

where we have used Eq. 9 in the first equality and Eq. 11 in the second. If a real number equals its square then it can only be zero or one. The expectation value is a continuous function in η and so it must be a constant, namely,

$$E(\psi_{HV}, P_\eta) = \{0, 1\}. \tag{14}$$

For any projection operator, P_η,

$$\begin{aligned} E(\psi, P_\eta) &= \langle\psi|P_\eta|\psi\rangle \\ &= \langle\psi|\eta\rangle\langle\eta|\psi\rangle \\ &= \langle\eta|P_\psi|\eta\rangle, \end{aligned} \tag{15}$$

where $P_\psi = \rho_\psi$, the density operator. Now, for a hidden variable state this expectation value can only be zero or one.

If the expectation value is zero then,

$$\langle\eta|\rho_\psi|\eta\rangle \quad = \quad 0, \quad \forall\ |\eta\rangle \in \mathcal{H}, \tag{16}$$

such that the density operator is just the null operator in the set $\mathcal{O}$, $\rho_\psi = \oslash$.

On the other hand, if the expectation value is one then,

$$\langle\eta|\rho_\psi|\eta\rangle \quad = \quad 1, \quad \forall\ |\eta\rangle \in \mathcal{H}. \tag{17}$$

This is satisfied if the density operator is the unit operator in $\mathcal{O}$, $\rho_\psi = \mathbb{I}$.

In either case, $\rho_\psi = \oslash$ and $\rho_\psi = \mathbb{I}$, the density operator cannot satisfy the properties required of it to be consistent with the expectation values of quantum mechanics, i.e. $\mathrm{Tr}(\rho_\psi) = 1$. In the case of the null operator, $\mathrm{Tr}(\rho_\psi) = 0$ and in the case of the identity operator, $\mathrm{Tr}(\rho_\psi) = d$, where d is the dimension of the Hilbert space.

From this contradiction von Neumann concludes that such a hidden variable state, with zero variance, cannot exist and reproduce the predictions (expectation values) of quantum mechanics.

Hermann-Bell critique

Hermann (1935) and Bell (1965) noticed that this argument against hidden variables was overly restrictive because it contained an implicit assumption about the hidden variable theory. In constructing his theorem (stated above), von Neumann had made three assumptions:

1. $E(|\psi\rangle, \mathbb{I}) = 1, \quad \forall\ |\psi\rangle \in \mathcal{H}$,
2. $E(|\psi\rangle, \alpha A + \beta B) = \alpha E(|\psi\rangle, A) + \beta E(|\psi\rangle, B), \quad \{\alpha, \beta\} \in \mathbb{R}, \{A, B\} \in \mathcal{O}$,

3. $E(|\psi\rangle, P) \geqslant 0, \quad \forall\ |\psi\rangle \in \mathcal{H},\ P \in \mathcal{P}$.

The second assumption is true in quantum mechanics, however it does not need to be true for a hidden variable theorem for each hidden variable state and in the case of non-commuting operators A and B cannot be true for each hidden variable state.

When considering a hidden variable description von Neumann assumed that the expectation value for a given hidden variable state is simply an eigenvalue of that observable. To assume that assumption 2 applies to hidden variable states as well is then to assume that eigenvalues add linearly, in the same way that quantum mechanical expectation values do. This is due to the identification of the expectation value with the eigenvalue for hidden variable states (they have no uncertainty). However, for non-commuting operators A and B, this is simply not true; their expectation values add linearly, but their eigenvalues do not.

For example consider the non-commuting observables of spin along the x and z axes. Let the quantum state of the system be given by a state which is not an eigenstate of any operator considered here,

$$\psi = \frac{1}{\sqrt{5}}\begin{pmatrix}2\\1\end{pmatrix}. \tag{18}$$

Now we consider the expectation value of the operator,

$$O = \frac{1}{\sqrt{2}}(S_x + S_z) \tag{19}$$

$$= \frac{\hbar}{2\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix} \tag{20}$$

which is given by,

$$E(\psi, O) = \frac{\hbar}{10\sqrt{2}}\begin{pmatrix}2 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix} \tag{21}$$

$$= \frac{7\hbar}{10\sqrt{2}} \tag{22}$$

We can also calculate this using the linearity of the expectation value (assumption 2),

