

Relativistic Electrodynamics

L1p1

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office hours: Mon 12-14.

Job is to guide you through an extremely satisfying course on theoretical physics. Satisfying: hard. But rewarding.

Books: "Introduction to Electrodynamics". David J Griffiths.
Sec. 10.1, 10.2, 11.1, 11.2, 12.1, 12.2, 12.3

Also: SR "Introduction to Special Relativity" W Rindler

ED "Classical Electrodynamics" J D Jackson.

Structure:

1. Scalar and Vector potential, Gauge transformations.
2. Radiation from special sources (point charge, dipole)
3. Basics of Relativity, space-time, Tensor Calculus
4. Relativistic kinematics and dynamics
5. Relativistic Fields, ...

RED builds on 1st/2nd year SR and EM courses

Prepares you for final year particle theory and GR

Introduction to the theoretical concepts that led to the postulation of the Higgs Boson!

Introduces notation used by the experts to effectively work with theories for relativistic particles and fields.

Modern physics exposes the underlying symmetries of Nature. Make sure to use notation and language that does not hide the symmetries of your theory!

Scalar and Vector potentials

L1, p2

Maxwells Equations: 1

$$(i) \underbrace{\vec{\nabla} \cdot \vec{E}}_{\text{Divergence}} = \rho / \epsilon_0$$

$$(iii) \underbrace{\vec{\nabla} \times \vec{E}}_{\text{Curl}} = - \frac{\partial \vec{B}}{\partial t}$$

$$(ii) \vec{\nabla} \cdot \vec{B} = 0$$

$$(iv) \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$

Set ourself a difficult problem: Solve these for general, time-dependent distribution of sources $\rho(\vec{r}, t)$, $\vec{J}(\vec{r}, t)$.

It will pay off to use the language of potentials rather than the fields themselves: (fields: 2×3 Dof. Pot: 4).

In Electrostatics: $\vec{\nabla} \times \vec{E} = 0$, so $\vec{E} = -\vec{\nabla} V$.

Not so in Electrodynamics!

But still have $\vec{\nabla} \cdot \vec{B} = 0$, so can still write $\vec{B}$ in terms of an $\vec{A}$:

$$(a) \boxed{\vec{B} = \vec{\nabla} \times \vec{A}(\vec{r}, t)} \quad \left\{ \vec{A}: \text{Vector potential, 3 comp.} \right\}$$

If we put this into Faradays law (iii) we find

$$\vec{\nabla} \times \vec{E} = - \frac{\partial}{\partial t} (\vec{\nabla} \times \vec{A})$$

$$\therefore \vec{\nabla} \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right) = 0.$$

i.e. $\vec{E} + \frac{\partial \vec{A}}{\partial t}$ is curl-less and can be written in terms of a scalar potential V :

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla} V \quad \text{such that}$$

$$(b) \boxed{\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}} \quad \begin{array}{l} \text{V: Scalar potential, one component} \\ \vec{A}(\vec{r}, t) \end{array}$$

This representation (a), (b) obviously automatically fulfills the two homogeneous Maxwell Eqⁿ (ii) and (iii).

Constraint from (i) and (iv) are found as:

(b) into (i):

$$-\vec{\nabla} \cdot \vec{E} = \nabla^2 V + \frac{\partial}{\partial t} (\vec{\nabla} \cdot \vec{A}) = -\frac{1}{\epsilon_0} \rho \quad (c)$$

Laplacian interchange order of diff. & for scalar functions.

(a) ^{and (b)} into (iv) gives

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \mu_0 \vec{J} - \mu_0 \epsilon_0 \vec{\nabla} \left(\frac{\partial V}{\partial t} \right) - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2}$$

interchange diff orders

Use $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla} (\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$ gives

$$\left(\nabla^2 \vec{A} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} \right) - \vec{\nabla} \left(\vec{\nabla} \cdot \vec{A} + \mu_0 \epsilon_0 \frac{\partial V}{\partial t} \right) = -\mu_0 \vec{J}. \quad (d)$$

Maxwell's Eq^s for the potential : (c), (d)

Satisfying? Well, not so nice equation...

Gauge Transformations

L1, p4

Notice that $(V, \bar{A})$ are not uniquely defined by (a) and (b).
[Just like the gravitational potential is not unique - a constant can be added without changing the dynamics.]

We can, with some constraints, change $(V, \bar{A})$ without changing $\bar{E}$ and $\bar{B}$: Gauge Freedom

What are the constraints?

$$\text{Let } \bar{A}' = \bar{A} + \bar{\alpha}, \quad V' = V + \beta.$$

Require $\bar{E}, \bar{B}$ unchanged:

$$\bar{B} = \bar{\nabla} \times \bar{A}$$

"

$$\bar{\nabla} \times \bar{A}' \quad \therefore \bar{\nabla} \times \bar{\alpha} = 0 \therefore \bar{\alpha} = \bar{\nabla} \lambda, \quad \lambda \text{ arbitrary scalar.}$$

Also $\bar{E}$ unchanged:

$$\bar{\nabla} \beta + \frac{\partial \bar{\alpha}}{\partial t} = 0.$$

Use $\bar{\alpha} = \bar{\nabla} \lambda$ and interchange $\bar{\nabla}$ and $\frac{\partial}{\partial t}$:

$$\bar{\nabla} \left(\beta + \frac{\partial \lambda}{\partial t} \right) = 0$$

I.e. $(\beta + \frac{\partial \lambda}{\partial t})$ is independent of the position $\bar{r}$.

We ~~can~~ call it $k(t) = \beta + \frac{\partial \lambda}{\partial t}$. Actually, just define a

new λ by adding $\int_0^t k(t') dt'$ to the old one. This does not affect the gradient of λ since it does not depend on $\bar{r}$.

With this new λ we have $\bar{A}' = \bar{A} + \bar{\nabla} \lambda$

$$V' = V - \frac{\partial \lambda}{\partial t}.$$

Conclusion: For an arbitrary λ , we can add $\vec{\nabla}\lambda$ to $\vec{A}$ and subtract $\frac{\partial \lambda}{\partial t}$ from V without changing $\vec{B}, \vec{E}$. L1p5

Such changes are called Gauge Transformations.

Exploit to simplify equations.

In magnetostatics, often choose $\vec{\nabla} \cdot \vec{A} = 0$.

In Electrodynamics, the most convenient gauge depends on problem at hand.

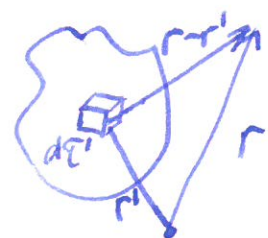
Coulomb Gauge

Pick $\vec{\nabla} \cdot \vec{A} = 0$.

Eqⁿ (b) then becomes

$$\nabla^2 V = -\frac{1}{\epsilon_0} \rho.$$

Poisson's Eqⁿ. Setting $V=0$ at ∞ , we have



$$V(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(r', t)}{|\vec{r} - \vec{r}'|} d\tau'$$

However: Still need to find $\vec{A}$, to get $\vec{E}$! (and of course $\vec{B}$).

Note: only one time t : V everywhere depends on charge distribution at the same time.

Change this, and V changes everywhere instantaneously. NOT in contr. with SR!

Only $\vec{E}/\vec{B}$ are measurable, and somehow it is built into $\vec{A}$ to correct for the instant change in V s.t. signal ^{of change} only arrives at speed c .

Coulomb Gauge, cont^d

L1, p6

Advantage: Easy V .

Disadv. : Hard $\vec{A}$.

(d) In Coulomb gauge reads:

$$\nabla^2 \vec{A} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \nabla \left(\frac{\partial V}{\partial t} \right).$$

Lorentz Gauge

$$\text{Pick } \vec{\nabla} \cdot \vec{A} = -\mu_0 \epsilon_0 \frac{\partial V}{\partial t}$$

to eliminate the middle term in (d).

This gives for (d):

$$\nabla^2 \vec{A} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}.$$

(c) becomes

$$\nabla^2 V - \mu_0 \epsilon_0 \frac{\partial^2 V}{\partial t^2} = -\frac{1}{\epsilon_0} \rho.$$

Look very similar!

write/define $\nabla^2 - \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} \equiv \square^2$ (d'Alembertian).

Then $\square^2 V = -\frac{1}{\epsilon_0} \rho$ $\square^2$ acts on each component independently.

$$\square^2 \vec{A} = -\mu_0 \vec{J}.$$

could even write

Nice for treatment in SR! $\square^2 (V, \vec{A}) = -(\frac{1}{\epsilon_0} \rho, \mu_0 \vec{J})$.

Use from now on.

Lecture 1 Summary.

Maxwell's Equations

$$M1 \quad \vec{\nabla} \cdot \vec{E} = \rho / \epsilon_0$$

$$M3 \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$M2 \quad \vec{\nabla} \cdot \vec{B} = 0$$

$$M4 \quad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$

Want to solve for arbitrary, time-dep. dist of ρ and $\vec{J}$.

Math-box: 1) For vector-field $\vec{F}$ with $\vec{\nabla} \cdot \vec{F} = 0$: Can write
 $\vec{F} = \vec{\nabla} \times \vec{A}$.
2) For vector-field $\vec{F}$ with $\vec{\nabla} \times \vec{F} = 0$: Can write
 $\vec{F} = \vec{\nabla} V$.

Introduce scalar and vector potentials

$$\vec{B}(\vec{r}, t) = \vec{\nabla} \times \vec{A}(\vec{r}, t)$$

$$[\text{used } M2, M3] \quad \vec{E}(\vec{r}, t) = -\vec{\nabla} V(\vec{r}, t) - \frac{\partial \vec{A}(\vec{r}, t)}{\partial t}$$

M1, M4 for the scalar and vector potential:

$$\nabla^2 V + \frac{\partial}{\partial t} (\vec{\nabla} \cdot \vec{A}) = -\frac{1}{\epsilon_0} \rho$$

$$\left(\nabla^2 \vec{A} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} \right) - \vec{\nabla} \left(\vec{\nabla} \cdot \vec{A} + \mu_0 \epsilon_0 \frac{\partial V}{\partial t} \right) = -\mu_0 \vec{J}$$

Does not fix $V, \vec{A}$ uniquely for $\vec{E}, \vec{B}$: Gauge-freedom!

For any (smooth) scalar function λ

$$\vec{A}' = \vec{A} + \vec{\nabla} \lambda, \quad V' = V - \frac{\partial \lambda}{\partial t}$$

(Coulomb: $\vec{\nabla} \cdot \vec{A} = 0$.
Easy $V, \vec{A}$ diff.
Not a bad idea

$(V', \vec{A}')$ gives same $(\vec{E}, \vec{B})$ as $(V, \vec{A})$.

Relativistic Electrodynamics

L2, p1

Solve Maxwell's equations for the potential for several cases.

Continuous Distributions (10.2).

