Supersymmetry

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1 The Coleman-Mandula no-go theorem

1.1 Introduction

The Coleman-Mandula no-go theorem [1] is a powerful theorem that essentially states that, given some reasonable (physical assumptions) that the only possible Lie algebra (as opposed to super-algebra or graded Lie algebra) of symmetry generators consist of the generators P_μ and $J_{\mu\nu}$ of the Poincaré group, and internal symmetry generators who commute with the Poincaré group and act on physical states by multiplying them with spin-independent, momentum-independent Hermitian matrices.

By a symmetry generator, we mean any Hermitian operator that commutes with the S-matrix; whose commutators are also symmetry generators; that takes single particle states into single particle states; and whose action on multiparticle states is a direct sum of actions on single-particle states.

Presented here is a filled out version of the proof of the theorem heavily lifted from Weinberg's Supersymmetry [2]. I do so as I have not been able to find any more accessible documents that discuss the theorem in any detail. After proving the theorem, we will then see how Supersymmetry manages to evade it by the relaxing one of the assumptions made in the theorem.

1.2 Statement of the theorem

If $\mathfrak{G}$ is a symmetry group of the S-Matrix, and the following assumptions hold:

1. For any M there are only a finite number of particle types with mass less than M,
2. Scattering occurs at almost all energies (except for perhaps some isolated set of energies),
3. The amplitudes for elastic two-body scattering are analytic functions of the scattering angle at almost all energies and angles,

then the generators of $\mathfrak{G}$ consist of only the generators of the Poincaré group $\mathfrak{P}$, and the generators of internal symmetries.

1.3 The B_α subalgebra spanned by generators that commute with P_μ

Define the subalgebra of $\mathfrak{G}$ to be generated by the symmetry generators B_α who commute with the four-momentum operator P_μ

$$[B_\alpha, P_\mu] = 0. \tag{1}$$

Let B_α have a momentum-dependent matrix representation b_α when acting upon single particle states

$$B_\alpha |p\, m\rangle = \sum_{m'} \Big(b_\alpha(p)\Big)_{m'm} |p\, m'\rangle. \tag{2}$$

m is a discrete index labelling spin z-components and particle type for particles of a definite mass $\sqrt{p_\mu p^\mu}$. Now, if the generators B_α obey a Lie algebra

$$[B_\alpha, B_\beta] = i \sum_\gamma C^\gamma_{\alpha\beta} B_\gamma, \tag{3}$$

then their matrix representations obey the same Lie algebra, since

$$\begin{aligned}
[B_\alpha, B_\beta]|p\, m\rangle &= \sum_{m',m''} \left[\Big(b_\alpha(p)\Big)_{m'm''}\Big(b_\beta(p)\Big)_{m''m} - (\alpha \leftrightarrow \beta)\right] |p\, m'\rangle \\
&= \sum_{m'} \left[\Big(b_\alpha(p) b_\beta(p)\Big)_{m'm} - (\alpha \leftrightarrow \beta)\right] |p\, m'\rangle \\
&= \sum_{m'} [b_\alpha(p), b_\beta(p)]_{m'm} |p\, m'\rangle \\
&= i \sum_\gamma C^\gamma_{\alpha\beta} B_\gamma |p\, m\rangle \\
&= i \sum_\gamma \sum_{m'} C^\gamma_{\alpha\beta} \Big(b_\gamma(p)\Big)_{m'm} |p\, m'\rangle.
\end{aligned}$$

Rearranging the above gives

$$\sum_{m'} \left([b_\alpha(p), b_\beta(p)]_{m'm} - i \sum_\gamma C^\gamma_{\alpha\beta} \left(b_\gamma(p)\right)_{m'm} \right) |p\, m'\rangle = 0,$$

and since $|p\, m'\rangle \neq 0$, then

$$[b_\alpha(p), b_\beta(p)]_{m'm} = i \sum_\gamma C^\gamma_{\alpha\beta} \Big(b_\gamma(p)\Big)_{m'm},$$

and so

$$[b_\alpha(p), b_\beta(p)] = i \sum_\gamma C^\gamma_{\alpha\beta} b_\gamma(p).$$

I will be honest, I'm not sure how to get rid of the sum over m'. My guess would be having to make each element of a new sum a modulus squared by perhaps considering the hermitian square of the above expression. Then instead of just the sum over m' being zero, each element of the sum would also be zero. Perhaps look at this again at some point.

As the operators B_α and the matrices $b_\alpha(p)$ share commutation relations, there is a homeomorphism between B_α and $b_\alpha(p)$. A homeomorphism is a mapping between two sets that preserves some mathematical structure. What we would really like is for there to be an isomorphism between B_α and $b_\alpha(p)$, i.e. a one-to-one correspondence between the elements B_α and $b_\alpha(p)$. This would be a bijective homeomorphism. Group theory aside, why should we want to do this? A powerful theorem proved in Section 15.2 of Weinberg's Applications [3] states that any Lie algebra of finite Hermitian matrices (like $b_\alpha(p)$) must be a direct sum of a semi-simple Lie algebra and $U(1)$ algebras. If we would show an isomorphism between B_α and $b_\alpha(p)$, then the B_α algebra would also have to be a direct sum of a semi-simple Lie algebra and $U(1)$ algebras.

1.4 Finding the isomorphism

Naïvely, since for a given four-momentum p, B_α acting on a state $|p\rangle$ is represented by the matrix $b_\alpha(p)$, it is expected that there are the same number N of matrices B_α as there are $b_\alpha(p)$. Therefore, you might expect that the mapping

$$
\begin{aligned}
B_1 &\longrightarrow b_1(p) \\
B_2 &\longrightarrow b_2(p) \\
\vdots \quad &\qquad \vdots \\
B_N &\longrightarrow b_N(p)
\end{aligned}
$$

is an isomorphism. Issues can arise however, if e.g. the $b(k)$ are degenerate. Consider the action of B_1 and B_2 on a one-particle state of four-momentum p

$$
\begin{aligned}
B_1|p\rangle &= b_1(p)|p\rangle \\
B_2|p\rangle &= b_2(p)|p\rangle.
\end{aligned}
$$

Now imagine there is a degeneracy in the $b(k)$ such that

$$b_1(p) = b_2(p).$$

We can work with a different set of matrices who are linear combinations of $b_\alpha(p)$. For convenience, we will just use a new set where $\tilde{b}_1 = b_1$, $\tilde{b}_2 = 0$, and $\tilde{b}_i = b_i$ for $i = 3, 4 \ldots N$. For this choice of four-momentum p, we would then have a mapping

$$
\begin{aligned}
B_1, B_2 &\longrightarrow \tilde{b}_1(p) \\
B_3 &\longrightarrow \tilde{b}_3(p) \\
\vdots \quad &\qquad \vdots \\
B_N &\longrightarrow \tilde{b}_N(p),
\end{aligned}
$$

which is not an isomorphism, since it is not bijective; given $\tilde{b}_1(p)$ we cannot tell whether B_1 or B_2 acted on the state $|p\rangle$ to produce it. We could fix this problem, however, if we could show that if any degeneracy occured in the $b_\alpha(p)$, then the B_α shared such a degeneracy. This would mean that, if for some four-momentum p, then such that if there are some coefficients c^α such that

$$\sum_\alpha c^\alpha b_\alpha(p) = 0, \tag{4}$$

that is, the $b_\alpha(p)$ are not linearly independant, then for the same set of coefficients c^α, then

$$\sum_\alpha c^\alpha B_\alpha = 0, \tag{5}$$

i.e. the B_α are not linearly independant in the same way. Showing $\sum_\alpha c^\alpha b_\alpha(k) = 0$ for all four-momentum k, is equivalent to the condition $\sum_\alpha c^\alpha B_\alpha = 0$. Therefore, if we can show that whenever $\sum_\alpha c^\alpha b_\alpha(p) = 0$ for some coefficients c^α and four-momentum p, and it is is also true that $\sum_\alpha c^\alpha b_\alpha(k) = 0$ for any four-momentum k, then $b_\alpha(p)$ and B_α share the same degeneracies. Back to the example we considered, this would imply that if $b_1(p) = b_2(p)$, then this would be the same as saying e.g. $c^1 = 1, c^2 - 1, c^i = 0$ for $i = 3, 4, \ldots N$, and therefore $B_1 = B_2$. Then, using the same linear combinations as before, we would have the mapping (dropping tildes)

$$\begin{aligned} B_1 &\longrightarrow b_1(p) \\ B_3 &\longrightarrow b_3(p) \\ \vdots & \qquad\quad \vdots \\ B_N &\longrightarrow b_N(p), \end{aligned}$$

which is an isomorphism. This is all very lovely, but what we need to show for this to be true is that whenever $\sum_\alpha c^\alpha b_\alpha(p) = 0$ for some coefficients c^α and four-momentum p, and it is is also true that $\sum_\alpha c^\alpha b_\alpha(k) = 0$ for any four-momentum k, which is what we shall do now.

Consider the action of B_α on two-particle states

$$B_\alpha |p\, m, q\, n\rangle = \sum_{m'} \Big(b_\alpha(p)\Big)_{m'm} |p\, m', q\, n\rangle + \sum_{n'} \Big(b_\alpha(p)\Big)_{n'n} |p\, m, q\, n'\rangle. \tag{6}$$

We can then define a matrix representation $b_\alpha(p, q)$ of the action of B_α on two-particle states

$$B_\alpha |p\, m, q\, n\rangle = \sum_{m'n'} \Big(b_\alpha(p, q)\Big)_{m'n', mn} |p\, m', q\, n'\rangle, \tag{7}$$

where

$$\Big(b_\alpha(p, q)\Big)_{m'n', mn} = \Big(b_\alpha(p)\Big)_{m'm} \delta_{n'n} + \Big(b_\alpha(q)\Big)_{n'n} \delta_{m'm}. \tag{8}$$

Now we need to consider the invariance of the S-Matrix S for elastic or quasi-elastic scattering for two-particle states into two-particle states. Before we do that, we shall quickly review what we mean by the scattering and the S-Matrix.

In scattering, we consider physical states to be asymptotic in the sense that states in the distant past ($t \to -\infty$) are denoted $|\text{in}\rangle$ states, and states in the distant future ($t \to -\infty$) are denoted $|\text{out}\rangle$ states. We can choose our physical states to be in an orthonormal

$$\langle m, \text{in}|n, \text{in}\rangle = \langle m, \text{out}|n, \text{out}\rangle = \delta_{mn}. \tag{9}$$

They also form a complete set of states

$$\sum_m |m, \text{in}\rangle\langle m, \text{in}| = \sum_m |m, \text{out}\rangle\langle m, \text{out}| = \mathbf{1}. \tag{10}$$

The matrix elements of S give the overlap between configurations of in and out states, and defines a unitary operator on states

$$S(\psi \to \phi) = \langle \phi, \text{out}|\psi, \text{in}\rangle = \langle \phi, \text{out}|S|\psi, \text{out}\rangle = \langle \phi, \text{in}|S|\psi, \text{in}\rangle, \tag{11}$$

so S satisfies the relation

$$S = \sum_m |m, \text{in}\rangle\langle m, \text{out}| \tag{12}$$

$$S^\dagger = \sum_m |m, \text{out}\rangle\langle m, \text{in}|, \tag{13}$$

and then it is easy to see that

$$S^\dagger S = SS^\dagger = 1. \tag{14}$$

Now, to isolate the interesting part of the S-Matrix, i.e. the connected part, it is usual to define the *T-matrix*

$$S = \mathbf{1} + iT, \tag{15}$$

where the **1** corresponds to particle states not interacting. T is also known as the connected part of the S-matrix.

If we consider the scattering of particles with four-momenta p and q into particles with four-momenta p' and q' with masses $\sqrt{p_\mu p^\mu} = \sqrt{p'_\mu p'^\mu}$ and $\sqrt{q_\mu q^\mu} = \sqrt{q'_\mu q'^\mu}$, then the connected part of the S-matrix can be written

$$\begin{aligned} S(p\,m, q\,n \to p'\,m', q'\,n')_{\text{connected}} &= \langle p'\,m', q'\,n'; \text{out}|p\,m, q\,n; \text{in}\rangle_{\text{connected}} \\ &= \langle p'\,m', q'\,n'|S|p\,m, q\,n\rangle_{\text{connected}} \\ &= \langle p'\,m', q'\,n'|iT|p\,m, q\,n\rangle. \end{aligned}$$

It is then convinient to extract the momentum conserving delta function from S (and T) to define the *invariant matrix element* $\mathcal{M}$

$$\langle p'\,m', q'\,n'|iT|p\,m, q\,n\rangle = (2\pi)^4\delta^{(4)}(p'+q'-p-q)i\Big(\mathcal{M}(p',q';p,q)\Big)_{m'n',mn},$$

and so ignoring numerical prefactors,

$$S(p\,m, q\,n \to p'\,m', q'\,n')_{\text{connected}} = \delta^{(4)}(p'+q'-p-q)\Big(\mathcal{M}(p',q';p,q)\Big)_{m'n',mn}, \tag{16}$$

in agreement with Eq. 24.B.6 in Weinberg. Now, if the B_α are symmetry generators, they must commute with the S-matrix

$$[B_\alpha, S] = 0. \tag{17}$$

It follows that for the scattering situation above,

$$\langle p'\,m', q'\,n'|[B_\alpha, S]|p\,m, q\,n\rangle = 0. \tag{18}$$

Rearranging gives

$$\langle p'\,m', q'\,n'|B_\alpha S|p\,m, q\,n\rangle = \langle p'\,m', q'\,n'|SB_\alpha|p\,m, q\,n\rangle.$$

Inserting a complete set of states, noting the relativistic one-particle identity operator

$$(\mathbf{1})_{1-\text{particle}} = \int \frac{d^3p}{(2\pi)^3}|p\rangle\frac{1}{2E_p}\langle p| \tag{19}$$

we find (this needs to be updated to include the factor of $1/2E_p$), althought the overall result does not depend on this)

$$\begin{aligned}&\sum_{m''n''}\int\frac{d^3p''}{(2\pi)^3}\int\frac{d^3q''}{(2\pi)^3}\langle p'\,m', q'\,n'|B_\alpha|p''m'', q''n''\rangle\langle p''m'', q''n''|S|p\,m, q\,n\rangle\\ =&\sum_{m''n''}\int\frac{d^3p''}{(2\pi)^3}\int\frac{d^3q''}{(2\pi)^3}\langle p'\,m', q'\,n'|S|p''m'', q''n''\rangle\langle p''m'', q''n''|B_\alpha|p\,m, q\,n\rangle.\end{aligned}$$