$$E(\psi, O) = \underbrace{\frac{\hbar}{10\sqrt{2}}\begin{pmatrix}2 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix}}_{E(\psi,\frac{1}{\sqrt{2}}S_x)=\frac{4\hbar}{10\sqrt{2}}} + \underbrace{\frac{\hbar}{10\sqrt{2}}\begin{pmatrix}2 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix}}_{E(\psi,\frac{1}{\sqrt{2}}S_z)=\frac{3\hbar}{10\sqrt{2}}} \tag{23}$$

$$= \frac{7\hbar}{10\sqrt{2}}. \tag{24}$$

This confirms that assumption 2 is valid when applied to quantum mechanical expectation values. Now consider the eigenvalues of these operators,

$$S_x \sim \pm\frac{\hbar}{2}, \tag{25}$$

$$S_z \sim \pm\frac{\hbar}{2}, \tag{26}$$

$$O \sim \pm\frac{\hbar}{2}, \tag{27}$$

note the operator O is just measuring the spin along the direction bisecting the x and z axes. A hidden variable state would have a definite value for each of these observables equal to one of the eigenvalues. von Neumann's assumption 2, applied to eigenvalues, would give us the relation,

$$\pm\frac{\hbar}{2} = \frac{1}{\sqrt{2}}\left(\pm\frac{\hbar}{2}\pm\frac{\hbar}{2}\right) \tag{28}$$

which is clearly false as the right hand side is either $\pm\frac{\hbar}{\sqrt{2}}$ or 0. There is some debate as to whether von Neumann understood this or not. Bell was astounded by an assumption which is clearly false for such a simple example. Others, such as Bubb (`arXiv:1006.0499`), have argued that von Neumann understood this perfectly well and Bell misinterpreted what he wrote.

Bell argued that the linearity condition need not apply per hidden variable state, but only when averaged over the hidden variables. We say that the hidden variable state ψ_{HV} has definite values for the operators A, B and O and that these values are eigenvalues of those operators. The specific value taken depends on the hidden variable. e.g. consider the values of the observables, $v(S_x, \lambda)$, $v(S_z, \lambda)$, $v(O, \lambda)$, which are eigenvalues of the operators determined by the unknown value of the hidden variable, λ.

If we look at a specific sub-ensemble corresponding to a given value of λ we see that the linearity assumption does not apply,

$$v(O, \lambda) \neq \frac{1}{\sqrt{2}}\big(v(S_x, \lambda) + v(S_z, \lambda)\big). \tag{29}$$

because eigenvalues do not generally add linearly. However, the hidden variables are just that, hidden, so to reproduce the predictions of quantum mechanics we only need to consider how this hidden variable state behaves once summed over the hidden variables. We do this by considering the sum over the sub-ensembles which make up the quantum state, weighted by a density of states over the sub-ensembles, $\omega(\lambda)$,

$$E(\psi_{HV}, O) = \sum_{\lambda} \omega(\lambda) v(O, \lambda). \tag{30}$$

So we see that the assumption of linearity does not need to hold per hidden variable state and that if it merely holds when averaging over the hidden variables then that is sufficient to evade von Neumann's argument. This extra freedom in the hidden variable description allows the predictions of QM to be reproduced.

Specifically, in von Neumann's argument he assumed that for a hidden variable state, ψ_{HV}, expectation values are eigenvalues and so have zero variance. Consider an observable O with eigenvalue $v(O)$. If we assume that expectation values are eigenvalues then,

$$E(\psi_{HV}, O)^2 = v(O)^2, \tag{31}$$

$$E(\psi_{HV}, O^2) = v(O^2) = v(O)^2, \tag{32}$$

such that $\Delta O = 0$. However, if we take into account the density of states over hidden variables then,

$$E(\psi_{HV}, O)^2 = \sum_{\lambda,\lambda'} \omega(\lambda)\omega(\lambda')v(O,\lambda)v(O,\lambda'), \tag{33}$$

$$E(\psi_{HV}, O^2) = \sum_{\lambda} \omega(\lambda)v(O^2,\lambda). \tag{34}$$

These are clearly not the same and so von Neumann's assumption about hidden variable expectation values is false. We retain some quantum mechanical uncertainty due to our ignorance of the hidden variables, even though the state is in fact completely determined.