[Using Lorentz Gauge]

$$(i) \square^2 V = -\frac{1}{\epsilon_0} \rho \quad \square^2 \equiv \nabla^2 - \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2}$$

$$(ii) \square^2 \vec{A} = -\mu_0 \vec{J}$$

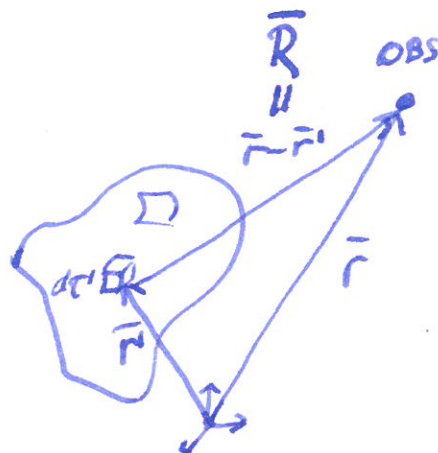
In the static case we find 4 copies of Poisson's Equation

$$\nabla^2 V = -\frac{1}{\epsilon_0} \rho, \quad \nabla^2 \vec{A} = -\mu_0 \vec{J}$$

With solutions

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} d\tau'$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|} d\tau'$$



Now, it would be tempting to suggest that since information travels at speed of light, it is not the state of the source at time t that matters, but at an earlier time

t_r when the imaginary message about the state of the source left. Since the message travels distance

$$|\vec{r} - \vec{r}'|, \text{ we have } t_r \equiv t - \frac{|\vec{r} - \vec{r}'|}{c}$$

Retarded time t_r
varies with position
~~even~~ for fixed t !
Sim. to distant star obs.

The natural generalisation from the static case would then be

$$V(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\vec{r}', t_r)}{R} d\tau' \quad \bar{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}', t_r)}{R} d\tau'$$

$\rho(\vec{r}', t_r)$: Charge density at point $\vec{r}'$ at the time t_r (pos. dep.).

Reasonable - but right? Test if this Ansatz fulfills MEqn for the potentials!

Pick V (or A_x, A_y, A_z). Show/test $\nabla^2 V - \mu_0\epsilon_0 \frac{\partial^2 V}{\partial t^2} = -\frac{1}{\epsilon_0} \rho$.

Difficulty: V depends on r through R , and through t_r !

So we start:

$$\bar{\nabla} V = \frac{1}{4\pi\epsilon_0} \int \left[(\bar{\nabla} \rho) \frac{1}{R} + \rho \bar{\nabla} \left(\frac{1}{R} \right) \right] d\tau'$$

wrt r By the chain rule we find

$$\bar{\nabla} \rho = \frac{\partial \rho}{\partial t} \bar{\nabla} t_r = -\frac{1}{c} \frac{\partial \rho}{\partial t} \bar{\nabla} R \quad \text{since } \frac{\partial}{\partial t} = \frac{\partial}{\partial t_r} \left[\bar{R} \text{ is indep of } t \right]$$

Math-box: $\bar{\nabla} R = \hat{\bar{R}}, \quad \bar{\nabla} \left(\frac{1}{R} \right) = -\frac{\hat{\bar{R}}}{R^2}$ (unit vector in dir. $\bar{R}$).

$$\text{So } \bar{\nabla} V = \frac{1}{4\pi\epsilon_0} \int \left[-\frac{\dot{\rho}}{c} \frac{\hat{\bar{R}}}{R} - \rho \frac{\hat{\bar{R}}}{R^2} \right] d\tau' \quad (a)$$

We now calculate the divergence of this ($\vec{\nabla} \cdot \vec{\nabla} V = \nabla^2 V$)

$$\nabla^2 V = \frac{1}{4\pi\epsilon_0} \int \left\{ -\frac{1}{c} \left[\frac{\hat{\mathbf{R}}}{R} \cdot (\vec{\nabla} \dot{\rho}) + \dot{\rho} \vec{\nabla} \cdot \left(\frac{\hat{\mathbf{R}}}{R} \right) \right] \right. \\ \left. - \left[\frac{\hat{\mathbf{R}}}{R^2} \cdot (\vec{\nabla} \rho) + \rho \vec{\nabla} \cdot \left(\frac{\hat{\mathbf{R}}}{R^2} \right) \right] \right\} d\tau'$$

But $\nabla \dot{\rho} = -\frac{1}{c} \ddot{\rho} \vec{\nabla} R = -\frac{1}{c} \ddot{\rho} \hat{\mathbf{R}}$.

$$\vec{\nabla} \cdot \left(\frac{\hat{\mathbf{R}}}{R} \right) = \frac{1}{R^2}, \quad \vec{\nabla} \cdot \left(\frac{\hat{\mathbf{R}}}{R^2} \right) = 4\pi \delta^3(\vec{R})$$

so

$$\nabla^2 V = \frac{1}{4\pi\epsilon_0} \int \left[\frac{1}{c^2} \frac{\ddot{\rho}}{R} - 4\pi \rho \delta^3(\vec{R}) \right] d\tau' =$$

$$= \frac{1}{c^2} \frac{\partial^2 V}{\partial t^2} - \frac{1}{\epsilon_0} \rho(\vec{r}, t)$$

Problem 10.8 Example Class shows the ret. potentials fulfill the Lorentz Gauge cond.

Now can ~~use~~ derive $\vec{E}, \vec{B}$

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}, \quad \vec{B} = \vec{\nabla} \times \vec{A}.$$

↑ this already calc.

$$\frac{\partial \vec{A}}{\partial t} = \frac{\mu_0}{4\pi} \int \frac{\dot{\mathbf{j}}}{R} d\tau'$$

$$A(\vec{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{j}(\vec{r}', t_r)}{R} d\tau'$$

$$V(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}', t_r)}{R} d\tau'$$

Putting together (a) and this, and using $c^2 = \frac{1}{\mu_0 \epsilon_0}$ L2, p4
 we have

$$(b) \quad \boxed{\vec{E}(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int \left[\frac{\rho(\vec{r}', t_r)}{R^2} \hat{R} + \frac{\dot{\rho}(\vec{r}', t_r)}{cR} \hat{R} - \frac{\ddot{\rho}(\vec{r}', t_r)}{c^2 R} \right] d\vec{r}'}$$

Similarly to
~~A bit more complicated to~~ get eqn. for $\vec{B}$:

$$(c) \quad \boxed{\vec{B}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int \left[\frac{\vec{J}(\vec{r}', t_r)}{R^2} + \frac{\dot{\vec{J}}(\vec{r}', t_r)}{cR} \right] \times \hat{R} d\vec{r}'}$$

Note: Jefimenko's Eqn.

1) (b) and (c) are not just the solution from the static case, with the retarded time inserted!
 This worked for Potentials, but would not give the right solⁿ if performed on the fields themselves.

2). Actually, a perfectly (?) valid solⁿ. to Max. Eqn. for pot. would also have been obtained by using advanced time $t_a \equiv t + \frac{R}{c}$.

But violates causality. Disregard for physics.
 Both solⁿ. arise basically because ∇^2 ~~involves~~ involves $\frac{\partial^2}{\partial t^2}$ and therefore does not care about the direction of time.

Summary of previous lecture 10.2

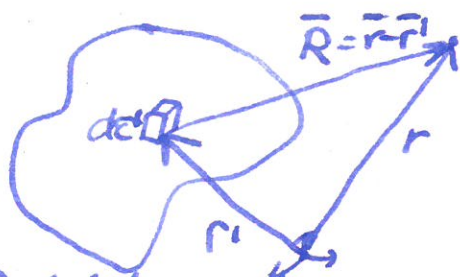
L3, P1

Lorentz Gauge: $\vec{\nabla} \cdot \vec{A} = -\mu_0 \epsilon_0 \frac{\partial V}{\partial t}$

so A1: $\nabla^2 V - \mu_0 \epsilon_0 \frac{\partial^2 V}{\partial t^2} = -\frac{1}{\epsilon_0} \rho$

A4: $\nabla^2 \vec{A} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}$

Find $V(\vec{r}), \vec{A}(\vec{r})$ for continuous, time-dependent charges and currents.



$$t_r = t - \frac{R}{c}$$

Postulate:

$$V(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}', t_r)}{R} d\tau' \quad \vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}', t_r)}{R} d\tau' \quad \textcircled{PL}$$

Proof: Calculate $\nabla^2 V$. First $\nabla V = \frac{1}{4\pi\epsilon_0} \int \left[-\frac{\vec{\rho}}{c} \frac{\hat{R}}{R} - \rho \frac{\hat{R}}{R^2} \right] d\tau' \quad (a)$

$$\nabla^2 V = \dots = \frac{1}{4\pi\epsilon_0} \int \left[\frac{1}{c^2} \frac{\partial^2 \rho(\vec{r}', t_r)}{\partial t^2} - 4\pi \rho(\vec{r}', t_r) \delta^3(\vec{r}) \right] d\tau'$$

$$= \cancel{\frac{\partial^2}{\partial t^2} \left\{ \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}', t_r)}{R} d\tau' \right\}} + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \left\{ \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}', t_r)}{R} d\tau' \right\}$$

$$- \frac{1}{\epsilon_0} \int \rho(\vec{r}', t_r) \delta^3(\vec{r}) d\tau'$$

$$= \frac{1}{c^2} \frac{\partial^2}{\partial t^2} V(\vec{r}, t) - \frac{1}{\epsilon_0} \rho(\vec{r}, t) \quad \square$$

The fields can then be derived from

L3, P2

$$\vec{E} = -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t}, \quad \vec{B} = \vec{\nabla} \times \vec{A}.$$

Again, complicated derivatives because of dependence of $\vec{r}$ through both $\vec{R}$ and the retarded time $t_r = t - \frac{R}{c}$.

Already know $\vec{\nabla}V$.

$$\frac{\partial \vec{A}}{\partial t} = \frac{\partial}{\partial t} \left\{ \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}', t_r)}{R} d\tau \right\} = \frac{\mu_0}{4\pi} \int \frac{\dot{\vec{J}}(\vec{r}', t_r)}{R} d\tau$$

Putting together and using $c^2 = \frac{1}{\mu_0 \epsilon_0}$:

$$\vec{E}(\vec{r}, t) = \frac{1}{4\pi \epsilon_0} \int \left[\frac{\rho(\vec{r}', t_r)}{R^2} \hat{R} + \underbrace{\frac{\dot{\rho}(\vec{r}', t_r)}{cR} \hat{R}}_{\text{ Jefimenko's Eq'n. }} - \frac{\dot{\vec{J}}(\vec{r}', t_r)}{c^2 R} \right] d\tau \quad J1$$

Similarly

$$\vec{B}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int \left[\frac{\vec{J}(\vec{r}', t_r)}{R^2} + \underbrace{\frac{\dot{\vec{J}}(\vec{r}', t_r)}{cR}}_{\text{ Jefimenko's Eq'n. }} \right] \times \hat{R} d\tau \quad J2$$

Note that these are not just the static solutions evaluated at the retarded time, as ~~was~~ the case for the potentials in the Lorentz gauge. ~~un~~: extra terms.

The use of ~~pot~~ potentials allowed us to guess the solⁿ in the dynamic, time-dependent case from the static solⁿ.

Will skip the discussion of the field from moving point charges (10.3) until after we have done some advanced topics of relativity. Much more satisfying discussion than the one in the book by Griffiths.