Now using the matrix representation of B_α acting on two-particle states in Eq. (7),

$$\begin{aligned}&\sum_{\hat{m}\hat{n}}\sum_{m''n''}\int\frac{d^3p''}{(2\pi)^3}\int\frac{d^3q''}{(2\pi)^3}\langle p'\,m', q'\,n'|\Big(b_\alpha(p'',q'')\Big)_{\hat{m}\hat{n},m''n''}|p''\hat{m}, q''\hat{n}\rangle\langle p''m'', q''n''|S|p\,m, q\,n\rangle\\ =&\sum_{\hat{m}\hat{n}}\sum_{m''n''}\int\frac{d^3p''}{(2\pi)^3}\int\frac{d^3q''}{(2\pi)^3}\langle p'\,m', q'\,n'|S|p''m'', q''n''\rangle\langle p''m'', q''n''|\Big(b_\alpha(p,q)\Big)_{\hat{m}\hat{n},mn}|p\,\hat{m}, q\,\hat{n}\rangle.\end{aligned}$$

The matrices b_α are a set of summed coefficients, and so in the usual way, states can be moved past them inside a sum

$$
\begin{aligned}
&\sum_{\hat{m}\hat{n}}\sum_{m''n''}\int\frac{d^3p''}{(2\pi)^3}\int\frac{d^3q''}{(2\pi)^3}\Big(b_\alpha(p'',q'')\Big)_{\hat{m}\hat{n},m''n''}\langle p'\,m',q'\,n'|p''\hat{m},q''\hat{n}\rangle\langle p''m'',q''n''|S|p\,m,q\,n\rangle\\
=&\sum_{\hat{m}\hat{n}}\sum_{m''n''}\int\frac{d^3p''}{(2\pi)^3}\int\frac{d^3q''}{(2\pi)^3}\langle p'\,m',q'\,n'|S|p''m'',q''n''\rangle\Big(b_\alpha(p,q)\Big)_{\hat{m}\hat{n},mn}\langle p''m'',q''n''|p\,\hat{m},q\,\hat{n}\rangle. \qquad (20)
\end{aligned}
$$

Now, the relativistic normalisation of two-particle states is (see Peskin and Schröeder page 109, Eq. 4.91)

$$
\langle p'q'|pq\rangle = 2E_p 2E_q(2\pi)^3\left(\delta^{(3)}(\mathbf{p}-\mathbf{p}')\delta^{(3)}(\mathbf{q}-\mathbf{q}')+\delta^{(3)}(\mathbf{p}-\mathbf{q}')\delta^{(3)}(\mathbf{q}-\mathbf{p}')\right), \qquad (21)
$$

where

$$
E_p = p^0 = \sqrt{\mathbf{p}^2+m^2}. \qquad (22)
$$

The addition of particle type orthonormality gives

$$
\begin{aligned}
\langle p'\,m',q'\,n'|p''\hat{m},q''\hat{n}\rangle =&2E_{p''}2E_{q''}(2\pi)^6\Big(\delta^{(3)}(\mathbf{p}''-\mathbf{p}')\delta^{(3)}(\mathbf{q}''-\mathbf{q}')\delta_{m'\hat{m}}\delta_{n'\hat{n}}+\\
&\delta^{(3)}(\mathbf{p}''-\mathbf{q}')\delta^{(3)}(\mathbf{q}''-\mathbf{p}')\delta_{m'\hat{n}}\delta_{n'\hat{m}}\Big) \qquad (23)\\
\langle p''m'',q''n''|p\,\hat{m},q\,\hat{n}\rangle =&2E_p 2E_q(2\pi)^6\Big(\delta^{(3)}(\mathbf{p}-\mathbf{p}'')\delta^{(3)}(\mathbf{q}-\mathbf{q}'')\delta_{m''\hat{m}}\delta_{n''\hat{n}}+\\
&\delta^{(3)}(\mathbf{p}-\mathbf{q}'')\delta^{(3)}(\mathbf{q}-\mathbf{p}'')\delta_{m''\hat{n}}\delta_{n''\hat{m}}\Big). \qquad (24)
\end{aligned}
$$

Evaluating the S-Matrix elements gives (up to constant numerical prefactors)

$$
\langle p''m'',q''n''|S|p\,m,q\,n\rangle = \delta^{(4)}\big((p''+q'')-(p+q)\big)\Big(\mathcal{M}(p'',q'';p,q)\Big)_{m''n'',mn} \qquad (25)
$$

$$
\langle p'\,m',q'\,n'|S|p''m'',q''n''\rangle = \delta^{(4)}\big((p''+q'')-(p'+q')\big)\Big(\mathcal{M}(p',q';p'',q'')\Big)_{m'n',m''n''}. \qquad (26)
$$

Now looking at the form of the delta functions in (23) and (24), the delta function in (26) becomes redundant when inserted into (20). In addition, the factors of $2E_{p''}2E_{q''}$ and $2E_p2E_q$ will cancel due when inserted into (20) due to the

$$
\begin{aligned}
&p\to p'',q\to q'' \text{ and}\\
&p\to q'',q\to p''
\end{aligned}
$$

symmetry of the parts of Eq. (24). Putting this all into (20) and ignoring mentioned cancelling prefactors yields

$$
\begin{aligned}
&\sum_{\hat{m}\hat{n}}\sum_{m''n''}\int d^3p''\int d^3q''\Big(b_\alpha(p'',q'')\Big)_{\hat{m}\hat{n},m''n''}\Bigg(\delta^{(3)}(\mathbf{p}''-\mathbf{p}')\delta^{(3)}(\mathbf{q}''-\mathbf{q}')\delta_{m'\hat{m}}\delta_{n'\hat{n}}+\\
&\qquad\delta^{(3)}(\mathbf{p}''-\mathbf{q}')\delta^{(3)}(\mathbf{q}''-\mathbf{p}')\delta_{m'\hat{n}}\delta_{n'\hat{m}}\Bigg)\Bigg(\mathcal{M}(p'',q'';p,q)\Bigg)_{m''n'',mn}\\
=&\sum_{\hat{m}\hat{n}}\sum_{m''n''}\int d^3p''\int d^3q''\Bigg(\mathcal{M}(p',q';p'',q'')\Bigg)_{m'n',m''n''}\Big(b_\alpha(p,q)\Big)_{\hat{m}\hat{n},mn}\times\\
&\qquad\Bigg(\delta^{(3)}(\mathbf{p}-\mathbf{p}'')\delta^{(3)}(\mathbf{q}-\mathbf{q}'')\delta_{m''\hat{m}}\delta_{n''\hat{n}}+\delta^{(3)}(\mathbf{p}-\mathbf{q}'')\delta^{(3)}(\mathbf{q}-\mathbf{p}'')\delta_{m''\hat{n}}\delta_{n''\hat{m}}\Bigg).
\end{aligned}
$$

Now perform the sums over the 'hat' indices, as well as the integrals over momentum space

$$
\begin{aligned}
&\sum_{m''n''}\Bigg(\Big(b_\alpha(p',q')\Big)_{m'n',m''n''}\Bigg(\mathcal{M}(p',q';p,q)\Bigg)_{m''n'',mn}+\Big(b_\alpha(q',p')\Big)_{n'm',m''n''}\Bigg(\mathcal{M}(q',p';p,q)\Bigg)_{m''n'',mn}\Bigg)\\
=&\sum_{m''n''}\Bigg(\Bigg(\mathcal{M}(p',q';p,q)\Bigg)_{m'n',m''n''}\Big(b_\alpha(p,q)\Big)_{m''n'',mn}+\Bigg(\mathcal{M}(p',q';q,p)\Bigg)_{m'n',m''n''}\Big(b_\alpha(p,q)\Big)_{n''m'',mn}\Bigg).
\end{aligned}
\tag{27}
$$

Okay, now we need to deal with these indices. Let's look a bit more closely at the symmetry of $\Big(b_\alpha(p,q)\Big)_{m'n',mn}$ under indice and arguement interchange.

$\Big(b_\alpha(p,q)\Big)_{m'n',mn}$ is the matrix representation of B_α acting on two-particle states

$$
\begin{aligned}
B_\alpha|pm,qn\rangle &= \sum_{m'}\Big(b_\alpha(p)\Big)_{m',m}|pm',qn\rangle+\sum_{m'}\Big(b_\alpha(q)\Big)_{n',n}|pm,qn'\rangle\\
&=\sum_{m'}\sum_{n'}\left[\Big(b_\alpha(p)\Big)_{m',m}\delta_{n'n}+\Big(b_\alpha(q)\Big)_{n',n}\delta_{m'm}\right]|pm',qn'\rangle \qquad (28)\\
&=\sum_{n'}\sum_{m'}\left[\Big(b_\alpha(p)\Big)_{n',m}\delta_{m'n}+\Big(b_\alpha(q)\Big)_{m',n}\delta_{n'm}\right]|pn',qm'\rangle, \qquad (29)
\end{aligned}
$$

where from the second to third line we have just relabelled the free indices. Now using the definition in Eq. (7), we can relate the sums (28) and (29) in terms of $b_\alpha(p,q)$

$$
\begin{aligned}
\sum_{m'n'}\Big(b_\alpha(p,q)\Big)_{m'n',mn}|pm',qn'\rangle &= \sum_{n'm'}\Big(b_\alpha(p,q)\Big)_{n'm',mn}|pn',qm'\rangle \qquad (30)\\
&=\sum_{m'n'}\Big(b_\alpha(p,q)\Big)_{n'm',mn}|pn',qm'\rangle, \qquad (31)
\end{aligned}
$$

where we can reorder the sums over nm to over mn as we have assumed particle finiteness. Now consider the action of B_α on the same state, but then use the symmetries of of the state to get some more information

$$B_\alpha|pm,qn\rangle = (-1)^{\text{spin}} B_\alpha|qn,pm\rangle \tag{32}$$
$$= (-1)^{\text{spin}} \sum_{m'n'} \Big(b_\alpha(q,p)\Big)_{n'm',nm} |qn',pm'\rangle \tag{33}$$
$$= (-1)^{\text{spin}} \sum_{m'n'} \Big(b_\alpha(q,p)\Big)_{m'n',nm} |qm',pn'\rangle. \tag{34}$$

Now, for 'nice' behaviour, I will make the assumption that, since $B_\alpha|pm,qn\rangle$ is a spin-preserving symmetry generator (unlike the supersymmetry generators we will meet later), then each summed state $|qm',pn'\rangle$ possesses the same spin-symmetries as the initial state $|pm,qn\rangle$. Therefore

$$(-1)^{\text{spin}} \sum_{m'n'} \Big(b_\alpha(q,p)\Big)_{m'n',nm} |qm',pn'\rangle = \left((-1)^{\text{spin}}\right)^2 \sum_{m'n'} \Big(b_\alpha(q,p)\Big)_{m'n',nm} |pn',qm'\rangle \tag{35}$$
$$= \sum_{m'n'} \Big(b_\alpha(q,p)\Big)_{m'n',nm} |pn',qm'\rangle. \tag{36}$$

Putting Eqs. (36) and (31) together

$$B_\alpha|pm,qn\rangle = \sum_{m'n'} \Big(b_\alpha(p,q)\Big)_{n'm',mn} |pn',qm'\rangle \tag{37}$$
$$= \sum_{m'n'} \Big(b_\alpha(q,p)\Big)_{m'n',nm} |pn',qm'\rangle. \tag{38}$$

We can then replace $\Big(b_\alpha(q',p')\Big)_{n'm',m''n''}$ in Eq. (27) (when they acted on a state)

$$\sum_{m'n'} \Big(b_\alpha(q',p')\Big)_{n'm',m''n''} \to \sum_{m'n'} \Big(b_\alpha(p',q')\Big)_{m'n',n''m''}, \tag{39}$$

and so Eq. (27) becomes

$$\begin{aligned}&\sum_{m''n''} \left(\Big(b_\alpha(p',q')\Big)_{m'n',m''n''} \Big(\mathcal{M}(p',q';p,q)\Big)_{m''n'',mn} + \Big(b_\alpha(p',q')\Big)_{m'n',n''m''} \Big(\mathcal{M}(q',p';p,q)\Big)_{m''n'',mn} \right) \\ &= \sum_{m''n''} \left(\Big(\mathcal{M}(p',q';p,q)\Big)_{m'n',m''n''} \Big(b_\alpha(p,q)\Big)_{m''n'',mn} + \Big(\mathcal{M}(p',q';q,p)\Big)_{m'n',m''n''} \Big(b_\alpha(p,q)\Big)_{n''m'',mn} \right).\end{aligned} \tag{40}$$

Finally, se need to deal with the elements of the connected S-Matrix elements under particle interchange. The non-trivial part of the S-matrix can be written

$$i\mathcal{M}(2\pi)^4\delta^{(4)}(p'+q'-p-q) = \begin{pmatrix}\text{sum of all connected, amputated Feynman} \\ \text{diagrams with } p,q \text{ incoming, } p',q' \text{ outgoing}\end{pmatrix}. \tag{41}$$

This means that the matrix element between two states is given by the sum of all possible processes between them, and will be independent of the state ordering

$$S(p\,m, q\,n \to p'\,m', q'\,n') = S(p\,m, q\,n \to q'\,n', p'\,m'). \tag{42}$$

and so, ignoring the trivial part of the S-Matrix

$$S(p\,m, q\,n \to p'\,m', q'\,n')_{\text{connected}} = \delta^{(4)}(p' + q' - p - q)\Big(\mathcal{M}(p', q'; p, q)\Big)_{m'n',mn} \tag{43}$$

$$= \delta^{(4)}(q' + p' - p - q)\Big(\mathcal{M}(q', p'; p, q)\Big)_{n'm',mn}. \tag{44}$$

Finally, we find the relation between connected S-Matrix elements

$$\Big(\mathcal{M}(q', p'; p, q)\Big)_{n'm',mn} = \Big(\mathcal{M}(p', q'; p, q)\Big)_{m'n',mn}. \tag{45}$$

Similarly,

$$\Big(\mathcal{M}(p', q'; q, p)\Big)_{m'n',nm} = \Big(\mathcal{M}(p', q'; p, q)\Big)_{m'n',mn}. \tag{46}$$

Substituting these relations into Eq. (40) we find

$$\sum_{m''n''}\left(\Big(b_\alpha(p', q')\Big)_{m'n',m''n''}\Big(\mathcal{M}(p', q'; p, q)\Big)_{m''n'',mn} + \Big(b_\alpha(p', q')\Big)_{m'n',n''m''}\Big(\mathcal{M}(p', q'; p, q)\Big)_{n''m'',mn}\right)$$
$$= \sum_{m''n''}\left(\Big(\mathcal{M}(p', q'; p, q)\Big)_{m'n',m''n''}\Big(b_\alpha(p, q)\Big)_{m''n'',mn} + \Big(\mathcal{M}(p', q'; p, q)\Big)_{m'n',n''m''}\Big(b_\alpha(p, q)\Big)_{n''m'',mn}\right). \tag{47}$$

Noting that the order of summation is not important (again from particle finiteness), we can rewrite this as

$$\sum_{m''n''}\Big(b_\alpha(p', q')\Big)_{m'n',m''n''}\Big(\mathcal{M}(p', q'; p, q)\Big)_{m''n'',mn} = \sum_{m''n''}\Big(\mathcal{M}(p', q'; p, q)\Big)_{m'n',m''n''}\Big(b_\alpha(p, q)\Big)_{m''n'',mn}, \tag{48}$$

which in matrix notation is

$$b_\alpha(p', q')\mathcal{M}(p', q'; p, q) = \mathcal{M}(p', q'; p, q)b_\alpha(p, q), \tag{49}$$

and is in agreement with Weinberg Eq. 24.B.5.