Radiation from accelerated charges and changing currents L3, p3 Chap 11

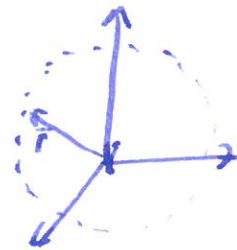
The generation of electromagnetic waves.

Propagate to infinity. Carry energy away from the source.

Consider "only" localised sources.

The total energy passing out through an imaginary spherical shell surrounding the source is given by the surface integral of the Poynting vector

$$P(r) = \oint \vec{S} \cdot d\vec{a} = \frac{1}{\mu_0} \oint (\vec{E} \times \vec{B}) \cdot d\vec{a}$$



The power radiated:

$$P_{\text{rad}} \equiv \lim_{r \rightarrow \infty} P(r).$$

Energy (per unit time) transported out to infinity.

Area of sphere grows like r^2 , so for the limit to be non-zero,

must have the Poynting vector decreasing no faster than $1/r^2$.

Inspect J1 and J2. All the terms from ~~static charges and currents~~ ^{in the fields $\vec{E}, \vec{B}$} situations decrease like $1/r^2$, so $\vec{S} \sim 1/r^4$. Static sources do not radiate!

Find parts of $\vec{E}, \vec{B}$ that decrease at most as $1/r$, construct the terms in the Poynting vector decreasing (at most) as $1/r^2$, integrate over surface, take limit $r \rightarrow \infty$.

Electric Dipole Radiation

Two tiny metal spheres separated by a distance d . Connected with wire.

At time t the charge on upper sphere is $q(t)$. Lower sphere: $-q(t)$.

charge driven back and forth at an angular frequency ω :

$$q(t) = q_0 \cos(\omega t)$$

Oscillating electric dipole. Dipole moment:

$$\vec{p}(t) = p_0 \cos(\omega t) \hat{z}, \quad p_0 = q_0 d.$$

The retarded potential is (PL)

$$V(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \left\{ \frac{q_0 \cos(\omega(t - R_+/c))}{R_+} - \frac{q_0 \cos(\omega(t - R_-/c))}{R_-} \right\}$$

Law of cosines: $R_{\pm} = \sqrt{r^2 \mp rd \cos\theta + (d/2)^2}$

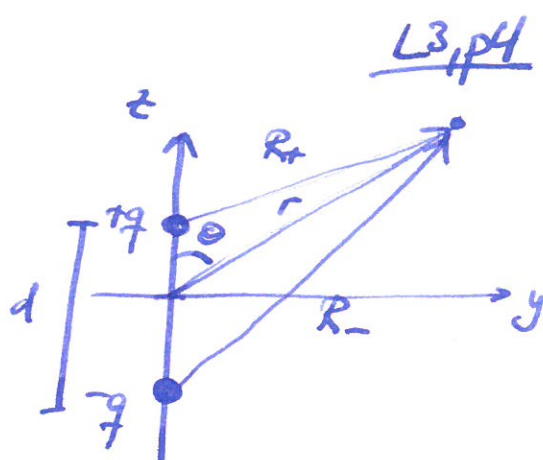
A series of approximations will allow us to calculate the power radiated.

Approx 1: $d \ll r$ (observer "far" from dipole)

Keep terms to first order in d/r . [no rad. to 0th order].

$$R_{\pm} \approx r \left(1 \mp \frac{d}{2r} \cos\theta \right) \quad \text{so}$$

$$\frac{1}{R_{\pm}} \approx \frac{1}{r} \left(1 \pm \frac{d}{2r} \cos\theta \right)$$



$$\text{Also } \cos[\omega(t - R/c)] \approx \cos[\omega(t - r/c) \pm \frac{\omega d}{2c} \cos\theta] \quad \underline{\text{L3, P5}}$$

$$= \cos[\omega(t - r/c)] \cos\left[\frac{\omega d}{2c} \cos\theta\right] \mp \sin[\omega(t - r/c)] \sin\left[\frac{\omega d}{2c} \cos\theta\right].$$

Approx 2:

"Perfect" dipole limit: $d \ll c/\omega$ (or $d \ll \lambda$)

$$\cos[\omega(t - R/c)] \approx \cos[\omega(t - r/c)]$$

$$\mp \frac{\omega d}{2c} \cos\theta \sin[\omega(t - r/c)].$$

$$\boxed{\begin{aligned} \cos\left[\frac{\omega d}{2c} \cos\theta\right] &\approx 1 \\ \sin\left[\frac{\omega d}{2c} \cos\theta\right] &\approx \frac{\omega d}{2c} \cos\theta. \end{aligned}}$$

4 Scalar Potential of oscillating perfect dipole

$$V(r, \theta, t) = \frac{p_0 \cos\theta}{4\pi\epsilon_0 r} \left\{ -\frac{\omega}{c} \sin[\omega(t - r/c)] + \frac{1}{r} \cos[\omega(t - r/c)] \right\}.$$

approx. 3

Radiation zone $r \gg \lambda$ (ie. $r \gg c/\omega$).

$$V(r, \theta, t) = -\frac{p_0 \omega}{4\pi\epsilon_0 c} \left(\frac{\cos\theta}{r} \right) \sin[\omega(t - r/c)].$$

Vector potential determined from the current in the wire

$$\vec{I}(t) = \frac{dq}{dt} \hat{z} = -q_0 \omega \sin(\omega t) \hat{z}.$$

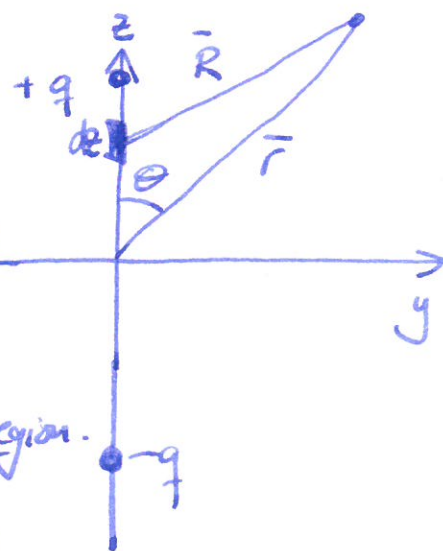
$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int_{-d/2}^{d/2} \frac{-q_0 \omega \sin[\omega(t - r/c)]}{R} \hat{z} dz$$

Use $d \ll r$ and $d \ll \lambda$ (ie. $d \ll c/\omega$)

$1/R$ varies slowly over int. region.

$\sin$ varies slowly on int. region.

Want first order in d/r . Integration introduces dependence on d . Approximate Integral by integrand at $z=0$ times integration length.



$$\vec{A}(r, \theta, t) = -\frac{\mu_0 P_0 \omega}{4\pi r} \sinh[\omega(t-r/c)] \hat{z}.$$

L3, p6

We can then calculate the most significant parts of the fields.

$$\vec{\nabla} V = \frac{\partial V}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial V}{\partial \theta} \hat{\theta}$$

$$= -\frac{P_0 \omega}{4\pi \epsilon_0 c} \left\{ \cos \theta \left(-\frac{1}{r^2} \sinh[\omega(t-r/c)] - \frac{\omega}{rc} \cos[\omega(t-r/c)] \right) \hat{r} - \frac{\sinh \theta}{r^2} \sinh[\omega(t-r/c)] \hat{\theta} \right\}$$

$$\approx \frac{P_0 \omega^2}{4\pi \epsilon_0 c^2} \left(\frac{\cos \theta}{r} \right) \cos(\omega(t-r/c)) \hat{r}$$

$$\frac{\partial A}{\partial t} = -\frac{\mu_0 P_0 \omega^2}{4\pi r} \cos[\omega(t-r/c)] (\cos \theta \hat{r} - \sinh \theta \hat{\theta})$$

$$\text{so } [\epsilon_0 c^2 = 1/\mu_0]$$

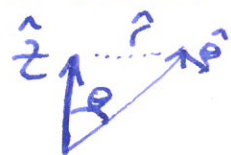
$$\vec{E} = -\vec{\nabla} V - \frac{\partial A}{\partial t} = -\frac{\mu_0 P_0 \omega^2}{4\pi} \left(\frac{\sinh \theta}{r} \right) \cos[\omega(t-r/c)] \hat{\theta}.$$

[notice how the $\vec{E}$ is generated from $\frac{\partial \vec{A}}{\partial t}$ in this gauge].

$\vec{\nabla} \times \vec{A}$ use spherical coordinates despite $\hat{z}$ direction, in order to be able to take zero form Poynting vector.

No ϕ dependence $\rightarrow$ simple curl in spherical coord.

$$\vec{\nabla} \times \vec{A} = \frac{1}{r} \left[\frac{\partial}{\partial r} (r A_\theta) - \frac{\partial A_r}{\partial \theta} \right] \hat{\phi}$$



$$A_\theta = A_z \sinh \theta$$

$$A_r = A_z \cos \theta$$

$$= -\frac{\mu_0 P_0 \omega}{4\pi r} \left\{ \frac{\omega}{c} \sinh \theta \cos[\omega(t-r/c)] + \frac{\sinh \theta}{r} \sinh[\omega(t-r/c)] \right\} \hat{\phi}$$

Appr. 3 $r \gg \omega/c$ (ie $r \gg \lambda$)

3pt

$$\vec{B} = \vec{V} \times \vec{A} = - \frac{\mu_0 p_0 \omega^2}{4\pi c} \left(\frac{\sin \theta}{r} \right) \cos[\omega(t - r/c)] \hat{\phi}.$$

Monochromatic (one ω) waves, radial direction, speed of light. $\vec{E}$ and $\vec{B}$ are in phase, mutually perpendicular. Transverse to direction of propagation. $E/B_0 = c$.

Energy radiated:

$$S = \frac{1}{\mu_0} (\vec{E} \times \vec{B}) = \frac{\mu_0}{c} \left\{ \frac{p_0 \omega^2}{4\pi} \left(\frac{\sin \theta}{r} \right) \cos[\omega(t - r/c)] \right\}^2 \hat{r}$$

Average over time cycle: $\langle \cos^2[\omega(t - r/c)] \rangle = 1/2$

$$\langle S \rangle = \left(\frac{\mu_0 p_0^2 \omega^4}{32 \pi^2 c} \right) \frac{\sin^2 \theta}{r^2} \hat{r}.$$

No radiation along axis of dipole. ($\theta = 0$ or π).

Donut shape intensity profile.

Total power is found by integrating over sphere

$$\langle P \rangle = \int \langle S \rangle \cdot d\vec{a} = \frac{\mu_0 p_0^2 \omega^4}{32 \pi^2 c} \int \frac{\sin^2 \theta}{r^2} r^2 \sin \theta d\theta d\phi$$

$$= \frac{\mu_0 p_0^2 \omega^4}{8 \pi c} \quad [\text{Indep of } r].$$

as expected.
(once $r \gg \omega$ used).

Summary of last lecture

Electric dipole radiation (11.1)

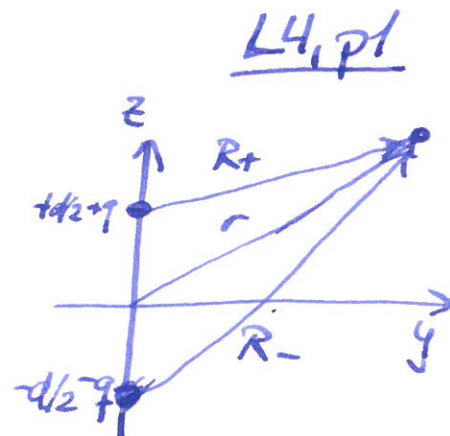
$q(t) = q_0 \cos(\omega t)$. Find the power radiated at large distances.
Calc. Poynting, $\int \vec{S} \cdot d\vec{a}$, $\lim_{r \rightarrow \infty}$.

Approx 1: $d \ll r$ (keep first term in d/r) | stems of $1/r^2$.

Approx 2: $d \ll \lambda$ ($d \ll c/\omega$) "perfect dipole"

Approx 3: $r \gg \lambda$ ($r \gg c/\omega$) "radiation zone" | $P_0 = q_0 d$

$$V(r, \theta, t) = -\frac{P_0 \omega}{4\pi \epsilon_0 c} \left(\frac{\cos \theta}{r} \right) \sin[\omega(t - r/c)]$$



Then calculate the Vector potential as on L3, p5.-p7.

Magnetic Dipole Radiation

L4, p2

Wire loop of radius b with alternating current

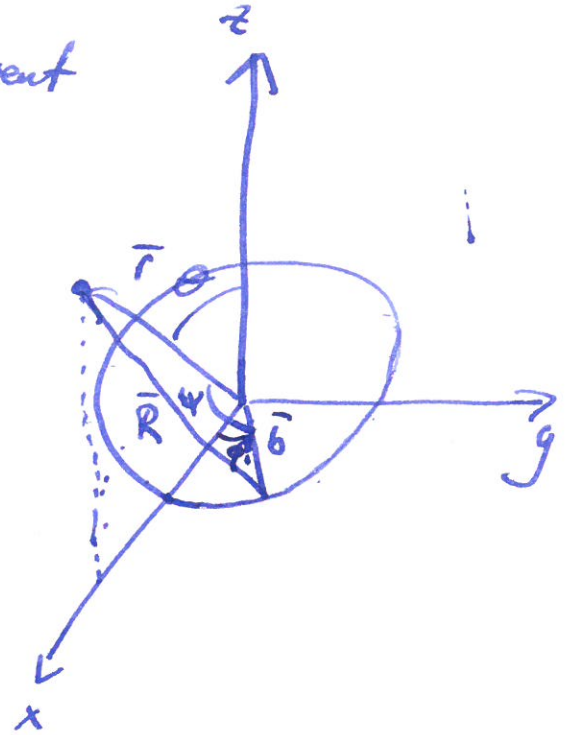
$$I(t) = I_0 \cos(\omega t)$$

Oscillating magnetic dipole

$$\vec{m}(t) = \pi b^2 I(t) \hat{z} = m_0 \cos \omega t \hat{z}$$

$$m_0 = \pi b^2 I_0$$

magnetic dipole moment.



Loop is uncharged, so the scalar potential V is zero!

Retarded vector potential is $\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}', t_r)}{R} d\vec{r}'$

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int \frac{I_0 \cos[\omega(t - R/c)]}{R} d\vec{l}'$$

In cylindrical coordinates / plane polar.

For a point (r, ϕ)

$\vec{A}$ must point in $\hat{\phi}$ -direction. No contribution in $\hat{\theta}$ -direction.

For any contribution in $+\hat{r}$, there is an equal and opposite contribution in $-\hat{r}$.
projection of $\hat{r}$ on $\hat{\phi}$.

$$\vec{A}(\vec{r}, t) = \frac{\mu_0 I_0 b}{4\pi} \hat{\phi} \int_0^{2\pi} \frac{\cos[\omega(t - R/c)]}{R} \cos \phi' d\phi' \quad \textcircled{a}$$

Law of cosines $R = \sqrt{r^2 + b^2 - 2rb \cos \psi}$

$$\vec{r} = r \sin \theta \hat{x} + r \cos \theta \hat{z} \quad \vec{b} = b \cos \phi' \hat{x} + b \sin \phi' \hat{y}$$

So $rb \cos \Psi = \vec{r} \cdot \vec{b} = r b \sin \theta \cos \phi'$ and so

$$R = \sqrt{r^2 + b^2 - 2rb \sin \theta \cos \phi'}$$

Approx 1: $b \ll r$

$$R \approx r \left(1 - \frac{b}{r} \sin \theta \cos \phi' \right)$$

$$\text{so } \frac{1}{R} \approx \frac{1}{r} \left(1 + \frac{b}{r} \sin \theta \cos \phi' \right) \quad \textcircled{b}$$

$$\text{And } \cos[\omega(t - R/c)] \approx \cos\left[\omega(t - R/c) + \frac{\omega b}{c} \sin \theta \cos \phi'\right]$$

$$= \cos[\omega(t - R/c)] \cos\left(\frac{\omega b}{c} \sin \theta \cos \phi'\right) - \sin[\omega(t - R/c)] \sin\left(\frac{\omega b}{c} \sin \theta \cos \phi'\right)$$

Again, doesn't really appear to help much on its own

Approx 2: $b \ll \frac{c}{\omega}$ (or $b \ll \lambda$)

In that case

$$= \cos[\omega(t - R/c)] - \frac{\omega b}{c} \sin \theta \cos \phi' \sin[\omega(t - R/c)] \quad \textcircled{c}$$

Insert into @

$$\bar{A}(\vec{r}, t) \approx \frac{\mu_0 I_0 a b}{4\pi r} \hat{\phi} \int_0^{2\pi} \left\{ \cos[\omega(t - R/c)] \right.$$

$$\left. + b \sin \theta \cos \phi' \left(\frac{b}{r} \cos[\omega(t - R/c)] - \frac{\omega}{c} \sin[\omega(t - R/c)] \right) \right\} \cos \phi' d\phi'$$

Drop ^{2nd order} ~~cross~~ term b^2 .

The first term integrates to zero $\int_0^{2\pi} \cos \phi' d\phi' = 0$.

L4, p4

2nd term: $\int_0^{2\pi} \cos^2 \phi' d\phi' = \pi$.

We then get

$$A(r, \theta, t) = \frac{\mu_0 m_0}{4\pi} \left(\frac{\sin \theta}{r} \right) \left\{ \frac{1}{r} \cos[\omega(t - r/c)] - \frac{\omega}{c} \sin[\omega(t - r/c)] \right\} \hat{\phi}$$

[static limit $\omega=0$ reduces to familiar result]

Approx 3
Radiation zone $r \gg \frac{c}{\omega}$ ($r \gg \lambda$)

First term suppressed, and get

$$\bar{A}(r, \theta, t) = - \frac{\mu_0 m_0 \omega}{4\pi c} \left(\frac{\sin \theta}{r} \right) \sin[\omega(t - r/c)] \hat{\phi}$$

can now derive the physical fields

$$\bar{E} = -\nabla V - \frac{\partial \bar{A}}{\partial t} = -\frac{\partial \bar{A}}{\partial t} = \frac{\mu_0 m_0 \omega^2}{4\pi c} \left(\frac{\sin \theta}{r} \right) \cos[\omega(t - r/c)] \hat{\phi}$$

$$\bar{B} = \nabla \times \bar{A} = - \frac{\mu_0 m_0 \omega^2}{4\pi c^2} \left(\frac{\sin \theta}{r} \right) \cos[\omega(t - r/c)] \hat{\theta}$$

Use spherical form of curl. $\bar{A}$ has only A_ϕ component.

$$\bar{\nabla} \times \bar{A} = \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A_\phi) \hat{r} - \frac{1}{r} \frac{\partial}{\partial r} (r A_\phi) \hat{\theta}$$

1/r suppressed.

Perpendicular, transverse to the direction of propagation, Ratio of amplitudes $E_0/B_0 = c$. Very similar to electric dipoles but now $\bar{B} \sim \hat{\theta}, \bar{E} \sim \hat{\phi}$.

$$\vec{S} = \frac{1}{\mu_0} (\vec{E} \times \vec{B}) = \frac{\mu_0}{c} \left\{ \frac{m_0 \omega^2}{4\pi c} \left(\frac{\sin \theta}{r} \right) \cos[\omega(t - r/c)] \right\}^2 \hat{r} \quad \underline{4.4.5}$$

$$\text{Intensity } \langle \vec{S} \rangle = \left(\frac{\mu_0 m_0^2 \omega^4}{32 \pi^2 c^3} \right) \frac{\sin^2 \theta}{r^2} \hat{r}.$$

Total radiated power

$$\begin{aligned} \langle P \rangle &= \int \langle S \rangle \cdot d\vec{a} = \frac{\mu_0 m_0^2 \omega^4}{32 \pi^2 c^3} \int \frac{\sin^2 \theta}{r^2} r^2 \sin \theta d\theta d\phi \\ &= \frac{\mu_0 m_0 \omega^4}{12 \pi c^3}. \end{aligned}$$

Power radiated by electric and magnetic dipoles of similar size

$$\frac{P_{\text{mag}}}{P_{\text{el}}} = \left(\frac{m_0}{q_0 c} \right)^2.$$

$$m_0 = \pi b^2 I_0, \quad q_0 = q_0 d, \quad I_0 = q_0 \omega \quad (\text{current in electric case})$$

set $d = \pi b$ for sake of comparison we get

$$\frac{P_{\text{mag}}}{P_{\text{el}}} = \left(\frac{\omega b}{c} \right)^2$$

small by appr. 2, and then squared!

End of EM for a while.

Now: Relativity. Chapter 12.1 next time.

Relativistic Electrodynamics

Special Relativity

LS, p1

The Maxwell equations predict electromagnetic waves which propagate at the fixed speed (in vacuum)

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

speed of light. μ_0 permeability. ϵ_0 permittivity.

But relative to what? Big question at the end of 1800s. EM predicts the speed of light to be the same indep. of the frame of the observer or the speed of the emitter. EM not invariant under Galilean transformations, simple addition of velocities.

However, EM gives remarkably consistent answers:

As loop rides through magnetic field a emf (electromotive force) is generated

Flux rule: $\mathcal{E} = -\frac{d\Phi}{dt}$ due to magnetic force on the charges in the wire loop.

Consider instead stationary wire loop

No magnetic force, since loop at rest. But since the magnet is moving, for all points $\frac{\partial \vec{B}}{\partial t} \neq 0$, and an electric field is generated (M3) $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

The electric force generates an emf of $\mathcal{E} = \frac{\partial \Phi}{\partial t}$

(Curl Theorem) $\oint \vec{E} \cdot d\vec{\ell} = \int (\vec{\nabla} \times \vec{E}) \cdot d\vec{a}$

Maxwell's EM fully consistent, ~~but~~ violates Galilean idea for transformations between moving frames.