This is interesting, as it says that for two-particle scattering that conserves total four-momentum $p' + q' = p + q$, then the matrix representations of B_α acting on two-particle in and out states are similar, i.e. related by the similarity transformation

$$b_\alpha(p', q') = S(p', q'; p, q)b_\alpha(p, q)S^{-1}(p', q'; p, q). \tag{50}$$

This is useful for trying to find the isomorphism between B_α and $b_\alpha(p, q)$, since if I can find a set of coefficients c^α and four-momenta p, q such that

$$\sum_\alpha c^\alpha b_\alpha(p,q) = 0, \tag{51}$$

I can then conclude that, for any four momenta p', q' on the same mass shells that satisfy $p'+q' = p+q$ (for the similarity transformation to exist) then

$$\begin{aligned}
\sum_\alpha c^\alpha b_\alpha(p',q') &= \sum_\alpha c^\alpha S(p',q';p,q) b_\alpha(p,q) S^{-1}(p',q';p,q) & (52)\\
&= S(p',q';p,q)\left(\sum_\alpha c^\alpha b_\alpha(p,q)\right) S^{-1}(p',q';p,q) & (53)\\
&= 0. & (54)
\end{aligned}$$

This does not, however, tell us that $\sum c^\alpha b_\alpha(p') = \sum c^\alpha b_\alpha(q') = 0$, which would be our condition for an isomorphism. Instead, this tells us that

$$\sum_\alpha c^\alpha b_\alpha(p',q') = \sum_\alpha c^\alpha \left(\Big(b_\alpha(p')\Big)_{m',m} \delta_{n'n} + \Big(b_\alpha(q')\Big)_{n',n} \delta_{m'm} \right) = 0, \tag{55}$$

and so

$$\sum_\alpha c^\alpha \Big(b_\alpha(p')\Big)_{m',m} \delta_{n'n} = -\sum_\alpha c^\alpha \Big(b_\alpha(q')\Big)_{n',n} \delta_{m'm}. \tag{56}$$

Therefore, $\sum c^\alpha b_\alpha(p')$ and $\sum c^\alpha b_\alpha(q')$ are both proportional to the identity matrix, but have opposite coefficients. We're not quite there yet. Let's consider just the traceless parts of the matrices $b_\alpha(p)$ and $b_\alpha(p,q)$.

Now, similar matrices have the same trace, and so $b_\alpha(p,q)$ and $b_\alpha(p',q')$ have the same trace if the similarity transformation (50) exists (i.e. if there is four-momentum conserving scattering). This is easy to see

$$\begin{aligned}
\mathrm{Tr}\left[b_\alpha(p',q')\right] &= \mathrm{Tr}\left[S(p',q';p,q) b_\alpha(p,q) S^{-1}(p',q';p,q)\right]\\
&= \mathrm{Tr}\left[S^{-1}(p',q';p,q) S(p',q';p,q) b_\alpha(p,q)\right]\\
&= \mathrm{Tr}\left[b_\alpha(p,q)\right].
\end{aligned} \tag{57}$$

Now, using the definition for $b_\alpha(p,q)$ in Eq. (8), we can use the trace properties of $b_\alpha(p,q)$ and $b_\alpha(p',q')$ to relate the traces of $b_\alpha(p), b_\alpha(q), b_\alpha(p')$, and $b_\alpha(q')$

$$\begin{aligned}
\mathrm{Tr}\left[\Big(b_\alpha(p,q)\Big)_{m'n',mn}\right] &= \mathrm{Tr}\left[\Big(b_\alpha(p)\Big)_{m'm}\delta_{n'n} + \Big(b_\alpha(p)\Big)_{n'n}\delta_{m'm}\right]\\
&= \mathrm{tr}\left[\Big(b_\alpha(p)\Big)_{m'm}\right]\mathrm{tr}\left[\delta_{n'n}\right] + \mathrm{tr}\left[\Big(b_\alpha(p)\Big)_{n'n}\right]\mathrm{tr}\left[\delta_{m'm}\right]\\
&= N(\sqrt{q_\mu q^\mu})\mathrm{tr}\left[\Big(b_\alpha(p)\Big)_{m'm}\right] + N(\sqrt{p_\mu p^\mu})\mathrm{tr}\left[\Big(b_\alpha(p)\Big)_{n'n}\right],
\end{aligned}$$

where we have used the trace of a tensor product of two matrices is the product of the traces of the individual matrices. $N(m)$ is the multiplicity of particles with mass m. The lower case 'tr' indicates a trace over one particle labels, and upper case 'Tr' indicates a trace over two-particle labels. Now using the trace relation (57), we can write

$$N(\sqrt{q_\mu q^\mu})\,\mathrm{tr}\,b_\alpha(p') + N(\sqrt{p_\mu p^\mu})\,\mathrm{tr}\,b_\alpha(q') = N(\sqrt{q_\mu q^\mu})\,\mathrm{tr}\,b_\alpha(p) + N(\sqrt{p_\mu p^\mu})\,\mathrm{tr}\,b_\alpha(q),$$

which rearranged gives

$$\frac{\mathrm{tr}\,b_\alpha(p')}{N(\sqrt{p_\mu p^\mu})} + \frac{\mathrm{tr}\,b_\alpha(q')}{N(\sqrt{q_\mu q^\mu})} = \frac{\mathrm{tr}\,b_\alpha(p)}{N(\sqrt{p_\mu p^\mu})} + \frac{\mathrm{tr}\,b_\alpha(q)}{N(\sqrt{q_\mu q^\mu})}. \tag{58}$$

This relation needs to be satisfied for almost all mass-shell four-momenta for which $p' + q' = p + q$. This is satisfied by having $\mathrm{tr}\,b_\alpha(p)/N(\sqrt{p_\mu p^\mu})$ linear in p, that is

$$\frac{\mathrm{tr}\,b_\alpha(p)}{N(\sqrt{p_\mu p^\mu})} = a^\mu_\alpha\, p_\mu, \tag{59}$$

where $a^\mu_\alpha \neq a^\mu_\alpha(p)$. Now let us define new symmetry generators by subtracting terms linear in the momentum operator P_μ

$$B^\sharp_\alpha := B_\alpha - a^\mu_\alpha P_\mu. \tag{60}$$

The action of $B^\sharp_\alpha$ on single particle states is

$$\begin{aligned} B^\sharp_\alpha|pm\rangle &= (B_\alpha - a^\mu_\alpha P_\mu)\,|pm\rangle \\ &= \sum_{m'} \left(\Big(b_\alpha(p)\Big)_{m',m} - a^\mu_\alpha p_\mu\, \delta_{m'm} \right) |pm'\rangle. \end{aligned}$$

Now if our scattering satisfies four-momentum conservation, then we can substitute Eq. (59) to give

$$\begin{aligned} \sum_{m'} \left(\Big(b_\alpha(p)\Big)_{m',m} - a^\mu_\alpha p_\mu\, \delta_{m'm} \right) |pm'\rangle &= \sum_{m'} \left(\Big(b_\alpha(p)\Big)_{m',m} - \frac{\mathrm{tr}\,b_\alpha(p)}{N(\sqrt{p_\mu p^\mu})}\, \delta_{m'm} \right) |pm'\rangle \\ &= \sum_{m'} \Big(b^\sharp_\alpha(p)\Big)_{m',m} |pm'\rangle, \end{aligned}$$

where the $b^\sharp_\alpha(p)$ are traceless matrices

$$\begin{aligned} \mathrm{tr}\left[\Big(b^\sharp_\alpha(p)\Big)_{m',m} \right] &= \mathrm{tr}\left[\Big(b_\alpha(p)\Big)_{m',m} - \frac{\mathrm{tr}\,b_\alpha(p)}{N(\sqrt{p_\mu p^\mu})}\, \delta_{m'm} \right] \\ &= \mathrm{tr}\left[\Big(b_\alpha(p)\Big)_{m',m} \right] - \frac{\mathrm{tr}\,b_\alpha(p)}{N(\sqrt{p_\mu p^\mu})}\, \mathrm{tr}\left[\delta_{m'm}\right] \\ &= \mathrm{tr}\,b_\alpha(p) - \frac{\mathrm{tr}\,b_\alpha(p)}{N(\sqrt{p_\mu p^\mu})}\, N(\sqrt{p_\mu p^\mu}) \\ &= 0, \end{aligned}$$

and represent the action of $B^\sharp_\alpha$ on single partice states. Now, as P_μ commutes with B_α, then it also commutes with $B^\sharp_\alpha$ as the identity matrix commutes with everything

$$[P_\mu, B^\sharp_\alpha] = 0. \tag{61}$$

It follows that the commutators of $B^\sharp_\alpha$ are the same as those of B_α. Using the Lie algebra in Eq. (3) we can write

$$[B^\sharp_\alpha, B^\sharp_\beta] = i\sum_\gamma C^\gamma_{\alpha\beta} B_\gamma = i\sum_\gamma C^\gamma_{\alpha\beta}\left(B^\sharp_\gamma + a^\mu_\gamma P_\mu\right). \tag{62}$$

Similarly, the commutators of $b^\sharp_\alpha(p)$ are the same as those of $b_\alpha(p)$, and so

$$[b^\sharp_\alpha(p), b^\sharp_\beta(p)] = i\sum_\gamma C^\gamma_{\alpha\beta} b_\gamma(b) = i\sum_\gamma C^\gamma_{\alpha\beta}\left(b^\sharp_\gamma(p) + a^\mu_\gamma p_\mu\right). \tag{63}$$

Since $b^\sharp_\alpha(p)$ are finite matrices, the trace of their commutators is zero. Consequently

$$\begin{aligned}
\mathrm{tr}\,[b^\sharp_\alpha(p), b^\sharp_\beta(p)] &= i\,\mathrm{tr}\sum_\gamma C^\gamma_{\alpha\beta}\left(b^\sharp_\gamma(p) + a^\mu_\gamma p_\mu\right) \\
&= iN(\sqrt{p_\mu p^\mu})\sum_\gamma C^\gamma_{\alpha\beta} a^\mu_\gamma p_\mu \\
&= 0.
\end{aligned}$$

For non-zero particle multiplicity and arbitrary four-momenta, this reduces to

$$\sum_\gamma C^\gamma_{\alpha\beta} a^\mu_\gamma = 0, \tag{64}$$

and implies that the $B^\sharp_\alpha$ obey a Lie algebra

$$[B^\sharp_\alpha, B^\sharp_\beta] = i\sum_\gamma C^\gamma_{\alpha\beta} B^\sharp_\gamma, \tag{65}$$

and are therefore also generators of a symmetry

$$\langle p'\,m', q'\,n'|[B^\sharp_\alpha, S]|p\,m, q\,n\rangle = 0. \tag{66}$$

Consequently, the $b^\sharp_\alpha(p', q')$ are related to $b^\sharp_\alpha(p, q)$ by a similarity transformation for four-momentum conserving scattering

$$b^\sharp_\alpha(p', q') = S(p', q'; p, q) b^\sharp_\alpha(p, q) S^{-1}(p', q'; p, q). \tag{67}$$

in the same way as in (50) for the $b_\alpha(p)$. $b^\sharp_\alpha(p', q')$ are matrices representing $B^\sharp_\alpha$ acting on two-particle states

$$B^\sharp_\alpha|pm, qn\rangle = \sum_{m'n'}\left(b^\sharp_\alpha(p, q)\right)_{m'n',mn}|pm', qn'\rangle, \tag{68}$$

where

$$\left(b^\sharp_\alpha(p,q)\right)_{m'n',mn} = \left(b^\sharp_\alpha(p)\right)_{m',m}\delta_{n'n} + \left(b^\sharp_\alpha(q)\right)_{n',n}\delta_{m'm}. \tag{69}$$

The $b^\sharp_\alpha(p,q)$ also obey the same commutation relations as $B^\sharp_\alpha$[1]

$$[b^\sharp_\alpha(p,q), b^\sharp_\beta(p,q)] = i\sum_\gamma C^\gamma_{\alpha\beta} b^\sharp_\gamma(p,q). \tag{70}$$

So why are we dealing with two-particle states instead of single particle states? As $\mathcal{M}(p',q';p,q)$ is a non-singular matrix (as we have asserted it is analytic), then if we can find some coefficients c^α such that

$$\sum_\alpha c^\alpha b^\sharp_\alpha(p,q) = 0,$$

for some fixed mass-shell momenta p and q, then

$$\sum_\alpha c^\alpha b^\sharp_\alpha(p',q') = 0$$

for almost all p' and q' on the same respective mass shells and also satisfy $p'+q' = p+q$ such that the similarity transformation (67) exists. Again, this tells us that

$$\sum_\alpha c^\alpha b^\sharp_\alpha(p',q') = \sum_\alpha c^\alpha \left(\left(b^\sharp_\alpha(p')\right)_{m',m}\delta_{n'n} + \left(b^\sharp_\alpha(q')\right)_{n',n}\delta_{m'm}\right) = 0, \tag{71}$$

and rearranged gives

$$\sum_\alpha c^\alpha \left(b^\sharp_\alpha(p')\right)_{m',m}\delta_{n'n} = -\sum_\alpha c^\alpha \left(b^\sharp_\alpha(q')\right)_{n',n}\delta_{m'm}. \tag{72}$$

The matrices $\sum_\alpha c^\alpha b^\sharp_\alpha(p')$ and $\sum_\alpha c^\alpha b^\sharp_\alpha(q')$ are again proportional to the identity matrix with opposite coefficients. However, we know that the $b^\sharp_\alpha(p)$ are traceless, and so

$$\sum_\alpha c^\alpha b^\sharp_\alpha(p') = \sum_\alpha c^\alpha b^\sharp_\alpha(q') = 0. \tag{73}$$

Just to quickly summarise what we have just found. If some set of coefficients c^α can be found, such that $\sum_\alpha c^\alpha b^\sharp_\alpha(p,q) = 0$ for some fixed mass shell four-momenta p and q, then $\sum_\alpha c^\alpha b^\sharp_\alpha(p') = \sum_\alpha c^\alpha b^\sharp_\alpha(q') = 0$ for all p' and q' on the same respective mass shells that satisfy four-momentum conservation $p'+q' = p+q$. Our goal is still to show an isomorphism from finite Hermitian matrices to B_α. This would require that we show $\sum_\alpha c^\alpha b^\sharp_\alpha(k) = 0$ for all mass-shell momenta k to have the mapping taking B_α into $b_\alpha(p,q)$ to be an isomorphism. Do not fret, we are almost there.

Currently we have only shown that

$$\sum_\alpha c^\alpha b^\sharp_\alpha(p') = 0$$

[1] This isn't obvious from an explicit calculation using the form of $b^\sharp_\alpha(p,q)$ in terms of single particle actions. Something along the lines done earlier for $b_\alpha(p)$ is fairly convincing, however.

for four-momenta p' and q' satisfying

$$q' = p + q - p',$$

where p' and q' are on mass shells. Now we will use the clever trick that Coleman and Mandula noted. If it is true that

$$\sum_\alpha c^\alpha b^\sharp_\alpha(p,q) = \sum_\alpha c^\alpha b^\sharp_\alpha(p',q') = 0,$$

then we know

$$\sum_\alpha c^\alpha b^\sharp_\alpha(p) = \sum_\alpha c^\alpha b^\sharp_\alpha(q) = \sum_\alpha c^\alpha b^\sharp_\alpha(p') = \sum_\alpha c^\alpha b^\sharp_\alpha(q')$$

under the circumstances mentioned. Using the expansion of $b^\sharp_\alpha(p,q)$ in terms of single-particle matrices (69), we can write

$$\left(b^\sharp_\alpha(p,q')\right)_{m'n',mn} = \left(b^\sharp_\alpha(p)\right)_{m',m}\delta_{n'n} + \left(b^\sharp_\alpha(q')\right)_{n',n}\delta_{m'm}$$

and so

$$\sum_\alpha c^\alpha b^\sharp_\alpha(p,q') = 0. \tag{74}$$

The similarity transformation (67) will take Eq. (74) to a similar relation for matrices $b^\sharp_\alpha$ effectively acting on states of total momentum $p+q'$. Eq. (74) then implies

$$\sum_\alpha c^\alpha b^\sharp_\alpha(k, p+q'-k) = 0 \tag{75}$$

noting that the above appears to be a state of the required total momenta. The similarity transformation only exists for mass-shell four-momenta k where both k and $p+q'-k$ are on the mass shell. Both q' and $p' = p+q-q'$ need to also be on the mass shell. This means

$$\begin{aligned} m_1 &= p'_\mu p'^\mu = (p+q-q')_\mu (p+q-q')^\mu \\ m_2 &= q'_\mu q'^\mu, \end{aligned}$$

and removes two degrees of freedom in q', leaving two. The requirement that both k and q' are on the mass shell

$$\begin{aligned} m_1 &= k_\mu k^\mu \\ m_2 &= q'_\mu q'^\mu \end{aligned}$$

removes one degree of freedom from k and one degree of freedom from q'. Finally, the requirement that $p+q'-k$ also be on the mass shell

$$m_2 = (p+q'-k)_\mu (p+q'-k)^\mu$$

removes one degree of freedom from q' and leaves us enough freedom to choose $\mathbf{k}$ to be anything we want within a finite volume of momentum space. This volume can be increased by making $\mathbf{p}$ and $\mathbf{q}$

sufficiently large. So now, what have we shown? If

$$\sum_\alpha c^\alpha b^\sharp_\alpha(p,q) = 0$$

for some fixed mass-shell four momenta, then

$$\sum_\alpha c^\alpha b^\sharp_\alpha(k) = 0 \tag{76}$$

for almost all mass-shell four-momenta k. Note, this statement is now unconstrained by the four-momentum conservation requirement, since k has nothing directly to do with the scattering kinematics.

Okay, we're almost there. Now suppose I find for some mass-shell four-momenta p and q that $\sum_\alpha c^\alpha b^\sharp_\alpha(p,q) = 0$. What happens if for some k_0 that $\sum_\alpha c^\alpha b^\sharp_\alpha(k_0) \neq 0$? If this were the case, a scattering process where particles with four-momenta k_0 and k scatter into particles of four-momenta k' and k'' will be forbidden by the symmetry generated by $\sum_\alpha c^\alpha B^\sharp_\alpha$, since if the symmetry allowed such a scattering process, a similarity transform would exist between $b^\sharp_\alpha(k_0,k)$ and $b^\sharp_\alpha(p,q)$, where $\sum_\alpha c^\alpha b^\sharp_\alpha(p,q) = 0$. One of our initial assumptions what that the scattering amplitude is an analytic functions of the scattering angle at almost all energies and angles. The scattering amplitude to a particle with momentum k_0 could not just jump to zero under the symmetry imposed by $B^\sharp_\alpha$ in an analytic way, so the existence of such a state is in contradiction with one of our assumptions. We must therefore conclude that if $\sum_\alpha c^\alpha b^\sharp_\alpha(p,q) =$ for some fixed mass-shell four-momentum p and q, then

$$\sum_\alpha c^\alpha b^\sharp_\alpha(k) = 0 \tag{77}$$

for all k, and consequently

$$\sum_\alpha c^\alpha B_\alpha = 0. \tag{78}$$

The mapping that takes B_α into $b^\sharp_\alpha(p,q)$ is therefore an isomorpism. A consequence of this is that, since matrix representation of the action of $B^\sharp_\alpha$ on two particle states is

$$\left(b^\sharp_\alpha(p,q)\right)_{m'n',mn} = \left(b^\sharp_\alpha(p)\right)_{m',m}\delta_{n'n} + \left(b^\sharp_\alpha(q)\right)_{n',n}\delta_{m'm}, \tag{79}$$

then for a given mn, $b^\sharp_\alpha(p,q)$ is a column vector with $N(\sqrt{p_\mu p^\mu})N(\sqrt{q_\mu q^\mu})$ entries. Therefore, the number of independant $b^\sharp_\alpha(p,q)$ cannot exceed $N(\sqrt{p_\mu p^\mu})N(\sqrt{q_\mu q^\mu})$. Due to the isomorphism between B_α and $b^\sharp_\alpha(p,q)$, this means there is at most, a finite number of independent symmetry generators B_α. This was not one of our assumptions, we have actually shown that B_α must be finite-dimensional. We can now move on to applying the theorem for finite Hermitian matrices to the generators B_α.

1.5 Dealing with the $U(1)$ algebras

So, a theorem from Weinberg (which I may prove at some point) states that a Lie algebra of finite Hermitian matrices like $b_\alpha(p,q)$ for fixed p and q is at most the direct sum of a semi-simple compact Lie algebra and some number of $U(1)$ Lie algebras. Since the Lie algebra of $b^\sharp_\alpha(p,q)$ and $B^\sharp_\alpha$ are isomorphic (as we have shown), then the $B^\sharp_\alpha$ must also span the direct sum of at most a compact semi-simple Lie algebra and $U(1)$ Lie algebras.

Let us first deal with the $U(1)$ algebras. If p and q are any mass-shell four-momenta, then we can always find a Lorentz generator J that leaves both p and q invariant. There are two main situations to consider

1. If p and q are both lightlike and parallel then choose J to generate spatial rotations around the common direction of $\mathbf{p}$ and $\mathbf{q}$. E.g. if $\mathbf{p}$ and $\mathbf{q}$ are both in the z-direction, then let $p^\mu = q^\mu = (1,0,0,1)^\mu$. J is then chosen to generate spatial rotations in the $x-y$ plane (i.e. around the z-axis).

2. Otherwise, $p+q$ is timelike, and we can take J to be the generator of spatial directions around the common direction of $\mathbf{p}$ and $\mathbf{q}$ in the centre of momentum frame ($\mathbf{p}+\mathbf{q}=0$). This would mean jumping to the frame where the particles look like they are colliding head-on. J would then generate rotations about the axis of collision.

Since the J have been chosen to generate spatial rotations, they are Hermitian. Let us be in the basis of two-particle states that diagonalise J such that

$$J|pm,qn\rangle = \sigma(m,n)|pm,qn\rangle. \tag{80}$$

We already know that $[P_\mu, B^\sharp_\alpha] = 0$. From the Lorentz algebra we know that

$$[J, P_\mu] \sim (\text{linear combination of } P_\mu)\,. \tag{81}$$

Using the Jacobi identity, we know that

$$[P_\mu,[J,B^\sharp_\alpha]] + [J,[B^\sharp_\alpha,P_\mu]] + [B^\sharp_\alpha,[P_\mu,J]] = 0,$$

and therefore

$$[P_\mu,[J,B^\sharp_\alpha]] = 0. \tag{82}$$

Since we have defined all generators that commute with P_μ consists of generators B_α, it follows that $[J,B^\sharp_\alpha]$ must then be a linear combination of B_α. We know more, however. Since the trace of the commutator of finite matrices is zero, then the trace of this linear combination of B_α must also be zero, and hence $[J,B^\sharp_\alpha]$ must be a linear combination of $B^\sharp_\alpha$

$$[J,B^\sharp_\alpha] = \sum_\beta c_\alpha^\beta B^\sharp_\beta. \tag{83}$$

Now, call the generators of the $U(1)$ Lie algebra $B^\sharp_i$ (and take these to be Hermitian) in the algebra of $B^\sharp_\alpha$. These generators must commute with all of the $B^\sharp_\alpha$, since a $U(1)$ subalgebra of $\mathfrak{G}$ is one with just a single generator that commutes with the whole of the algebra $\mathfrak{G}$. If this is the case, then B_i must commute with $[J, B_i^\sharp]$

$$[B_i^\sharp, [J, B_i^\sharp]] = 0. \tag{84}$$

In the two-particle basis where J is diagonal, the expectation value of this double commutator is

$$\langle pm, qn|\, [B_i^\sharp, [J, B_i^\sharp]]\, |pm, qn\rangle = \langle pm, qn| \left(2B_i^\sharp J B_i^\sharp - B_i^\sharp B_i^\sharp J - J B_i^\sharp B_i^\sharp\right) |pm, qn\rangle = 0$$

Now since our choice of J is just spatial rotations, the generators of J are hermitian, and so we can rewrite the above as

$$2\langle pm, qn| \left(B_i^\sharp J B_i^\sharp - B_i^\sharp B_i^\sharp J\right) |pm, qn\rangle = 0.$$

Now inserting a complete set of states

$$\begin{aligned}
0 = &\sum_{m''n''} \int \frac{d^3p''}{(2\pi)^3} \int \frac{d^3q''}{(2\pi)^3} \Big[\langle pm, qn|B_i^\sharp|p''m'', q''n''\rangle\langle p''m'', q''n''|J B_i^\sharp|pm, qn\rangle \\
&\qquad - \langle pm, qn|B_i^\sharp|p''m'', q''n''\rangle\langle p''m'', q''n''|B_i^\sharp J|pm, qn\rangle \Big] \\
= &\sum_{m'',n''} \int \frac{d^3p''}{(2\pi)^3} \int \frac{d^3q''}{(2\pi)^3} \left[\Big(\sigma(m'', n'') - \sigma(m, n)\Big) \left|\langle pm, qn|B_i^\sharp|p''m'', q''n''\rangle\right|^2 \right].
\end{aligned}$$

Now using the matrix representations of $B_i^\sharp$ acting on two-particle states

$$\begin{aligned}
0 = &\sum_{m'n'} \sum_{m''n''} \int \frac{d^3p''}{(2\pi)^3} \int \frac{d^3q''}{(2\pi)^3} \Bigg[\Big(\sigma(m'', n'') - \sigma(m, n)\Big) \times \\
&\qquad \left|\langle pm, qn| \Big(b_i^\sharp(p'', q'')\Big)_{m'n', m''n''} |p''m', q''n'\rangle\right|^2 \Bigg]
\end{aligned}$$

Again moving the states past the matrix coefficients

$$\begin{aligned}
0 = &\sum_{m'n'} \sum_{m''n''} \int \frac{d^3p''}{(2\pi)^3} \int \frac{d^3q''}{(2\pi)^3} \Bigg[\Big(\sigma(m'', n'') - \sigma(m, n)\Big) \times \\
&\qquad \left| \Big(b_i^\sharp(p'', q'')\Big)_{m'n', m''n''} \langle pm, qn|p''m', q''n'\rangle\right|^2 \Bigg],
\end{aligned}$$

and using the state normalisation in Eq. (21) and dividing out coefficients

$$\begin{aligned}
0 = &\sum_{m'n'} \sum_{m''n''} \int d^3p'' \int d^3q'' \Bigg[\Big(\sigma(m'', n'') - \sigma(m, n)\Big) \times \\
&\left| \Big(b_i^\sharp(p'', q'')\Big)_{m'n', m''n''} \Big(\delta^{(3)}(\mathbf{p}'' - \mathbf{p})\delta^{(3)}(\mathbf{q}'' - \mathbf{q})\delta_{m'm}\delta_{n'n} + \delta^{(3)}(\mathbf{p}'' - \mathbf{q})\delta^{(3)}(\mathbf{q}'' - \mathbf{p})\delta_{m'n}\delta_{n'm}\Big)\right|^2 \Bigg].
\end{aligned}$$

Evaulating the momentum integrals

$$0 = \sum_{m'n'} \sum_{m''n''} \left[\Big(\sigma(m'', n'') - \sigma(m, n)\Big) \times \left| \Big(b_i^\sharp(p, q)\Big)_{m'n', m''n''} \delta_{m'm}\delta_{n'n} + \Big(b_i^\sharp(q, p)\Big)_{m'n', m''n''} \delta_{m'n}\delta_{n'm}\right|^2 \right].$$

and so performing the sum over $m'n'$ gives

$$0=\sum_{m''n''}\left[\Big(\sigma(m'',n'')-\sigma(m,n)\Big)\times\left|\Big(b_i^\sharp(p,q)\Big)_{mn,m''n''}+\Big(b_i^\sharp(q,p)\Big)_{nm,m''n''}\right|^2\right].$$

Splitting up the sum

$$\begin{aligned}0=&\sum_{m''n''}\left[\Big(\sigma(m'',n'')-\sigma(m,n)\Big)\left|\Big(b_i^\sharp(p,q)\Big)_{mn,m''n''}\right|^2\right]\\&+\sum_{m''n''}\left[\Big(\sigma(m'',n'')-\sigma(m,n)\Big)\Big(b_i^\sharp(q,p)\Big)_{nm,m''n''}\right|^2\Bigg].\end{aligned}$$

Relabelling the free indices in the second sum

$$\begin{aligned}0=&\sum_{m''n''}\left[\Big(\sigma(m'',n'')-\sigma(m,n)\Big)\left|\Big(b_i^\sharp(p,q)\Big)_{mn,m''n''}\right|^2\right]\\&+\sum_{m''n''}\left[\Big(\sigma(n'',m'')-\sigma(m,n)\Big)\Big(b_i^\sharp(q,p)\Big)_{nm,n''m''}\right|^2\Bigg],\end{aligned}$$

and then using the symmetry of $b_\alpha(p,q)$ in Eq. (39) and noting to also change the ordering of $\sigma(p,q)$, then

$$\begin{aligned}0=&\sum_{m''n''}\left[\Big(\sigma(m'',n'')-\sigma(m,n)\Big)\left|\Big(b_i^\sharp(p,q)\Big)_{mn,m''n''}\right|^2\right]\\&+\sum_{m''n''}\left[\Big(\sigma(m'',n'')-\sigma(m,n)\Big)\Big(b_i^\sharp(p,q)\Big)_{mn,m''n''}\right|^2\Bigg],\end{aligned}$$

implying

$$\sum_{m''n''}\left[\Big(\sigma(m'',n'')-\sigma(m,n)\Big)\left|\Big(b_i^\sharp(p,q)\Big)_{mn,m''n''}\right|^2\right]=0,$$

and relabelling the free indices and noting that $b_i^\sharp(p,q)$ is Hermitian, then

$$\sum_{m'n'}\left[\Big(\sigma(m',n')-\sigma(m,n)\Big)\left|\Big(b_i^\sharp(p,q)\Big)_{m'n',mn}\right|^2\right]=0, \tag{85}$$

and is in agreement with Weinberg Eq. 24.B.21. This equation is valid for all m, n.