Einstein: reject Galilean transformations and find new rules based on two postulates:

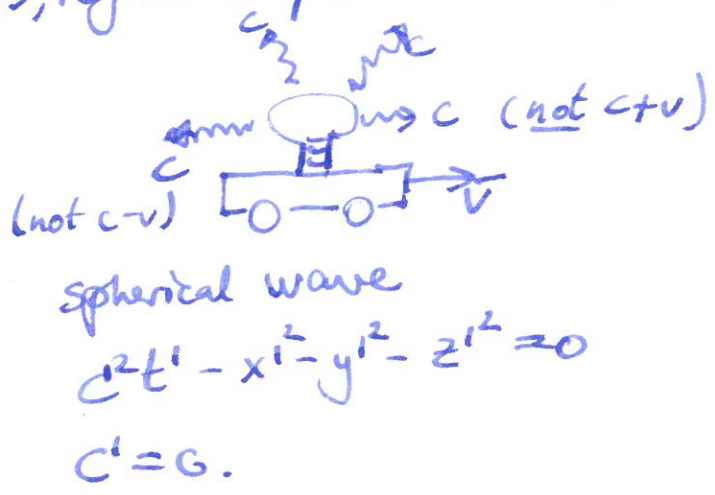
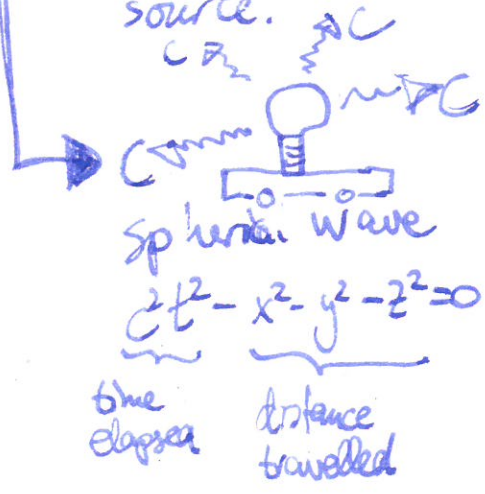
1. The principle of relativity

The laws of physics ^{are the same (have the same form)} apply in all inertial reference systems. (Inertial Frames) The results of exp are the same in all IF.

Inertial frames: Non-accelerated. (= special Rel. ^{accel frames → general rel})
Objects under influence of no forces move at constant velocity.

2. Universal speed of light

The speed of light in vacuum is the same for all inertial observers, regardless of the motion of the source.



Gives rise to seemingly surprising phenomena that you L5, p3
have already studied:

i) Relativity of simultaneity: Two events that are simultaneous in one ~~frame~~ ^{IF} are not in general simult. in another ~~frame~~ ^{IF}.

ii) Time Dilation: Moving clocks run slow

iii) Lorentz Contractions: Moving objects are shortened

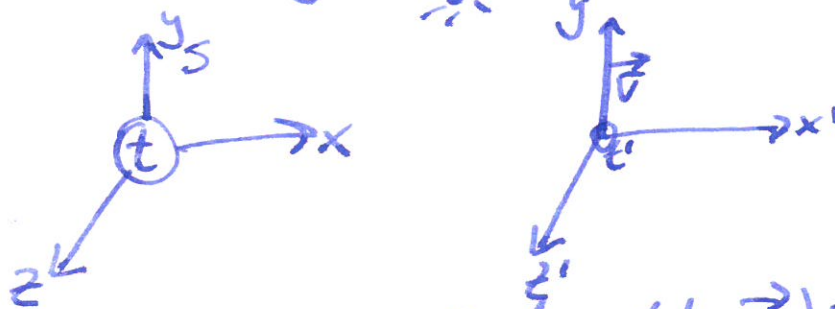
Will develop efficient formalism for the study of physics obeying the laws of SR.

Consider first Lorentz Transformations

An "event": An occurrence which can be ^adescribed a single set of coordinates $\vec{x}$ and time t in a given IF.
Eg. flash of light.

Consider two IF, S & S' in the standard configuration
(S' boosted along x -axis in S)

at $t=t'=0$ the two IF coincide.



An event has coordinates $(t, \vec{x}) = (t, x, y, z)$ in S
 $(t', \vec{x}') = (t', x', y', z')$ in S' .

We want to know the coordinates in S' as a function of those in S , i.e.

$$t'(t, x, y, z)$$

$$x'(t, x, y, z)$$

$$y'(t, x, y, z)$$

$$z'(t, x, y, z)$$

Derived at Level 1. Result:

Lorentz txf. for S & S' in standard. conf (boost along x -axis).

$$ct' = \gamma ct - \gamma \beta x$$

$$x' = -\gamma \beta ct + \gamma x$$

$$y' = y$$

$$z' = z$$

Derived from req. of lin. transf. and 2nd. postulate.

Notation: $\vec{\beta} = \vec{v}/c$, $\beta = |\beta| = v/c = |\vec{v}|/c$

$$\gamma = \gamma(v) = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - \beta^2}}$$

Inverse transf. can be found by letting $\vec{v} \rightarrow -\vec{v}$, i.e. $\vec{\beta} \rightarrow -\vec{\beta}$.

Check that

$$c^2 t^2 - x^2 - y^2 - z^2 = c^2 t'^2 - x'^2 - y'^2 - z'^2$$

i.e. $c^2 t^2 - x^2 - y^2 - z^2$ is invariant under LT. (boost)
(2nd postulate).

Def: LT: Any linear transformation that leaves $c^2 t^2 - x^2 - y^2 - z^2$ invariant.

Note: Rotations are a special LT (since they leave the length of the vector $\vec{x}$ invariant, and therefore $-x^2 - y^2 - z^2$ is invariant. And t does not change under a rotation.

$$\begin{aligned}
 c^2 t'^2 - x'^2 &= \gamma^2 c^2 t^2 - 2\gamma^2 c\beta x t + \gamma^2 \beta^2 x^2 \\
 &\quad - \gamma^2 \beta^2 c^2 t^2 + 2\gamma^2 c\beta x t - \gamma^2 x^2 \\
 &= c^2 t^2 \underbrace{\gamma^2(1-\beta^2)}_1 - x^2 \underbrace{\gamma^2(1-\beta^2)}_1 = ct^2 - x^2.
 \end{aligned}$$

Vectors and 4-Vectors

Consider "normal" Euclidean space (3 Dim, no time)

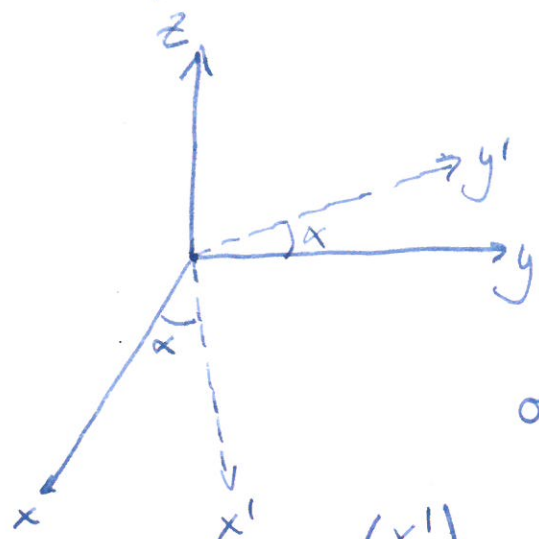
Fix Coordinate system (CS) K : Point $P \rightarrow$ Coordinates $\bar{x}(x, y, z)$

Make transformation (rotation, translation, ...), i.e. pick another CS: K' .

Same point $P \rightarrow$ different coordinates $\bar{x}' = (x', y', z')$.

Define coordinate transformation by giving $\bar{x}'$ as a function of $\bar{x}$ (or vice versa).

Example: Rotation around z -axis



$$\bar{x}'(\bar{x}) \begin{cases} x'(x, y, z) = \cos \alpha x + \sin \alpha y + 0 \cdot z \\ y'(x, y, z) = -\sin \alpha x + \cos \alpha y + 0 \cdot z \\ z'(x, y, z) = 0 \cdot x + 0 \cdot y + z \end{cases}$$

or:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$\uparrow$ Vector $\bar{x}'$ = matrix R
 $\uparrow$ Vector $\bar{x}$
matrix multiplication

In general:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \partial x'/\partial x & \partial x'/\partial y & \partial x'/\partial z \\ \partial y'/\partial x & \partial y'/\partial y & \partial y'/\partial z \\ \partial z'/\partial x & \partial z'/\partial y & \partial z'/\partial z \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

L5, p6

$$R = \frac{\partial \bar{x}'}{\partial \bar{x}}$$

Rotations mix ~~$\{x, y, z\}$~~ $\{x, y, z\}$ coordinates

Lorentz transformations mix $\{t, x, y, z\}$.

↳ 4-vector (ct, x, y, z)

↑
multiply by c to have
same ~~coordinates~~ units for
all coordinates.

The Lorentz boost from before can be written

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \partial ct'/\partial ct & \partial ct'/\partial x & \partial ct'/\partial y & \partial ct'/\partial z \\ \partial x'/\partial ct & \partial x'/\partial x & \partial x'/\partial y & \partial x'/\partial z \\ \partial y'/\partial ct & \partial y'/\partial x & \partial y'/\partial y & \partial y'/\partial z \\ \partial z'/\partial ct & \partial z'/\partial x & \partial z'/\partial y & \partial z'/\partial z \end{pmatrix} \cdot \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

For boost along x -axis Λ

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Define boost parameter Ψ
 $\Psi = \text{ArcTanh } \beta \therefore \cosh \Psi = \gamma$
 $\sinh \Psi = \gamma\beta$

$$= \begin{pmatrix} \cosh \Psi & -\sinh \Psi & 0 & 0 \\ -\sinh \Psi & \cosh \Psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Compare with standard
rotations!

Note: A rotation (around eg. z-axis) is also a LT.

LS, PT

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & +\sin\alpha & 0 \\ 0 & -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

An arbitrary (general) LT needs 6 parameters
3 rotation angles and 3 boost parameters.

A single boost along $\vec{n}$: $R^{-1} \cdot \Lambda \cdot R$

4 parameters
($\vec{n}$: 3 + velocity).

rotate back. boost along x-axis rotate s.t. new x-axis along $\vec{n}$

Relativistic EM

L7.p1

Summary of previous lecture:

Develop efficient formalism for expressing the laws of physics in accordance with the relativity principle.

Maxwell's eqn obey the rel. principle; but not straightforward to see this in their standard notation.

4-vector

$$x^\mu = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \stackrel{\text{obj not.}}{=} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

Lorentz T:

$\Lambda^\mu{}_\nu \equiv \frac{\partial x'^\mu}{\partial x^\nu}$ represented by 4x4 matrix.

$$x'^\mu = \sum \Lambda^\mu{}_\nu x^\nu \equiv \Lambda^\mu{}_\nu x^\nu$$

$\Lambda^\mu{}_\nu$ boost
p along
x-axis

$$\begin{pmatrix} \cosh \psi & -\sinh \psi & 0 & 0 \\ -\sinh \psi & \cosh \psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Any object that transforms under LT as $\Lambda^\mu{}_\nu$ is called a Contravariant 4-vector.

$$q'^\mu = \Lambda^\mu{}_\nu q^\nu$$

is called a Contravariant 4-vector.

[Forms a vector space]. Minkowski space-time.