Now if there is any m, n for which $\sigma=\sigma(m,n)$, and any m', n' for which $\sigma(m',n')=\sigma'\neq\sigma$, then there must be a choice of m, n for which σ is smallest, i.e. a choice of m, n such that

$$\sigma(m,n)=\sigma\leq\sigma' \qquad\qquad \forall\ m',n'$$

For this choice of m, n, the right hand side of Eq. (85) is

$$\sum_{m'n'}\left[\Big(\sigma(m',n')-\sigma(m,n)\Big)\left|\Big(b_i^\sharp(p,q)\Big)_{m'n',mn}\right|^2\right]\geq 0,$$

where the equality is obtained only when, for every $\sigma(m',n')\neq\sigma(m,n)$, $\Big(b_i^\sharp(p,q)\Big)_{m'n',mn}=0$. We must then conclude that $\Big(b_i^\sharp(p,q)\Big)_{m'n',mn}$ vanishes for every $\sigma(m',n')\neq\sigma(m,n)$. Then, calculating the commutator of B_i with J acting on a two particle state

$$\begin{aligned}[B_i^\sharp, J]|pm,qn\rangle &= \Big(B_i^\sharp J - J B_i^\sharp\Big)|pm,qn\rangle \\ &= \sum_{m'n'}\left[\Big(\sigma(m,n)\Big(b_i^\sharp(p,q)\Big)_{m'n',mn} - J\Big(b_i^\sharp(p,q)\Big)_{m'n',mn}\Big)|pm',qn'\rangle\right].\end{aligned}$$

$\Big(b_i^\sharp(p,q)\Big)_{m'n',mn}$ are just the elements of a Hermitian matrix, and so we can pass J to the right of it in the second term

$$\begin{aligned}&\sum_{m'n'}\left[\Big(\sigma(m,n)\Big(b_i^\sharp(p,q)\Big)_{m'n',mn} - J\Big(b_i^\sharp(p,q)\Big)_{m'n',mn}\Big)|pm',qn'\rangle\right] \\ =&\sum_{m'n'}\left[\Big(\sigma(m,n)\Big(b_i^\sharp(p,q)\Big)_{m'n',mn} - \Big(b_i^\sharp(p,q)\Big)_{m'n',mn}J\Big)|pm',qn'\rangle\right] \\ =&\sum_{m'n'}\left[\Big(\sigma(m,n)\Big(b_i^\sharp(p,q)\Big)_{m'n',mn} - \sigma(m',n')\Big(b_i^\sharp(p,q)\Big)_{m'n',mn}\Big)|pm',qn'\rangle\right] \\ =&\sum_{m'n'}\left[\Big(\sigma(m,n)-\sigma(m',n')\Big)\Big(b_i^\sharp(p,q)\Big)_{m'n',mn}|pm',qn'\rangle\right].\end{aligned}$$

Since we have showed that $\Big(b_i^\sharp(p,q)\Big)_{m'n',mn}$ vanishes for every $\sigma(m',n')\neq\sigma(m,n)$, it follows that

$$\sum_{m'n'}\left[\Big(\sigma(m,n)-\sigma(m',n')\Big)\Big(b_i^\sharp(p,q)\Big)_{m'n',mn}|pm',qn'\rangle\right]=0,$$

and so

$$[B_i^\sharp, J]|pm,qn\rangle = 0,$$

therefore

$$[B_i^\sharp, J] = 0. \tag{86}$$

So each of the $U(1)$ generators $B_i^\sharp$ commutes with J. Now, we have chosen J to be the generator that leaves p and q invariant. If p and q are chosen to be parallel in the x-plane, then J is simply $J_x = J^{23}$. We can also make choices of p and q to be parallel along any of the other axis, and therefore conclude that

$$[B_i^\sharp, J^{23}] = [B_i^\sharp, J^{31}] = [B_i^\sharp, J^{12}] = 0, \tag{87}$$

i.e. $B_i^\sharp$ commutes with all of the generators of spatial rotations. Wonderful. Now, since this is the case, then we also know from the Lorentz algebra

$$i\epsilon_{123}[B_i^\sharp, J^3] = [B_i^\sharp, [K_1, K_2]] = 0. \tag{88}$$

Using again the Bianchi identity (and repeating similarly for J^2 and J^3), this tells us that

$$[K_1, [K_2, B_i^\sharp]] - [K_2, [K_1, B_i^\sharp]] = 0 \tag{89}$$
$$[K_2, [K_3, B_i^\sharp]] - [K_3, [K_2, B_i^\sharp]] = 0 \tag{90}$$
$$[K_3, [K_1, B_i^\sharp]] - [K_1, [K_3, B_i^\sharp]] = 0. \tag{91}$$

Now just focussing on Eq. (89),

$$[K_1, [K_2, B_i^\sharp]] = [K_2, [K_1, B_i^\sharp]],$$

which when expanded shows

$$K_1K_2B_i^\sharp - K_1B_i^\sharp K_2 - K_2B_i^\sharp K_1 + B_i^\sharp K_2K_1 = K_2K_1B_i^\sharp - K_2B_i^\sharp K_1 - K_1B_i^\sharp K_2 + B_i^\sharp K_1K_2,$$

and cancelling leaves

$$[B_i^\sharp, K_1K_2] + [B_i^\sharp, K_2K_1] = 0,$$

but since

$$[B_i^\sharp, [K_1, K_2]] = 0,$$

then

$$[B_i^\sharp, K_1K_2] = 0. \tag{92}$$

Re-expanded, we find

$$\begin{aligned} B_i^\sharp K_1K_2 &= K_1B_i^\sharp K_2 - [K_1, B_i^\sharp]K_2 \\ &= K_1K_2B_i^\sharp - K_1[K_2, B_i^\sharp] - [K_1, B_i^\sharp]K_2 \\ &= K_1K_2B_i^\sharp, \end{aligned}$$

where the final line follows from Eq. (92). Consequently

$$K_1[K_2, B_i^\sharp] = -[K_1, B_i^\sharp]K_2. \tag{93}$$

Doing the same for Eqs. (90) and (91), and multipling on the left by the inverse of K_1 (or its analogue), we find

$$[K_1, B_i^\sharp] = -K_3^{-1}[K_3, B_i^\sharp]K_1 \tag{94}$$
$$[K_2, B_i^\sharp] = -K_1^{-1}[K_1, B_i^\sharp]K_2 \tag{95}$$
$$[K_3, B_i^\sharp] = -K_2^{-1}[K_2, B_i^\sharp]K_3. \tag{96}$$

We can then solve this by substitution

$$\begin{aligned}[K_3, B_i^\sharp] &= -K_2^{-1}[K_2, B_i^\sharp]K_3 \\ &= K_2^{-1}K_1^{-1}[K_1, B_i^\sharp]K_2K_3 \\ &= -K_2^{-1}K_1^{-1}K_3^{-1}[K_3, B_i^\sharp]K_1K_2K_3.\end{aligned}$$

The left hand side of this equation is traceless. The right hand side, however, is not in general, unless the commutator vanishes (I have not been able to show this rigorously, but it feels right). Therefore, for consistency, we require

$$[K_1, B_i^\sharp] = [K_2, B_i^\sharp] = [K_3, B_i^\sharp] = 0, \tag{97}$$

i.e. $B_i^\sharp$ commutes with the generators of Lorentz boosts. It then follows that $B_i^\sharp$ commutes with all the generators $J_{\mu\nu}$ of the Lorentz group

$$[B_i^\sharp, J_{\mu\nu}] = 0. \tag{98}$$

This is critical. Since the $B_i^\sharp$ commute with boosts, then $b_i(p)^\sharp_{n'n}$ are independent of three-momentum. Also, as the $B_i^\sharp$ commute with rotations, then $b_i(p)^\sharp_{n'n}$ act as unit matrices on spin indices. Because of this, we conclude that the $B_i^\sharp$ are the generators of an ordinary internal symmetry.

1.6 The remaining semi-simple compact Lie algebra

We are now left to deal with the remaining $B_\alpha^\sharp$, the generators of a semi-simple compact Lie algebra. Let's go back to the non-traceless B_α. Now, if B_α are the generators of a semi-simple compact Lie algebra, and if these generators commute with the four-momentum P_μ, then

$$[P_\mu, B_\beta] = i\sum_\alpha C^\alpha_{\mu\beta} B_\alpha = 0,$$

and therefore

$$C^\alpha_{\mu\beta} = -C^\alpha_{\beta\mu} = 0. \tag{99}$$

Now consider the effect of a proper Lorentz transformation

$$x^\mu \to \Lambda^\mu{}_\nu x^\nu,$$

which has a representation on the Hilbert space by the unitary operator $U(\Lambda)$

$$B_\alpha \to U(\Lambda)B_\alpha U^{-1}(\Lambda).$$

$U(\Lambda)B_\alpha U^{-1}(\Lambda)$ is a Hermitian symmetry generator that commutes with $\Lambda_\mu{}^\nu P_\nu$, and since Λ is well behaved (non-singular), then $U(\Lambda)B_\alpha U^{-1}(\Lambda)$ must also commute with P_μ

$$[U(\Lambda)B_\alpha U^{-1}(\Lambda), P_\mu] = 0.$$

Since we have said that all symmetry generators that commute with P_μ are spanned by the generators B_α, it follows that $U(\Lambda)B_\alpha U^{-1}(\Lambda), P_\mu$ is a linear combination of B_β

$$U(\Lambda)B_\alpha U^{-1}(\Lambda) = \sum_\beta D^\beta{}_\alpha(\Lambda)B_\beta, \tag{100}$$

where the $D^\beta{}_\alpha(\Lambda)$ are a set of real coefficients that 'furnish' a representation of the homogeneous Lorentz group

$$D(\Lambda_1)D(\Lambda_2) = D(\Lambda_1\Lambda_2). \tag{101}$$

The $U(\Lambda)B_\alpha U^{-1}(\Lambda)$ also satisfy the same commutation relations as the B_α

$$\begin{aligned}
[U(\Lambda)B_\alpha U^{-1}(\Lambda), U(\Lambda)B_\beta U^{-1}(\Lambda)] &= U(\Lambda)[B_\alpha, B_\beta]U^{-1}(\Lambda) \\
&= U(\Lambda)i\sum_\gamma C^\gamma_{\alpha\beta}B_\gamma U^{-1}(\Lambda) \\
&= i\sum_\gamma C^\gamma_{\alpha\beta}U(\Lambda)B_\gamma U^{-1}(\Lambda).
\end{aligned}$$

We can then rewrite the Lie algebra for the transformed B_α in terms of the $D^\beta{}_\alpha(\Lambda)$

$$\begin{aligned}
[U(\Lambda)B_\alpha U^{-1}(\Lambda), U(\Lambda)B_\beta U^{-1}(\Lambda)] &= \sum_{\alpha'\beta'} D^{\alpha'}{}_\alpha(\Lambda)D^{\beta'}{}_\beta(\Lambda)[B_{\alpha'}, B_{\beta'}] \\
&= i\sum_{\alpha'\beta'\delta} D^{\alpha'}{}_\alpha(\Lambda)D^{\beta'}{}_\beta(\Lambda)C^\delta_{\alpha'\beta'}B_\delta \\
&= i\sum_\gamma C^\gamma_{\alpha\beta}U(\Lambda)B_\gamma U^{-1}(\Lambda) \\
&= i\sum_{\gamma\gamma'} D^{\gamma'}{}_\gamma(\Lambda)C^\gamma_{\alpha\beta}B_{\gamma'},
\end{aligned}$$

therefore

$$\sum_{\gamma\gamma'} D^{\gamma'}{}_\gamma(\Lambda)C^\gamma_{\alpha\beta}B_{\gamma'} = \sum_{\alpha'\beta'\delta} D^{\alpha'}{}_\alpha(\Lambda)D^{\beta'}{}_\beta(\Lambda)C^\delta_{\alpha'\beta'}B_\delta. \tag{102}$$

It follows that

$$\begin{aligned}
0 &= \sum_{\gamma\gamma'} D^{\gamma'}{}_\gamma(\Lambda)C^\gamma_{\alpha\beta}B_{\gamma'} - \sum_{\alpha'\beta'\delta} D^{\alpha'}{}_\alpha(\Lambda)D^{\beta'}{}_\beta(\Lambda)C^\delta_{\alpha'\beta'}B_\delta \\
&= \sum_\delta \left(\sum_\gamma D^\delta{}_\gamma(\Lambda)C^\gamma_{\alpha\beta} - \sum_{\alpha'\beta'} D^{\alpha'}{}_\alpha(\Lambda)D^{\beta'}{}_\beta(\Lambda)C^\delta_{\alpha'\beta'}\right)B_\delta,
\end{aligned}$$

which for any B_δ implies

$$\sum_\delta \left(\sum_\gamma D^\delta{}_\gamma(\Lambda)C^\gamma_{\alpha\beta} - \sum_{\alpha'\beta'} D^{\alpha'}{}_\alpha(\Lambda)D^{\beta'}{}_\beta(\Lambda)C^\delta_{\alpha'\beta'}\right) = 0. \tag{103}$$

Now multiplying by $D^{\delta'}{}_{\delta}(\Lambda^{-1})$, we have

$$
\begin{aligned}
0 &= \sum_{\delta}\left(\sum_{\gamma} D^{\delta}{}_{\gamma}(\Lambda)D^{\delta'}{}_{\delta}(\Lambda^{-1})C^{\gamma}_{\alpha\beta} - \sum_{\alpha'\beta'} D^{\alpha'}{}_{\alpha}(\Lambda)D^{\beta'}{}_{\beta}(\Lambda)D^{\delta'}{}_{\delta}(\Lambda^{-1})C^{\delta}_{\alpha'\beta'}\right)\\
&= \sum_{\gamma}\delta^{\delta'}_{\gamma}C^{\gamma}_{\alpha\beta} - \sum_{\alpha'\beta'\delta} D^{\alpha'}{}_{\alpha}(\Lambda)D^{\beta'}{}_{\beta}(\Lambda)D^{\delta'}{}_{\delta}(\Lambda^{-1})C^{\delta}_{\alpha'\beta'}\\
&= C^{\delta'}_{\alpha\beta} - \sum_{\alpha'\beta'\delta} D^{\alpha'}{}_{\alpha}(\Lambda)D^{\beta'}{}_{\beta}(\Lambda)D^{\delta'}{}_{\delta}(\Lambda^{-1})C^{\delta}_{\alpha'\beta'},
\end{aligned}
$$

and so

$$
C^{\gamma}_{\alpha\beta} = \sum_{\alpha'\beta'\gamma'} D^{\alpha'}{}_{\alpha}(\Lambda)D^{\beta'}{}_{\beta}(\Lambda)D^{\gamma}{}_{\gamma'}(\Lambda^{-1})C^{\gamma'}_{\alpha'\beta'}. \tag{104}
$$