Covariant 4-vectors

For $q^\mu = (q^0, q^1, q^2, q^3)$ define

$$q_\mu = (q^0, -q^1, -q^2, -q^3).$$

Introduce the metric $g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ L7, p2

$$g_\mu = g_{\mu\nu} g^\nu$$

Also, $g^\mu = g^{\mu\nu} g_\nu$ ~~is~~

How to raise and lower an index!

Apply to the metric: $g^\mu{}_\nu = g^{\mu\rho} g_{\rho\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \delta^\mu{}_\nu$

Covariant 4-vector transforms as

$$X'_\mu = g_{\mu\nu} X^\nu = g_{\mu\nu} \Lambda^\nu{}_\rho X^\rho = \underbrace{g_{\mu\nu} \Lambda^\nu{}_\rho g^{\rho\sigma}}_{\Lambda_\mu{}^\sigma} X_\sigma$$

$$\equiv \Lambda_\mu{}^\sigma X_\sigma$$

Recall: $\Lambda^\mu{}_\nu = \frac{\partial x'^\mu}{\partial x^\nu}$ now $\Lambda_\mu{}^\sigma = \frac{\partial x'^\sigma}{\partial x_\mu}$ since it transf. Contravariant vectors

Why introduce Covariant 4-vector?

Consider $X^\mu X_\mu = X^0 X_0 + X^1 X_1 + X^2 X_2 + X^3 X_3$
 $= X^0 X^0 - X^1 X^1 - X^2 X^2 - X^3 X^3$
 $(= (ct)^2 - x^2 - y^2 - z^2)$

Minkowski
scalar product.

Lorentz Invariant.

Now, Any Contravariant 4-vector y^μ transforms like x^μ . So also

$y^\mu X_\mu$ is Lorentz Invariant.

$$x^\mu x_\mu = x'^\mu x'_\mu = (\Lambda^\mu_\nu x^\nu) (\Lambda_\mu^\rho x_\rho) \quad \underline{L7, p3}$$

$$= (\Lambda^\mu_\nu \Lambda_\mu^\rho) x^\nu x_\rho$$

$$\text{True } \forall x^\mu \therefore \Lambda^\mu_\nu \Lambda_\mu^\rho = \delta_\nu^\rho (=g_\nu^\rho).$$

$$\therefore \Lambda_\mu^\rho = \text{[scribbled out]}$$

$$\text{Remember } \Lambda^\rho_\mu = \frac{\partial x'^\rho}{\partial x^\mu}$$

$$\text{So } \Lambda^{-1\rho}_\mu = \frac{\partial x^\rho}{\partial x'^\mu} = \Lambda_\mu^\rho \quad (\text{Already knew } \Lambda_\mu^\rho = \frac{\partial x'^\rho}{\partial x^\mu})$$

$$\text{Also } \det(\Lambda) = \pm 1.$$

New result: For any Contravariant vector y^μ , Covariant x_μ :

$y^\mu x_\mu$ is Lorentz invariant.

$$\text{Proof: } y'^\mu x'_\mu = (\Lambda^\mu_\nu y^\nu) (\Lambda_\mu^\sigma x_\sigma)$$

$$= (\Lambda^\mu_\nu \Lambda_\mu^\sigma) y^\nu x_\sigma$$

$$= (\Lambda^\mu_\nu \Lambda^{-1\sigma}_\mu) y^\nu x_\sigma$$

$$= (\delta^\sigma_\nu \Lambda^\mu_\nu) y^\nu x_\sigma$$

$$= y^\nu x_\nu$$

$$\begin{aligned} \text{or: } &= \frac{\partial x'^\mu}{\partial x^\nu} \frac{\partial x^\sigma}{\partial x'^\mu} y^\nu x_\sigma \\ &= \frac{\partial x^\sigma}{\partial x^\nu} y^\nu x_\sigma \\ &= \delta^\sigma_\nu y^\nu x_\sigma \\ &= y^\nu x_\nu. \end{aligned}$$

Also:

L7, p4

$$y^\mu x_\mu = y^\mu (g_{\mu\nu} x^\nu) = (y^\mu g_{\mu\nu}) x^\nu = y_\nu x^\nu$$

So often write $y \cdot x$ or $x \cdot y$. Minkowski scalar product.

Tensors and the relativity principle

The laws of Physics should now be expressed in terms of quantities with the right properties under LT:

Scalar: A single quantity (number) that does not ~~transform~~ ^{change} under LT. Scalar: Tensor of rank 0. 0 indices. 4⁰ entries.

4-vector: an entity of 4 numbers that transforms as

$$\left. \begin{array}{l} \text{Contravariant: } x^\mu \rightarrow x'^\mu = \Lambda^\mu_\nu x^\nu \\ \text{covariant: } x_\mu \rightarrow x'_\mu = \Lambda_\mu^\nu x_\nu \end{array} \right\} \Lambda^\mu_\nu \Lambda_\mu^\rho = \delta_\nu^\rho$$

Tensor of rank 1. 1 index. 4¹ = 4 entries

Tensor of rank 2: 2 indices. 4² = 16 entries.

Contravariant $F^{\mu\nu} \rightarrow F'^{\mu\nu} = \Lambda^\mu_\rho \Lambda^\nu_\sigma F^{\rho\sigma}$

Covariant $F_{\mu\nu} \rightarrow F'_{\mu\nu} = \Lambda_\mu^\rho \Lambda_\nu^\sigma F_{\rho\sigma}$

mixed $F^\mu_\nu \rightarrow F'^\mu_\nu = \Lambda^\mu_\rho \Lambda_\nu^\sigma F^\rho_\sigma$

Note: Count only free indices (i.e. not repeated indices). L7, p5

eg $g^{\mu\nu}$ is a tensor of rank 2 (the metric tensor).

but $g^{\mu}_{\mu} = \delta^{\mu}_{\mu} = 4$ is a tensor of rank 0. (ie scalar).

Reason for this notation:

By writing the laws of physics in terms of (Minkowski) tensors, it is automatically ensured they fulfill rel. principle.

Eg. Maxwell eqn (M1 + M4) $\partial_{\mu} F^{\mu\nu} = j^{\nu}$

$$\begin{aligned}\vec{\nabla} \cdot \vec{E} &= \rho/\epsilon_0 \\ \vec{\nabla} \times \vec{B} &= \mu_0 \vec{j} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}\end{aligned}$$

Summary of last lecture

$$\text{Metric } g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

raises and lowers indices.

Meaning of upper or lower index:

Transforms as contravariant or covariant vector

$$q'^{\mu} = \Lambda^{\mu}_{\nu} q^{\nu}$$

$$q'_{\mu} = \Lambda_{\mu}^{\nu} q_{\nu}$$

Important result: For any contravariant 4-vector y^{μ} ,
Covariant x_{μ} :

$y^{\mu} x_{\mu}$ is Lorentz Invariant.

written as $y \cdot x = x \cdot y$

Tensors and the relativity principle

- Scalar (Tensor of rank 0): 0 indices.
- 4-vector (Tensor of rank 1): 1 index
- Rank 2 tensor: 2 indices. $4^2 = 16$ entries.

$$F^{\mu\nu} \rightarrow F'^{\mu\nu} = \Lambda^{\mu}_{\rho} \Lambda^{\nu}_{\sigma} F^{\rho\sigma}$$

Write the laws of Physics in terms of tensors to ensure they obey the relativity principle.

Basics of Relativity

L7, p6
L8, p2

Consider 2 events x^μ, y^μ (=0 w.o.l.g.).

The (LT-)invariant distance between the two events

$$s^2 \equiv (x-y)^2 = (x-y) \cdot (x-y) = (x-y)^\mu (x-y)_\mu$$

$$\begin{aligned} &= x^\mu x_\mu \\ & \quad y^\mu = (t, 0, 0, 0) \end{aligned}$$

Lorentz-invariant classification of the distance:

$s^2 = 0$: light-like

$s^2 < 0$: space-like (The space contribution to the Mink. prod. from the spat. components is biggest).

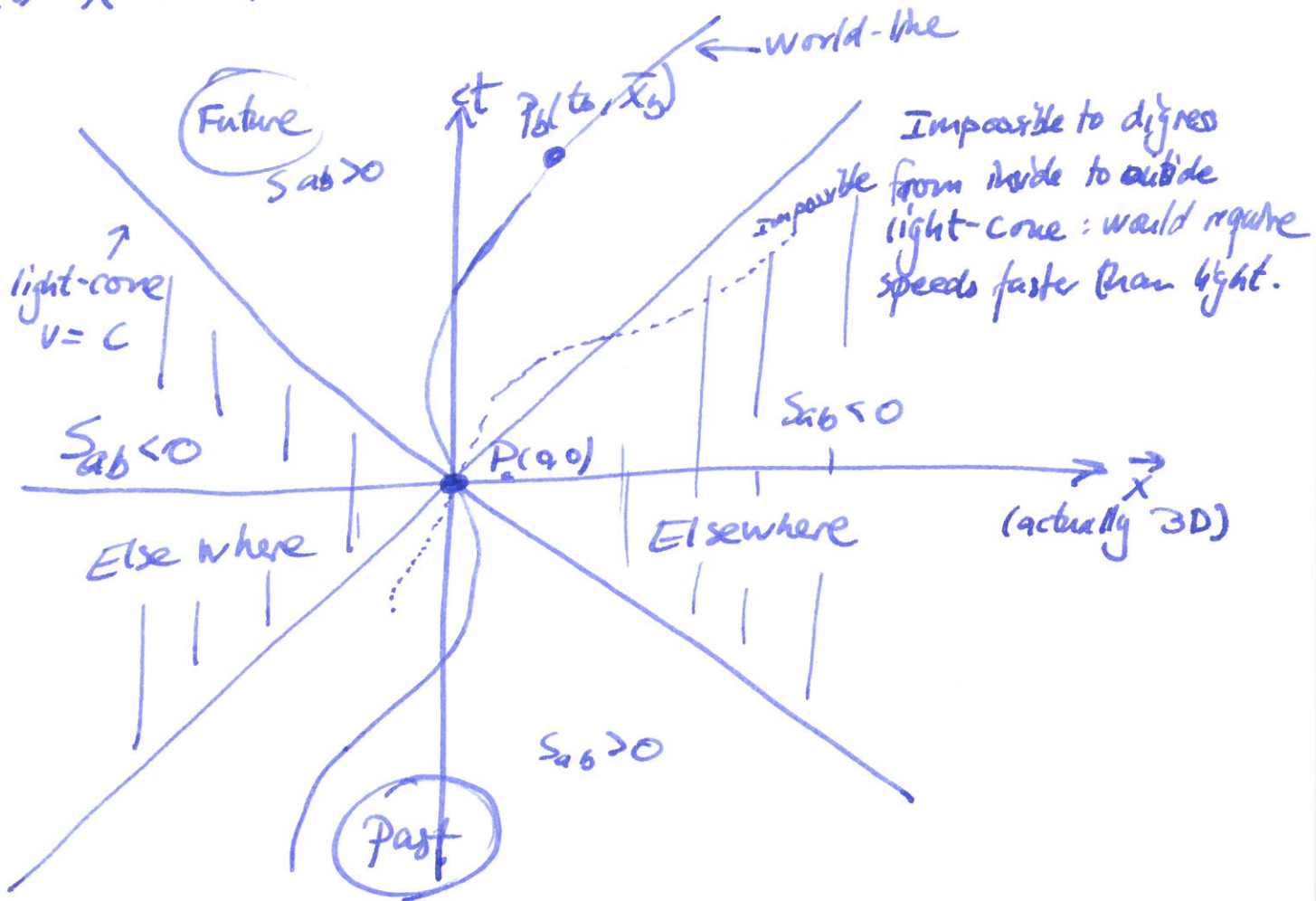
There exists a frame s' s.t. the events happen at the same time but at different positions.

$s^2 > 0$: Time-like (the contri. from the 0th component is largest).

There exists a frame s'' s.t. the events happen at the same point but at different times.

Consider a particle in a certain IF, where it has $\vec{x}=0$ at $t=0$.

L7, p7
L8, p3



Future: These events occur after $P(0,0)$ for all observers. Only these points in phase-space can be influenced by P_a (Causal connections)

Past: These events occur before $P(0,0)$ for all observers. Only these points can influence / have a causal effect on P_a

Elsewhere: For some obs., these happen before $P(0,0)$, for some after. There can be no causal connection with $P_a(0,0)$.

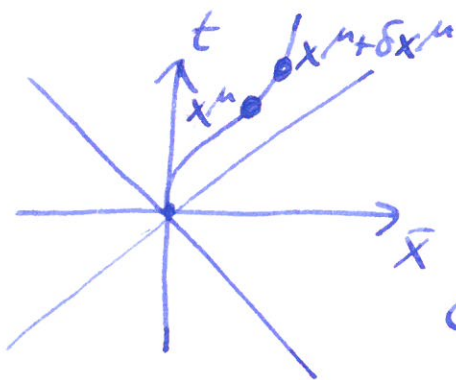
Proper time τ and time dilation

L8, p4

Consider a particle with a certain world line. The

Proper Time τ is defined as the time as measured by a clock moving with the particle.

Let the particle have an instantaneous velocity $\bar{v}(t)$ wrt. some IF S . In the time interval dt , its position changes $d\bar{x} = \bar{v}(t) dt$.



$$ds^2 = \delta x^\mu \delta x_\mu$$

$$ds^2 \text{ in } S = ds^2 \text{ in particle frame}$$

$$c^2 dt^2 - d\bar{x}^2 = c^2 d\tau^2$$

$$\therefore c^2 dt^2 - v^2 dt^2 = c^2 d\tau^2$$

$$\therefore d\tau = \sqrt{1 - v^2/c^2} dt = \frac{dt}{\gamma(v)}.$$

After a time difference $t_2 - t_1$ in S , the proper time difference

$$\text{is } \Delta\tau = \int_{t_1}^{t_2} d\tau = \int_{t_1}^{t_2} \frac{dt}{\gamma(v)} = \int_{t_1}^{t_2} \frac{dt}{\gamma(v(t))}.$$

$$\text{Note } \gamma(v) > 1 \therefore \Delta\tau < \Delta t$$

$\therefore$ A moving clock runs more slowly than a stationary clock.
Time dilation.

"Twin Paradox".

Strictly cannot treat the setup within SR since there is no way of bringing twins back together without acceleration. Life time of decaying particles.

Length Contraction

L8, p5

Consider a rod with length $l_0 = \Delta x = |\bar{x}_e - \bar{x}_b|$ at rest in IFS. l_0 is called the proper length of the rod.

What is the length of the rod as measured by an observer moving with velocity v along the direction of the rod (e.g. x -axis).

To measure the length in S' we measure the position of the two endpoints of the rod at the same time in S' (which does not correspond to simultaneous measurements in S !).

[old notation]

$$l_0 = x_e - x_b = \gamma(x'_e + vt'_e) - \gamma(x'_b + vt'_b)$$

$$[t'_e = t'_b] \quad = \gamma(x'_e - x'_b) = \gamma l'$$

l' is the length as measured by the moving observer.

$$l' = \frac{l_0}{\gamma(v)}.$$

$l' < l_0$, thus for the moving observer, the rod is contracted

→ length contraction (moving objects are shortened).

No effect if an observer moves perp. to the direction of the rod.

⇒ Volumes transform as $V' = \frac{V_0}{\gamma(v)}$

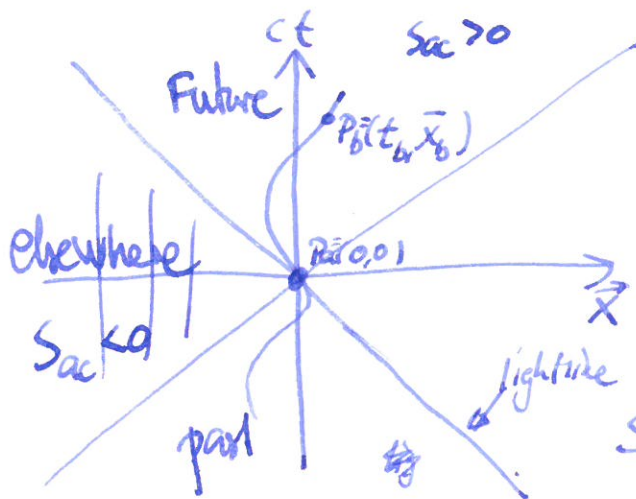
Pole & barn Paradox (~~exercise~~) why not a paradox?

Relativistic EM

L9, p1

Summary of last lecture.

Consider a particle path in a certain IF



The Minkowski product of any two 4-vectors is Lorentz-invariant.

Consider a third point C in space-time. Classify the distance

$$S_{ac} = (x_a - x_c)^2 = (x_a - x_c) \cdot (x_a - x_c)$$

$S_{ac} = 0$: separation is lightlike

$S_{ac} < 0$: separation is spacelike. No causal connection.

Time-ordering frame-dependent. $\exists S' : t'_a = t'_c$

$S_{ac} > 0$: Time-like separation. Time-ordering kept under LT.

Causal connection possible.

Proper time: Time measured by co-moving clock.

$$\Delta\tau = \int_{t_1}^{t_2} \frac{dt}{\gamma(v)}$$

A moving clock runs slow

Length Contraction:

$$l' = \frac{l^0}{\gamma(v)}$$

l^0 : length measured in rest-frame. [measurement of the ends of rod: spacelike separation]

rod contracted as measured by moving observer.

Volumes also transform as $V' = \frac{V_0}{\gamma(v)}$.

4-velocity

So far know of only two types of four-vectors: x^μ , $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$.

Now, define $u^\mu = \frac{dx^\mu}{d\tau}$ $\leftarrow$ proper time

[note $\frac{dx^\mu}{dt}$ not a 4-vector]

dx^μ : 4-vector

$d\tau$ scalar

dt transforms as 0th component of 4-vector

Relation to 3-velocity

$$u^\mu \equiv \frac{dx^\mu}{d\tau} = \frac{dx^\mu}{dt} \frac{dt}{d\tau} = \gamma \frac{dx^\mu}{dt} = \gamma (c, \vec{v})$$

What is the square of the 4-velocity? $u^\mu = \gamma(c, \vec{v})$
 $u_\mu = \gamma(c, -\vec{v})$

$$u^2 = u^\mu u_\mu = \gamma^2 (c^2 - v^2) = c^2 \gamma^2 (1 - v^2/c^2) = c^2$$

$\uparrow$
 L.I. choose frame where $v=0$, $\gamma=1$.

then $u^\mu = (c, 0) \therefore u^\mu u_\mu = c^2$.

General trick when calculating L.I. quantities: Pick frame that makes calc. easy!

[Note that if we had defined the 4-velocity as $\frac{dx^\mu}{dt}$, then the Minkowski scalar $\frac{dx^\mu}{dt} \cdot \frac{dx_\mu}{dt} = c^2 - v^2$, i.e. NOT L.I.]

4-Momentum

12.2

Probably the most important
4-vector!

L9, p3

Define $p^\mu = m u^\mu$

where m is the rest mass of the particle. [sometimes denoted by m_0 , but we will only ever refer to the rest masses, so no need to keep the subscript].

$$p^\mu = m \gamma (c, \vec{v}) = (m \gamma c, m \gamma \vec{v}) \equiv (m \gamma c, m \gamma \vec{v})$$

$m \gamma$: relativistic inertial mass (hardly ever used, since frame dependent).
(velocity -)

The spatial components of the 4-momentum are very similar to the newtonian 3-momentum (but $m \gamma \rightarrow m \gamma = m \cdot \gamma$).

What is the interpretation of the 0th component?

Will see: $\frac{1}{c} \times$ Relativistic Energy.

Consider a particle with rest mass m . Let an unbalanced force $\vec{F}$ act on it. Classically, the kinetic energy of the particle is

$$E_{\text{kin}} = \int \vec{F} \cdot d\vec{x} = \int \frac{d\vec{p}}{dt} \cdot d\vec{x} \stackrel{\text{let moment be in } \vec{x}}{=} \int \frac{d\vec{p}}{dt} \cdot \vec{v} dt$$

We want to keep this relation in S. relativity for the 4-momentum

$$E_{\text{kinetic}} = \int \frac{d}{dt} (m \gamma v) dx. \quad \text{Use } dx = v dt$$

$$= \int \frac{d}{dt} \left(\frac{m v}{\sqrt{1 - v^2/c^2}} \right) v dt$$

$$\text{Now } \frac{dt}{dt} = \frac{d}{dt} \left(\frac{1}{\sqrt{1-v^2/c^2}} \right) = -\frac{1}{2} \left(1 - \frac{v^2}{c^2} \right)^{-3/2} \left(-\frac{2v}{c^2} \right) \frac{dv}{dt} \quad \text{L9, p4}$$

$$= \frac{1}{c^2} \frac{1}{(1-v^2/c^2)^{3/2}} v \frac{dv}{dt}$$

$$\text{So } E_{\text{kinetic}} = \int \left(m \gamma \frac{dv}{dt} + m v \frac{d\gamma}{dt} \right) v dt$$

$$= \int \frac{m}{(1-v^2/c^2)^{3/2}} \frac{dv}{dt} \left((1-v^2/c^2) + v^2/c^2 \right) v dt$$

$$= \int m c^2 \cdot \underbrace{\frac{1}{c^2} v \frac{dv}{dt}}_{\frac{d}{dt}} \frac{1}{(1-v^2/c^2)^{3/2}} dt$$

$$= \int \frac{d}{dt} (m c^2 \gamma) dt \quad \text{if the particle was initially at rest.}$$

and let final velocity be v :

$$E_{\text{kinetic}} = [m c^2 \gamma(v)]_{v=0}^{v=v} = m c^2 (\gamma - 1).$$

Useful check: Consider Newtonian limit $v^2/c^2 \ll 1$.

$$\text{Taylor expand } \gamma(v) = \frac{1}{\sqrt{1-v^2/c^2}} = 1 + \frac{v^2}{2c^2} + O\left(\frac{v^4}{c^4}\right).$$

> for $v^2/c^2 \ll 1$ get

$$E_{\text{kin}} = m c^2 \left(1 + \frac{v^2}{2c^2} - 1 \right) = \frac{1}{2} m v^2$$

as expected.