This means that the structure constants of the Lie Algebra are 'invariant tensors' in the above sense. Contracting the above with $C^{\alpha}_{\gamma\delta}$ defines the Lie algebra metric or Cartan-Killing form $g_{\beta\delta}$

$$
\begin{aligned}
g_{\beta\delta} := \sum_{\alpha\gamma} C^{\gamma}_{\alpha\beta}C^{\alpha}_{\gamma\delta} &= \sum_{\alpha\gamma}\sum_{\alpha'\beta'\gamma'}\sum_{\alpha''\gamma''\delta'} D^{\alpha'}{}_{\alpha}(\Lambda)D^{\beta'}{}_{\beta}(\Lambda)D^{\gamma}{}_{\gamma'}(\Lambda^{-1})D^{\gamma''}{}_{\gamma}(\Lambda)D^{\delta'}{}_{\delta}(\Lambda)D^{\alpha}{}_{\alpha''}(\Lambda^{-1})C^{\gamma'}_{\alpha'\beta'}C^{\alpha''}_{\gamma''\delta'}\\
&= \sum_{\alpha\gamma}\sum_{\alpha'\beta'\gamma'}\sum_{\alpha''\gamma''\delta'} D^{\alpha'}{}_{\alpha}(\Lambda)D^{\alpha}{}_{\alpha''}(\Lambda^{-1})D^{\gamma}{}_{\gamma'}(\Lambda^{-1})D^{\gamma''}{}_{\gamma}(\Lambda)D^{\beta'}{}_{\beta}(\Lambda)D^{\delta'}{}_{\delta}(\Lambda)C^{\gamma'}_{\alpha'\beta'}C^{\alpha''}_{\gamma''\delta'}\\
&= \sum_{\alpha'\beta'\gamma'}\sum_{\alpha''\gamma''\delta'} \delta^{\alpha'}_{\alpha''}\delta^{\gamma''}_{\gamma'}D^{\beta'}{}_{\beta}(\Lambda)D^{\delta'}{}_{\delta}(\Lambda)C^{\gamma'}_{\alpha'\beta'}C^{\alpha''}_{\gamma''\delta'}\\
&= \sum_{\alpha'\gamma'}\sum_{\beta'\delta'} D^{\beta'}{}_{\beta}(\Lambda)D^{\delta'}{}_{\delta}(\Lambda)C^{\gamma'}_{\alpha'\beta'}C^{\alpha'}_{\gamma'\delta'}\\
&= \sum_{\beta'\delta'} D^{\beta'}{}_{\beta}(\Lambda)D^{\delta'}{}_{\delta}(\Lambda)g_{\beta'\delta'},
\end{aligned}
$$

and so we find

$$
g_{\beta\delta} = \sum_{\beta'\delta'} D^{\beta'}{}_{\beta}(\Lambda)D^{\delta'}{}_{\delta}(\Lambda)g_{\beta'\delta'}. \tag{105}
$$

Now since $C^{\alpha}_{\mu\beta} = -C^{\alpha}_{\beta\mu} = 0$, it follows that

$$
g_{\mu\alpha} = g_{\alpha\mu} = 0 \tag{106}
$$

We will now alter our notation to distinguish symmetry generators other than the P_μ by using subscripts A, B in the place of α, β etc. Now since $C^{A}_{\mu B} = -C^{A}_{B\mu} = 0$, it follows from the definition of the Lie algebra metric and Eq. (106) that

$$
g_{AB} = \sum_{CD} C^{D}_{AC}C^{C}_{BD}. \tag{107}
$$

Now, as we have assumed that the B_A span a compact semi-simple Lie algebra. Wikipedia tells me that a real Lie algebra is called compact if the Killing form is negative definite. This contradicts Weinberg's statement that, for similar reasons, g_{AB} is postive definite. If we look in Weinberg II in the footnote on page 9, he equivalently says that a simple or semi-simple Lie algebra is said to be compact if the matrix

$-\sum_{CD} C^D{}_{AC} C^C{}_{BD}$ is positive definite. Let's check for ourselves the 'positiveness' of g_{AB}. Now there is a basis for the Lie algebra for which the struture constants $C^A{}_{BC}$ are antisymmetric in all three indices, and so

$$\begin{aligned} g_{AB} &= \sum_{CD} C^D{}_{AC} C^C{}_{BD} \\ &= -\sum_{CD} C^C{}_{AD} C^C{}_{BD}. \end{aligned}$$

For the diagonal elements g_{AA}, we are looking at the negative of the sum of squared real elements, and so the diagonal entries of g_{AB} are positive or zero, and hence if g_{AB} is definite anything, it will be negative definite. Also, this agrees with both wikipedia and Weinberg II, where the matrix $-\sum_{CD} C^D{}_{AC} C^C{}_{BD}$ would then be positive definite, as would be true for a compact simple or semi-simple Lie algebra, as is the case with the algebra spanned by the generators B_α.

Now, if we consider the matrices $g^{1/2} D(\Lambda) g^{-1/2}$, we can see from Eq. (101), that they also furnish a representation of the homogeneous Lorentz group

$$g^{1/2} D(\Lambda_1) g^{-1/2} g^{1/2} D(\Lambda_2) g^{-1/2} = g^{1/2} D(\Lambda_1) D(\Lambda_2) g^{-1/2} = g^{1/2} D(\Lambda_1 \Lambda_2) g^{-1/2}.$$

We can also show that these matrices are orthogonal

$$\begin{aligned} \left((g^{1/2} D(\Lambda) g^{-1/2})^T (g^{1/2} D(\Lambda) g^{-1/2})\right)_{\beta\delta} &= \sum_\alpha \left(g^{1/2} D(\Lambda) g^{-1/2})_{\alpha\beta}\right)^T (g^{1/2} D(\Lambda) g^{-1/2})_{\alpha\delta} \\ &= \sum_\alpha \sum_{\alpha'\beta'} \sum_{\gamma'\delta'} \left(g^{1/2}_{\alpha\alpha'} D(\Lambda)_{\alpha'\beta'} g^{-1/2}_{\beta'\beta}\right)^T g^{1/2}_{\alpha\gamma'} D(\Lambda)_{\gamma'\delta'} g^{-1/2}_{\delta'\delta} \\ &= \sum_\alpha \sum_{\alpha'\beta'} \sum_{\gamma'\delta'} g^{-1/2}_{\beta\beta'} D(\Lambda)_{\beta'\alpha'} g^{1/2}_{\alpha'\alpha}\, g^{1/2}_{\alpha\gamma'} D(\Lambda)_{\gamma'\delta'} g^{-1/2}_{\delta'\delta} \\ &= \sum_{\alpha'\beta'} \sum_{\gamma'\delta'} g^{-1/2}_{\beta\beta'} D(\Lambda)_{\beta'\alpha'} g_{\alpha'\gamma'} D(\Lambda)_{\gamma'\delta'} g^{-1/2}_{\delta'\delta} \\ &= \sum_{\gamma'\delta'} g^{-1/2}_{\beta\beta'}\, g_{\beta'\delta'}\, g^{-1/2}_{\delta'\delta} \\ &= \delta_{\beta\delta}, \end{aligned}$$

therefore $g^{1/2} D(\Lambda) g^{-1/2}$ is an orthogonal matrix. Between lines 4 and 5, Eq. we have used (105). Since both the $D\Lambda$ and the g are real, then $g^{1/2} D(\Lambda) g^{-1/2}$ is an orthogonal matrix, and hence is a unitary matrix, since

$$O^T O = (O^*)^T O = O^\dagger O = 1.$$

It follows that the matrices $g^{1/2} D(\Lambda) g^{-1/2}$ furnish a unitary, finite-dimensional representation of the homogenous Lorentz group. The representation is finite dimensional, since the $g^{1/2}$ and $D(\Lambda)$ are matrices that are finite in size, of which there are a finite number of linearly independant combinations.

However, the Lorentz group is non-compact! The only finite-dimensional representation of the

Lorentz group is the trivial representation, i.e.

$$D(\Lambda) = 1.$$

And so, with $D(\Lambda) = 1$, then the generators B_A commute with $U(\Lambda)$ for all Λ, since

$$\begin{aligned} U(\Lambda) B_\alpha U^{-1}(\Lambda) &= \sum_\beta D^\beta{}_\alpha(\Lambda) B_\beta \\ &= \sum_\beta \delta^\beta{}_\alpha(\Lambda) B_\beta \\ &= B_\alpha, \end{aligned}$$

and so

$$[B_A, U(\Lambda)] = 0.$$

And so we have shown that, now reverting back to our original notation, that

$$[B_\alpha, U(\Lambda)] = 0, \tag{108}$$

where B_α are the generators of the semi-simple Lie algebra of $\mathfrak{G}$ that commute with P_μ. And so we have shown that, both the $B_i^\sharp$ which are the traceless $U(1)$ generators, and the B_α, which are the generators spanning the semi-simple Lie algebra, both commuting with the four-momentum P_μ, are either internal symmetry generators, or linear combinations of components of P_μ.

1.7 The subalgebra spanned by generators that don't commute with P_μ

Having dealt with the symmetry generators that commute with P_μ, let us look at the possible symmetry generators which do not. The action of a general symmetry generator A_α on a one-particle state $|p, n\rangle$ of four-momentum p would be

$$A_\alpha |p, n\rangle = \sum_{n'} \int d^4 p' \Big(\mathcal{A}_\alpha(p', p) \Big)_{n'n} |p', n'\rangle, \tag{109}$$

where $\mathcal{A}$ is the 'kernel,' and n and n' are aggain discrete indices labelling both spin z-components and particle types. The kernel must vanish unless both p and p' are on the mass shell. Now if A_α is a symmetry generator, then so is

$$A_\alpha^f := \int d^4 x \exp(iP \cdot x) A_\alpha \exp(-iP \cdot x) f(x), \tag{110}$$

where P_μ is the four-momentum operator, and $f(x)$ is a function of our choice. Let us now act on a one-particle state

$$\begin{aligned}
A^f_\alpha|p,n\rangle &= \int d^4x \exp(iP\cdot x)A_\alpha \exp(-iP\cdot x)f(x)|p,n\rangle \\
&= \int d^4x \exp(iP\cdot x)A_\alpha \exp(-ip\cdot x)f(x)|p,n\rangle \\
&= \sum_{n'}\int d^4x \int d^4p' \exp(-ip\cdot x)\exp(iP\cdot x)\Big(\mathcal{A}_\alpha(p',p)\Big)_{n'n} f(x)|p',n'\rangle \\
&= \sum_{n'}\int d^4x \int d^4p' \exp(-ip\cdot x)\exp(ip'\cdot x)\Big(\mathcal{A}_\alpha(p',p)\Big)_{n'n} f(x)|p',n'\rangle \\
&= \sum_{n'}\int d^4p' \int d^4x \exp(i(p'-p)\cdot x)f(x)\Big(\mathcal{A}_\alpha(p',p)\Big)_{n'n}|p',n'\rangle \\
&= \sum_{n'}\int d^4p' \tilde{f}(p'-p)\Big(\mathcal{A}_\alpha(p',p)\Big)_{n'n}|p',n'\rangle,
\end{aligned}$$

where $\tilde{f}(k)$ is the Fourier transform

$$\tilde{f}(k) = \int d^4x \exp(ik\cdot x)f(x). \tag{111}$$

And so

$$A^f_\alpha|p,n\rangle = \sum_{n'}\int d^4p' \tilde{f}(p'-p)\Big(\mathcal{A}_\alpha(p',p)\Big)_{n'n}|p',n'\rangle. \tag{112}$$

Now, suppose there is a pair of mass shell four-momenta p_1 and $p_1+\Delta$ with $\Delta \neq 0$. For a two-particle scattering process with four-momenta satisfying $p_1+q_1 \to p_2+q_2$, then in general, $q_1+\Delta$, $p_2+\Delta$, and $q_2+\Delta$ will not be mass shells. Now, since we can choose $\tilde{f}(k)$ to be anything we want, let us choose it to vanish outside of a sufficiently small region around $k=\Delta$. Then, for the above scattering process

$$\begin{aligned}
A^f_\alpha|p_1,n\rangle &= \sum_{n'}\int d^4p'_1 \tilde{f}(p'_1-p_1)\Big(\mathcal{A}_\alpha(p'_1,p_1)\Big)_{n'n}|p'_1,n'\rangle = \sum_{n'}\tilde{f}(\Delta)\Big(\mathcal{A}_\alpha(p_1+\Delta,p_1)\Big)_{n'n}|p_1+\Delta,n'\rangle \\
A^f_\alpha|q_1,n\rangle &= \sum_{n'}\int d^4q'_1 \tilde{f}(q'_1-q_1)\Big(\mathcal{A}_\alpha(q'_1,q_1)\Big)_{n'n}|q'_1,n'\rangle = \sum_{n'}\tilde{f}(\Delta)\Big(\mathcal{A}_\alpha(q_1+\Delta,q_1)\Big)_{n'n}|q_1+\Delta,n'\rangle,
\end{aligned}$$

and similarly for p_2 and q_2. We have established, however, that if $p_1+\Delta$ is on the mass shell, then a general $q_1+\Delta$ will not be, and similarly for p_2 and q_2. Consequently, the action of A^f_α upon $|p_1,n\rangle$ does not annihilate the state, but its action upon $|q_1,n\rangle$, $|p_2,n\rangle$, and $|q_2,n\rangle$ does

$$A^f_\alpha|p_1,n\rangle \neq 0 \tag{113}$$

$$A^f_\alpha|q_1,n\rangle = A^f_\alpha|p_2,n\rangle = A^f_\alpha|q_2,n\rangle = 0. \tag{114}$$

This is because everywhere we have chosen $\tilde{f}(k)$ to be non-zero, the kernel $\mathcal{A}_\alpha(q_1+\Delta,q_1)$ vanishes due to $q_1+\Delta$ not being on the mass shell. This is a bit of an issue, since we assumed that scattering occurs at almost all energies (except for perhaps some isolated set of energies). We have just shown that if a pair of four-momenta p_1 and $p_1 + \Delta$ are on the mass shell, then an appropriate $\tilde{f}(k)$ can be chosen such that A_α^f annihilates all states with momenta q_1, p_2, and q_2, and so the symmetry generated by A_α^f forbids a process to have the kinematics $p_1 + q_1 \to p_2 + q_2$. This would mean that scattering does not occur at almost all energies and angles, so contradits with our assumption. This problem is averted if A_α commutes with the four-momentum operator P_μ since

$$\begin{aligned} A_\alpha^f|p,n\rangle &= \int d^4x \exp(iP\cdot x)A_\alpha \exp(-iP\cdot x)f(x)|p,n\rangle \\ &= \left(\int d^4x f(x)\right) A_\alpha|p,n\rangle \propto A_\alpha|p,n\rangle, \end{aligned}$$

and so $\tilde{f}(k)$ does not enter into the action of A_α^f on a one-particle state, and cannot be appropriately chosen to cause problems. This option isn't great however, since if A_α commuted with the four-momentum operator P_μ, then the A_α would just be linear combinations of the B_α, which we have already discussed. Instead of allowing the A_α to commute with P_μ, consider a kernel $\mathcal{A}_\alpha(p',p)$ that contains a momentum space delta function

$$\Big(\mathcal{A}_\alpha(p',p)\Big)_{n'n} = \delta^{(4)}(p'-p)\Big(a_\alpha^0(p',p)\Big)_{n'n}.$$

Then, the action of A_α^f on a one particle state becomes

$$\begin{aligned} A_\alpha^f|p,n\rangle &= \sum_{n'}\int d^4p' \tilde{f}(p'-p)\Big(\mathcal{A}_\alpha(p',p)\Big)_{n'n}|p',n'\rangle \\ &= \sum_{n'}\int d^4p' \tilde{f}(p'-p)\delta^{(4)}(p'-p)\Big(a_\alpha^0(p',p)\Big)_{n'n}|p',n'\rangle \\ &= \tilde{f}(0)\sum_{n'}\Big(a_\alpha^0(p)\Big)_{n'n}|p,n'\rangle. \end{aligned}$$

Now, the function of our choice $\tilde{f}(0)$ is now evaluated independant of the momentum of the state that it acts on. Therefore, if A_α^f is a symmetry generator that does not annihilate a state with four-momentum p, then it will also not annihilate a state of four-momentum p', and so the symmetry generated by A_α would allow a scattering process with kinematics $p_1 + q_1 \to p_2 + q_2$ for p_1, q_1, p_2, and q_2 on the mass shell. This is now in-line with the assumptions about scattering we made at the beginning. Now let us consider something more general. Consider the kernels $\mathcal{A}_\alpha(p',p)$ to be distributions, that is, contain objects proportional to $\delta^{(4)}(p'-p)$ as well as at most a finite number D_α of derivatives $\partial/\partial p'_\mu$ of $\delta^{(4)}(p'-p)$. This leads to a kernel expansion

$$\begin{aligned} \Big(\mathcal{A}(p',p)\Big)_{n'n} =& \Big(a_\alpha^0(p',p)\Big)_{n'n}\delta^{(4)}(p'-p) + \Big(a_\alpha^1(p',p)\Big)_{n'n}^{\mu_1}\frac{\partial}{\partial p'^{\mu_1}}\delta^{(4)}(p'-p) \\ &+ \Big(a_\alpha^2(p',p)\Big)_{n'n}^{\mu_1\mu_2}\frac{\partial^2}{\partial p'^{\mu_1}\partial p'^{\mu_2}}\delta^{(4)}(p'-p) + \cdots \\ &+ \Big(a_\alpha^{D_\alpha}(p',p)\Big)_{n'n}^{\mu_1\mu_2\cdots\mu_{D_\alpha}}\frac{\partial^{D_\alpha}}{\partial p'^{\mu_1}\partial p'^{\mu_2}\cdots\partial p'^{\mu_{D_\alpha}}}\delta^{(4)}(p'-p). \end{aligned} \tag{115}$$

Note, for e.g. a^2, the 2 is just a label, i.e. $a^2 \neq a \times a$. Now note the property of delta functions

$$\int d^4p' \frac{\partial}{\partial p'^\mu} \delta^{(4)}(p'-p) f^\mu(p') = -\int d^4p' \delta^{(4)}(p'-p) \frac{\partial}{\partial p'^\mu} f^\mu(p'). \tag{116}$$

Now let us consider the first derivative term in Eq. (115)

$$\left(a^1_\alpha(p',p)\right)^{\mu_1}_{n'n} \frac{\partial}{\partial p'^{\mu_1}} \delta^{(4)}(p'-p)$$

in the action of A^f_α upon a one particle state

$$\begin{aligned}
\left(A^f_\alpha|p,n\rangle\right)_{\text{first derivative}} &= \sum_{n'} \int d^4p' \tilde{f}(p'-p) \left(a^1_\alpha(p',p)\right)^{\mu_1}_{n'n} \frac{\partial}{\partial p'^{\mu_1}} \delta^{(4)}(p'-p)|p',n'\rangle \\
&= -\sum_{n'} \int d^4p' \frac{\partial}{\partial p'^{\mu_1}} \left(\tilde{f}(p'-p)\left(a^1_\alpha(p',p)\right)^{\mu_1}_{n'n}\right) \delta^{(4)}(p'-p)|p',n'\rangle \\
&= -\sum_{n'} \frac{\partial}{\partial p^{\mu_1}} \left(\tilde{f}(0)\left(a^1_\alpha(p)\right)^{\mu_1}_{n'n}|p,n'\rangle\right) \\
&= -\tilde{f}(0) \sum_{n'} \frac{\partial}{\partial p^{\mu_1}} \left(\left(a^1_\alpha(p)\right)^{\mu_1}_{n'n}|p,n'\rangle\right).
\end{aligned}$$

From the first two terms in Eq. (115), we therefore have

$$A^f_\alpha|p,n\rangle = \tilde{f}(0) \sum_{n'} \left(\left(a^0_\alpha(p)\right)_{n'n}|p,n'\rangle - \frac{\partial}{\partial p^{\mu_1}}\left(\left(a^1_\alpha(p)\right)^{\mu_1}_{n'n}|p,n'\rangle\right)\right) + \text{higher derivatives}. \tag{117}$$

Expanding the first derivative term

$$A^f_\alpha|p,n\rangle = \tilde{f}(0) \sum_{n'} \left(\left(a^0_\alpha(p)\right)_{n'n} - \frac{\partial}{\partial p^{\mu_1}}\left(a^1_\alpha(p)\right)^{\mu_1}_{n'n} - \left(a^1_\alpha(p)\right)^{\mu_1}_{n'n}\frac{\partial}{\partial p^{\mu_1}}\right)|p,n'\rangle + \text{h.d.} \tag{118}$$

Then we can define

$$\left(a'^0_\alpha(p)\right)_{n'n} = \tilde{f}(0)\left(\left(a^0_\alpha(p)\right)_{n'n} - \frac{\partial}{\partial p^{\mu_1}}\left(a^1_\alpha(p)\right)^{\mu_1}_{n'n}\right),$$

so that, expecting we can do similarly for other terms,

$$A^f_\alpha|p,n\rangle = \sum_{n'} \left(\left(a'^0_\alpha(p)\right)_{n'n} + \left(a'^1_\alpha(p)\right)^{\mu_1}_{n'n}\frac{\partial}{\partial p^{\mu_1}} + \cdots + \left(a'^{D_\alpha}_\alpha(p)\right)^{\mu_1\cdots\mu_{D_\alpha}}_{n'n}\frac{\partial^{D_\alpha}}{\partial p^{\mu_1}\partial p^{\mu_2}\cdots\partial p^{\mu_{D_\alpha}}}\right)|p,n'\rangle. \tag{119}$$

We can see from this that if the kernels $a_\alpha(p',p)$ contain at most a finite number D_α of derivatives of $\delta^{(4)}(p'-p)$, then the action of the symmetry generator A^f_α, on one-particle states, is a polynomial of order D_α in the derivatives $\partial/\partial p^\mu$ with matrix coefficicents $a'_\alpha(p)\Big)_{n'n}$ that currently depend on momentum and spin. This also holds for A_α, since

$$A_\alpha|p,n\rangle = \sum_{n'}\left(\left(a^0_\alpha(p)\right)_{n'n} + \left(a^1_\alpha(p)\right)^{\mu_1}_{n'n}\frac{\partial}{\partial p^{\mu_1}} + \dots + \left(a^{D_\alpha}_\alpha(p)\right)^{\mu_1\cdots\mu_{D_\alpha}}_{n'n}\frac{\partial^{D_\alpha}}{\partial p^{\mu_1}\partial p^{\mu_2}\cdots\partial p^{\mu_{D_\alpha}}}\right)|p,n'\rangle, \quad (120)$$

where we no longer write the primes on the matrix coefficients for convenience. The coefficients themselves are in general different to those in A^f_α for a given $\tilde{f}(0)$, since A_α does not depend on the choice of $\tilde{f}(k)$.

Now if A_α are general symmetry generators, then they must contain the B_α as a subset (i.e. the B_α are the subset of A_α who commute with P_μ). The B_α, as we have shown, act only as matrices on one particle states as in Eq. (2). They do not act as a polynomial of momentum derivatives of the state as in Eq. (120). Therefore, the B_α can in principle be formed by the D_α-fold commutator of momentum operators with A_α

$$B^{\mu_1\cdots\mu_{D_\alpha}}_\alpha := [P^{\mu_1},[P^{\mu_2},\dots[P^{\mu_{D_\alpha}},A_\alpha]]\dots]. \quad (121)$$

To demonstrate, let's consider a A_α with $D_\alpha = 1$ acting on a one-particle state $|p,n\rangle$. Then

$$\begin{aligned}
B^\mu_\alpha|p,n\rangle &= [P^\mu, A_\alpha]|p,n\rangle \\
&= \sum_{n'}[P^\mu, \left(a^0(p)\right)_{n'n} + \left(a^1(p)\right)^\nu_{n'n}\frac{\partial}{\partial p^\nu}]|p,n'\rangle \\
&= \sum_{n'}[P^\mu, \left(a^1(p)\right)^\nu_{n'n}\frac{\partial}{\partial p^\nu}]|p,n'\rangle \\
&= \sum_{n'}\left(P^\mu\left(a^1(p)\right)^\nu_{n'n}\frac{\partial}{\partial p^\nu} - \frac{\partial}{\partial p^\nu}P^\mu\left(a^1(p)\right)^\nu_{n'n} - P^\mu\left(a^1(p)\right)^\nu_{n'n}\frac{\partial}{\partial p^\nu}\right)|p,n'\rangle \\
&= -\sum_{n'}\frac{\partial}{\partial p^\nu}P^\mu\left(a^1(p)\right)^\nu_{n'n}|p,n'\rangle \\
&= -\sum_{n'}\frac{\partial}{\partial p^\nu}p^\mu\left(a^1(p)\right)^\nu_{n'n}|p,n'\rangle \\
&= -\sum_{n'}\delta^\mu_\nu\left(a^1(p)\right)^\nu_{n'n}|p,n'\rangle \\
&= -\sum_{n'}\left(a^1(p)\right)^\mu_{n'n}|p,n'\rangle,
\end{aligned}$$

and so

$$B^\mu_\alpha|p,n\rangle = \sum_{n'}\left(b_\alpha(p)\right)^\mu_{m'm}|p,n'\rangle = -\sum_{n'}\left(a^1(p)\right)^\mu_{n'n}|p,n'\rangle. \quad (122)$$

Note that, although we have shown that the traceless $b^\sharp_\alpha$ are momentum independent, this does not necessarily mean that the $b_\alpha(p)$ are.

Now, since the B_α commute with the four-momentum operator P^μ, consider the commuator of $B^{\mu_1\cdots\mu_{D_\alpha}}_\alpha$ with P^μ between states with four-momentum p_1 and p_2. For simplicity, we will just consider the commutator of $B^{\mu_1}_\alpha$ with P^μ, i.e. the $D_\alpha = 1$ case, then consider the extension to general D_α

$$\begin{aligned}
\langle p_2|[B^{\mu_1}_\alpha, P^\mu]|p_1\rangle &= \langle p_2|[[P^{\mu_1}, A_\alpha], P^\mu]|p_1\rangle \\
&= \langle p_2|\left(P^{\mu_1}A_\alpha P^\mu - A_\alpha P^{\mu_1}P^\mu - P^\mu P^{\mu_1}A_\alpha + P^\mu A_\alpha P^{\mu_1}\right)|p_1\rangle \\
&= \langle p_2|\left(p_2^{\mu_1}A_\alpha p_1^\mu - A_\alpha p_1^{\mu_1}p_1^\mu - p_2^\mu p_2^{\mu_1}A_\alpha + p_2^\mu A_\alpha p_1^{\mu_1}\right)|p_1\rangle \\
&= \left(p_2^{\mu_1}p_1^\mu - p_1^{\mu_1}p_1^\mu - p_2^\mu p_2^{\mu_1} + p_2^\mu p_1^{\mu_1}\right)\langle p_2|A_\alpha|p_1\rangle \\
&= -\left(p_2 - p_1\right)^{\mu_1}\left(p_2 - p_1\right)^\mu \langle p_2|A_\alpha|p_1\rangle \\
&= -\left(p_2 - p_1\right)^{\mu_1}\left(p_2 - p_1\right)^\mu \int d^4p'\langle p_2|\mathcal{A}_\alpha(p', p_1)|p'\rangle \\
&= -\left(p_2 - p_1\right)^{\mu_1}\left(p_2 - p_1\right)^\mu \int d^4p'\Big(a^0_\alpha(p', p_1)\delta^{(4)}(p' - p_1) + \\
&\qquad\qquad a^1_\alpha(p', p_1)^\nu \frac{\partial}{\partial p'^\nu}\delta^{(4)}(p' - p_1)\Big)\langle p_2|p'\rangle \\
&= -\left(p_2 - p_1\right)^{\mu_1}\left(p_2 - p_1\right)^\mu \int d^4p'\delta^{(4)}(p' - p_1)\left[\left(a^0_\alpha(p', p_1) + \frac{\partial}{\partial p'^\nu}a^1_\alpha(p', p_1)^\nu\right) + \right. \\
&\qquad\qquad \left. a^1_\alpha(p', p_1)^\nu \frac{\partial}{\partial p'^\nu}\right]\langle p_2|p'\rangle \\
&= -\left(p_2 - p_1\right)^{\mu_1}\left(p_2 - p_1\right)^\mu \left(a'^0_\alpha(p_1) + a^1_\alpha(p_1)^\nu \frac{\partial}{\partial p'^\nu}\right)\langle p_2|p_1\rangle,
\end{aligned}$$

where

$$a'^0_\alpha(p_1) = a^0_\alpha(p_1) + \frac{\partial}{\partial p_1{}^\nu}a^1_\alpha(p_1)^\nu.$$

Using the one-particle state relativistic normalization,

$$\langle p|q\rangle = 2E_p(2\pi)^3\delta^{(3)}(\mathbf{p} - \mathbf{q}) \tag{123}$$

and noting that the four-momenta p_2 and p_1 must be on the same mass shell, then

$$\begin{aligned}
\langle p_2|[B^{\mu_1}_\alpha, P^\mu]|p_1\rangle &= -\left(p_2 - p_1\right)^{\mu_1}\left(p_2 - p_1\right)^\mu \left(a'^0_\alpha(p_1) + a^1_\alpha(p_1)^\nu \frac{\partial}{\partial p'^\nu}\right)\langle p_2|p_1\rangle \\
&= -\left(p_2 - p_1\right)^{\mu_1}\left(p_2 - p_1\right)^\mu \left(a'^0_\alpha(p_1) + a^1_\alpha(p_1)^\nu \frac{\partial}{\partial p'^\nu}\right)2E_{p_2}(2\pi)^3\delta^{(3)}(\mathbf{p_2} - \mathbf{p_1}) \\
&\propto \left(p_2 - p_1\right)^{\mu_1}\left(p_2 - p_1\right)^\mu \left(a'^0_\alpha(p_1) + a^1_\alpha(p_1)^\nu \frac{\partial}{\partial p'^\nu}\right)\delta^{(4)}(p_2 - p_1),
\end{aligned}$$

where in the last step, we note that if p_2 and p_1 are on the same mass shell, then the evaluation of the delta function $\delta^{(3)}(\mathbf{p_2} - \mathbf{p_1})$ is equivalent to the evaluation of $\delta^{(4)}(p_2 - p_1)$ up to some proportional constant. We have shown for $D_\alpha = 1$ that

$$\langle p_2|[B^{\mu_1}_\alpha, P^\mu]|p_1\rangle \propto \left(p_2 - p_1\right)^{\mu_1}\left(p_2 - p_1\right)^\mu \left(a'^0_\alpha(p_1) + a^1_\alpha(p_1)^\nu \frac{\partial}{\partial p'^\nu}\right)\delta^{(4)}(p_2 - p_1), \tag{124}$$

that is, the matrix elements of the commutators of $B^{\mu_1 \cdots \mu_{D_\alpha}}_\alpha$ with P^μ between states of four-momentum

p_2 and p_1 ae proportional to $D_\alpha + 1$ factors of $p_2 - p_1$ times a polynomial of order D_α in momentum derivatives acting on $\delta^{(4)}(p_2 - p_1)$. Can we motivate this as being true for general D_α? First, we can motivative the $D_\alpha+1$ factors of p_2-p_1 by working from the outside of the commutator inwards. Consider now general D_α, then

$$\begin{aligned}\langle p_2|[[P^{\mu_1},[P^{\mu_2},\ldots[P^{\mu_{D_\alpha}},A_\alpha]]\ldots],P^\mu]|p_1\rangle &= -(p_2-p_1)^\mu\langle p_2|[P^{\mu_1},[P^{\mu_2},\ldots[P^{\mu_{D_\alpha}},A_\alpha]]\ldots]|p_1\rangle \\ &= -(p_2-p_1)^\mu(p_2-p_1)^{\mu_1}\langle p_2|[P^{\mu_2},[P^{\mu_3},\ldots[P^{\mu_{D_\alpha}},A_\alpha]]\ldots]|p_1\rangle \\ &= -(p_2-p_1)^\mu(p_2-p_1)^{\mu_1}\ldots(p_2-p_1)^{\mu_{D_\alpha-1}}\langle p_2|[P^{\mu_{D_\alpha}},A_\alpha]|p_1\rangle \\ &= -(p_2-p_1)^\mu(p_2-p_1)^{\mu_1}\ldots(p_2-p_1)^{\mu_{D_\alpha}}\langle p_2|A_\alpha|p_1\rangle.\end{aligned}$$

So there are the $D_\alpha + 1$ factors of $p_2 - p_1$, for general D_α. Now what about the polynomial of order D_α in momentum derivatives?

$$\begin{aligned}\langle p_2|A_\alpha|p_1\rangle &= \int d^4p'\langle p_2|\mathcal{A}_\alpha(p',p_1)|p'\rangle \\ &= \int d^4p'\Big(a^0_\alpha(p',p_1)\delta^{(4)}(p'-p_1) + a^1_\alpha(p',p_1)^\nu\frac{\partial}{\partial p'^\nu}\delta^{(4)}(p'-p_1) \\ &\qquad + a^{D_\alpha}_\alpha(p',p_1)^{\nu_1\nu_2\cdots\nu_{D_\alpha}}\frac{\partial^{D_\alpha}}{\partial p'^{\nu_1}\partial p'^{\nu_2}\cdots\partial p'^{\nu_{D_\alpha}}}\delta^{(4)}(p'-p_1)\Big)\langle p_2|p'\rangle.\end{aligned}$$

Using another property of delta functions

$$\int dx\delta^{(n)}(x)f(x) = (-1)^n\int dx\delta(x)f^{(n)}(x), \tag{125}$$

where here (n) indicates the n-th derivative with respect to x, then we can see

$$\begin{aligned}\langle p_2|A_\alpha|p_1\rangle = \int d^4p'\delta^{(4)}(p'-p_1)\Big(&{a'}^0_\alpha(p',p_1) + {a'}^1_\alpha(p',p_1)^\nu\frac{\partial}{\partial p'^\nu} \\ &+\ldots+{a'}^{D_\alpha}_\alpha(p',p_1)^{\nu_1\nu_2\cdots\nu_{D_\alpha}}\frac{\partial^{D_\alpha}}{\partial p'^{\nu_1}\partial p'^{\nu_2}\cdots\partial p'^{\nu_{D_\alpha}}}\Big)\langle p_2|p'\rangle,\end{aligned}$$

where the primed coefficients have absorbed the factors of -1 and the derivatives of other coefficients from the product rule, as seen in the $D_\alpha = 1$ case. Performing the integral and using the same mass-shell trick as before, we find that

$$\langle p_2|A_\alpha|p_1\rangle \propto \left({a'}^0_\alpha(p_1) + a^1_\alpha(p_1)^\nu\frac{\partial}{\partial p'^\nu} + \ldots + {a'}^{D_\alpha}_\alpha(p',p_1)^{\nu_1\nu_2\cdots\nu_{D_\alpha}}\frac{\partial^{D_\alpha}}{\partial p'^{\nu_1}\partial p'^{\nu_2}\cdots\partial p'^{\nu_{D_\alpha}}}\right)\delta^{(4)}(p_2-p_1),$$

and so

$$\langle p_2 | [B_\alpha^{\mu_1 \cdots \mu_{D_\alpha}}, P^\mu] | p_1 \rangle \propto (p_2 - p_1)^\mu (p_2 - p_1)^{\mu_1} \ldots (p_2 - p_1)^{\mu_{D_\alpha}} \Bigg(a'^0_\alpha(p_1) + a^1_\alpha(p_1)^\nu \frac{\partial}{\partial p'^\nu} + \ldots + a'^{D_\alpha}_\alpha(p', p_1)^{\nu_1 \nu_2 \cdots \nu_{D_\alpha}} \frac{\partial^{D_\alpha}}{\partial p'^{\nu_1} \partial p'^{\nu_2} \cdots \partial p'^{\nu_{D_\alpha}}} \Bigg) \delta^{(4)}(p_2 - p_1). \tag{126}$$

We have then showed that for any D_α that the matrix elements of the commutators of $B_\alpha^{\mu_1 \cdots \mu_{D_\alpha}}$ with P^μ between states of four-momentum p_2 and p_1 ae proportional to $D_\alpha + 1$ factors of $p_2 - p_1$ times a polynomial of order D_α in momentum derivatives acting on $\delta^{(4)}(p_2 - p_1)$. This therefore vanishes, since the delta functions would take $p_2 \to p_1$, and the prefactors would go to zero. This is good, as then the $B_\alpha^{\mu_1 \cdots \mu_{D_\alpha}}$ automatically commute with the P^μ, as we would like.

Now from our definition of $B^\sharp_\alpha$ in Eq. (60), we then know that

$$\begin{aligned} B_\alpha |p, n\rangle = \sum_{n'} \Big(b_\alpha(p) \Big)_{n'n} |p, n'\rangle &= \left(B^\sharp_\alpha - a^\mu_\alpha P_\mu \right) |p, n\rangle \\ &= \sum_{n'} \left[\left(b^\sharp_\alpha \right)_{n'n} + a^\mu_\alpha p_\mu \delta_{n'n} \right] |p, n'\rangle, \end{aligned}$$

and so the action of B_α on one particle states can be written

$$\Big(b_\alpha(p) \Big)_{n'n} = \left(b^\sharp_\alpha \right)_{n'n} + a^\mu_\alpha p_\mu \delta_{n'n}. \tag{127}$$

It follows that for the generators $B_\alpha^{\mu_1 \cdots \mu_{D_\alpha}}$ that

$$\Big(b_\alpha(p) \Big)^{\mu_1 \cdots \mu_{D_\alpha}}_{n'n} = \left(b^\sharp_\alpha \right)^{\mu_1 \cdots \mu_{D_\alpha}}_{n'n} + a^{\mu \mu_1 \cdots \mu_{D_\alpha}}_\alpha p_\mu \delta_{n'n}, \tag{128}$$

where the $b^{\sharp\, \mu_1 \cdots \mu_{D_\alpha}}_\alpha$ are the momentum-independent, traceless Hermitian matrices generating an ordinary internal symmetry algebra (as shown in the previous section), and the $a^{\mu \mu_1 \cdots \mu_{D_\alpha}}_\alpha$ are momentum-independent numerical constants. It is clear from Eq. (126) that ${B_\alpha}^{\mu_1 \cdots \mu_{D_\alpha}}$ is symmetric in all of its indices, and therefore $b_\alpha(p)^{\mu_1 \cdots \mu_{D_\alpha}}$ must also be. $b^\sharp_\alpha(p)^{\mu_1 \cdots \mu_{D_\alpha}}$ and $a^{\mu \mu_1 \cdots \mu_{D_\alpha}}_\alpha$ may also then be taken to be symmetric in the indices $\mu_1 \cdots \mu_{D_\alpha}$.

Now, although A_α does not neccessarily commute with P_μ, we have asserted in our first assumption, that for any M, there are only a finite number of particle types with mass less than M. This means that the mass-squared operator $P_\mu P^\mu$ must have discrete eigenvalues, and since A_α cannot take one-particle states off the mass shell, this means that A_α commutes with the mass-squared operator

$$[P_\mu P^\mu, A_\alpha] = 0. \tag{129}$$

For $D_\alpha \geq 1$, this implies

$$\begin{aligned}
[P^{\mu_1}P_{\mu_1}, [P^{\mu_2}, \ldots [P^{\mu_{D_\alpha}}, A_\alpha]] \ldots] &= P_{\mu_1}[P^{\mu_1}, [P^{\mu_2}, \ldots [P^{\mu_{D_\alpha}}, A_\alpha]] + [P^{\mu_1}, [P^{\mu_2}, \ldots [P^{\mu_{D_\alpha}}, A_\alpha]]P_{\mu_1} \\
&= P_{\mu_1}B_\alpha{}^{\mu_1 \cdots \mu_{D_\alpha}} + B_\alpha{}^{\mu_1 \cdots \mu_{D_\alpha}}P_{\mu_1} \\
&= 2P_{\mu_1}B_\alpha{}^{\mu_1 \cdots \mu_{D_\alpha}} \\
&= 0,
\end{aligned}$$

where the last line follows from Eq. (129). This means that

$$p_{\mu_1}b_\alpha^{\mu_1 \cdots \mu_{D_\alpha}}(p) = 0. \tag{130}$$

In terms of the traceless and diagonal parts of $b_\alpha(p)$, this means

$$p_{\mu_1}b_\alpha^{\mu_1 \cdots \mu_{D_\alpha}}(p) = p_{\mu_1}\left(b_\alpha^{\sharp\ \mu_1 \cdots \mu_{D_\alpha}} + a_\alpha^{\mu\mu_1 \cdots \mu_{D_\alpha}}p_\mu\right) = 0. \tag{131}$$

If we have a theory which contains massive particles, then p can be any timelike direction we want. If the above is to hold for all such choices of four-momentum, then, comparing coefficients

$$b_\alpha^{\sharp\ \mu_1 \cdots \mu_{D_\alpha}} = 0, \tag{132}$$

and

$$a_\alpha^{\mu\mu_1 \cdots \mu_{D_\alpha}} = -a_\alpha^{\mu_1\mu \cdots \mu_{D_\alpha}}. \tag{133}$$

Combined with the symmetry in the indices $\mu_1 \cdots \mu_{D_\alpha}$, this implies that, for $D_\alpha \geq 2$

$$\begin{aligned}
a_\alpha^{\mu\mu_1\mu_2 \cdots \mu_{D_\alpha}} &= -a_\alpha^{\mu_1\mu\mu_2 \cdots \mu_{D_\alpha}} \\
&= -a_\alpha^{\mu_1\mu_2\mu \cdots \mu_{D_\alpha}} \\
&= a_\alpha^{\mu_2\mu_1\mu \cdots \mu_{D_\alpha}} \\
&= a_\alpha^{\mu_2\mu\mu_1 \cdots \mu_{D_\alpha}} \\
&= -a_\alpha^{\mu\mu_2\mu_1 \cdots \mu_{D_\alpha}} \\
&= -a_\alpha^{\mu\mu_1\mu_2 \cdots \mu_{D_\alpha}},
\end{aligned}$$

and so for $D_\alpha \geq 2$

$$a_\alpha^{\mu\mu_1\mu_2 \cdots \mu_{D_\alpha}} = -a_\alpha^{\mu\mu_1\mu_2 \cdots \mu_{D_\alpha}} = 0, \tag{134}$$

and therefore for $D_\alpha \geq 2$

$$B_\alpha^{\mu_1 \cdots \mu_{D_\alpha}} = 0. \tag{135}$$

So what are we left with? For $D_\alpha = 0$, we have the generators A_α who are the B_α that commute with P_μ. There must therefore be an internal symmetry generator or some linear combination of P_μ. We are also left with the $D_\alpha = 1$, for which

$$B_\alpha^\mu = [P^\mu, A_\alpha],$$

and

$$B_\alpha^\nu = a_\alpha^{\mu\nu} P_\mu,$$

where $a_\alpha^{\nu\mu}$ are some numerical constants antisymmetric in μ and ν. It follows that

$$[P^\nu, A_\alpha] = a_\alpha^{\mu\nu} P_\mu, \tag{136}$$

i.e. the commutator of A_α with the four-momentum operator produces linear combinations of the four momentum operator. Finally, it follows from the Poincaré algebra that

$$A_\alpha = -\frac{1}{2} i a_\alpha^{\mu\nu} J_{\mu\nu} + B_\alpha, \tag{137}$$

where $J_{\mu\nu}$ is the generator of proper Lorentz transformations satisfying

$$[P^\mu, J^{\rho\sigma}] = -i\eta^{\nu\rho} P^\sigma + i\eta^{\nu\sigma} P^\rho, \tag{138}$$

and B_α commutes with P_μ. The factors of i and $1/2$ are just conventional. Since A_α and $J_{\mu\nu}$ are symmetry generators, so is B_α.

We have therefore shown that, assuming particle type finiteness, the occurence of scattering at most energies, and the analycity of two-body scattering amplitudes, that the only possible Lie algebra of symmetry generators consists of the generators P_μ and $J_{\mu\nu}$ of translastions and homogenous Lorentz transformations, together will possible internal symmetry generators, which commute with P_μ and $J_{\mu\nu}$.

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