Define: Rest energy of particle: $E_0 = mc^2$

Total relativistic energy of particle: $E = E_0 + E_{kinetic}$
 $= \gamma mc^2$.

So the 4-momentum is given by

$$p^\mu = \left(\frac{E}{c}, \vec{p} \right)$$

Where E is the total relativistic energy and $\vec{p}$ is the relativistic 3-momentum $\vec{p} = \gamma m \vec{v}$.

Minkowski scalar product:

$$p^2 = p^\mu p_\mu = E^2/c^2 - \vec{p}^2$$

$\parallel \leftarrow$ rest frame

$$mu^\mu mu_\mu = m^2 c^2$$

$$\therefore \boxed{E^2 - (pc)^2 = (mc^2)^2}$$

4-acceleration

L9, p6

Define $a^\mu \equiv \frac{d^2 x^\mu}{d\tau^2} = \frac{du^\mu}{d\tau}$ (is really a 4-vector). L11, p2

Recall $u^\mu = (\gamma c, \gamma \vec{v})$ & $\frac{d}{d\tau} = \gamma \frac{d}{dt} \equiv \gamma \cdot$

$$a^\mu = \frac{du^\mu}{d\tau} = \gamma \cdot \frac{d}{dt} (\gamma c, \gamma \vec{v}) = \gamma \cdot (\dot{\gamma} c, \dot{\gamma} \vec{v} + \gamma \dot{\vec{v}})$$

$$= \gamma \left(\frac{d\gamma}{dt} c, \frac{d\gamma}{dt} \vec{v} + \gamma \vec{a} \right)$$

$\perp$ usual 3-acceleration

Def. Proper acceleration: The acceleration of the particle measured in its instantaneous rest frame.

In this frame $v=0 \therefore \gamma=1, \dot{\gamma}=0$ & so

$$a^\mu = (0, \vec{a})$$

For a uniformly accelerated particle the proper acceleration is constant. $a^\mu = (0, \vec{a}) \therefore a^\mu a_\mu = -a^2$. (in any frame).

Compute $u^\mu a_\mu$

a) Take general form and do lots of algebra...

b) Choose "best possible" frame $v=0, u^\mu = (c, \vec{0}), a^\mu = (0, \vec{a})$

$$\therefore u^\mu a_\mu = 0$$

c) use $u^\mu u_\mu = c^2$, so $\frac{d}{d\tau} (u^\mu u_\mu) = 2u^\mu a_\mu \therefore u^\mu a_\mu = 0$.

4-acceleration $\perp$ to 4 velocity!

4-force

Define $f^\mu = \frac{dp^\mu}{d\tau}$ (is a 4-vector!).

L9, p7
L11, p3

$$\frac{d}{d\tau} p^\mu = \frac{d}{d\tau} (E/c, \vec{p}) = \gamma \frac{d}{dt} (E/c, \vec{p})$$

$$= \left(\gamma \frac{dE}{c dt}, \gamma \vec{F} \right)$$

⚡ Normal 3-force. $\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(m\gamma\vec{v})$.

Another 4-vector $\tilde{f}^\mu = m a^\mu$ looks almost the same,
but only for constant m (rest mass). ~~Not true for all~~
~~general collision processes~~

$$f^\mu = \frac{dp^\mu}{d\tau} = \frac{d}{d\tau} (m u^\mu) \stackrel{\text{sometimes}}{=} m \frac{du^\mu}{d\tau} = m a^\mu$$

Not true for general processes.

Proper definition $f^\mu = \frac{dp^\mu}{d\tau}$.

Relativistic EM

L10, p1

Summary of last lecture

New 4-vectors:

4-velocity $u^\mu = \frac{dx^\mu}{d\tau} = \gamma(c, \vec{v})$

$$u^\mu u_\mu = u \cdot u = u^2 = c^2 \quad [\text{choose smart frame}]$$

4-momentum

$$p^\mu = m \cancel{u} u^\mu = m \gamma(c, \vec{v}) = (E/c, \vec{p})$$

E : Total relativistic energy

$\vec{p}$: Relativistic 3-momentum $\vec{p} = m \gamma \vec{v}$.

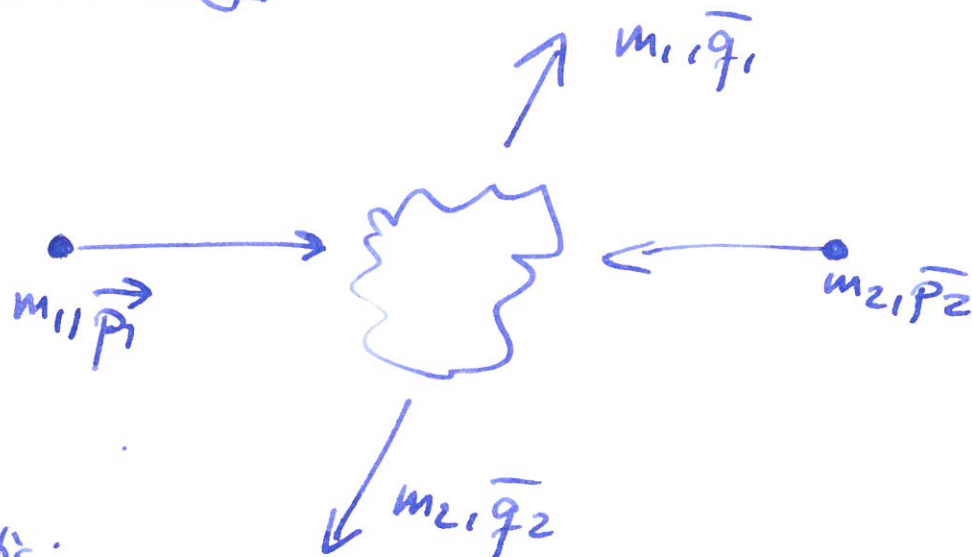
$$p^2 = p^\mu p_\mu = E^2/c^2 - \vec{p}^2 \text{ or}$$

$$\parallel \quad m^2 c^2 \quad \text{or} \quad \boxed{E^2 - (\vec{p}c)^2 = (mc^2)^2}$$

E and $\vec{p}$ conserved for closed systems

[Conserved: Time-indep.]

L. Invariant: indep of frame].

Elastic scattering

Elastic:

Incoming and outgoing particles are the same.
 Even when we know nothing about the details of the interaction
 we know the kinematic constraint off

4- momentum conservation

Encodes two conservation of

Energy
momentum

oth component
spatial components.

Eg 2 particles with equal masses $m_1 = m_2 = m$

In the centre of mass frame we have $\vec{p}_1 = -\vec{p}_2 \equiv \vec{p}$

$$\left. \begin{aligned} p_1^\mu &= \left(\frac{E_1}{c}, \vec{p} \right) \\ p_2^\mu &= \left(\frac{E_2}{c}, -\vec{p} \right) \end{aligned} \right\} \begin{aligned} &\text{as } m_1 = m_2 \text{ we have} \\ &E_1 = E_2 = m\gamma(v)c^2 = E \text{ in CMF.} \end{aligned}$$

After collision $q_1^\mu = (E_1'/c, \vec{q}_1)$

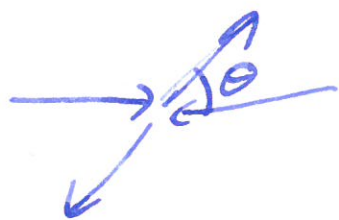
$$q_2^\mu = (E_2'/c, \vec{q}_2)$$

4-momentum cons: $p_1^\mu + p_2^\mu = q_1^\mu + q_2^\mu \therefore$

$$(2E/c, \vec{0}) = \left(\frac{E_1' + E_2'}{c}, \vec{q}_1 + \vec{q}_2 \right).$$

But since $m_1 = m_2$ and $|\vec{q}_1| = |\vec{q}_2|$, $E_1' = E_2' = E$. $|\vec{q}_1| = |\vec{q}_2| = |\vec{p}|$.

Scattering angle θ (and azimuthal angle ϕ) only degree of freedom



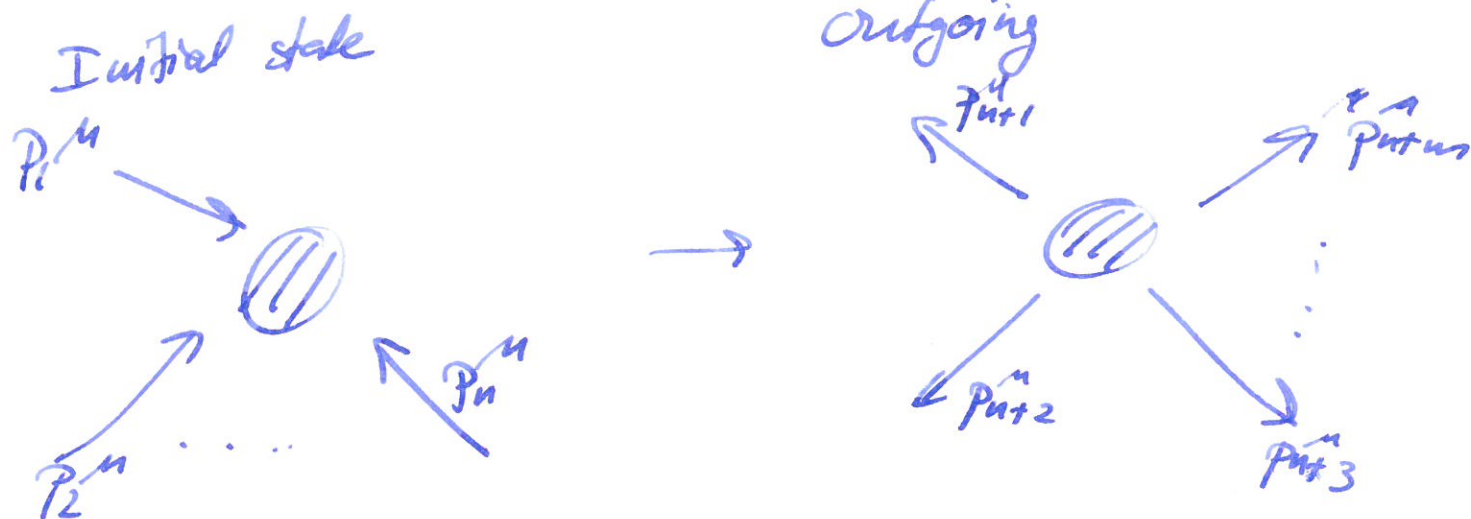
Check # dof:

$$\underbrace{2 \times 4}_{\substack{\text{2 x 4-momenta} \\ \text{of outgoing particles}}} - \underbrace{2}_{\substack{\uparrow \\ \text{onshell cond.} \\ E^2 - (pc)^2 = (mc^2)^2}} - \underbrace{4}_{\substack{\uparrow \\ \text{4-momentum} \\ \text{cons.}}} = 2.$$

Inelastic scattering and threshold energy

L10, p4

Consider now the a process where the number of particles and their masses can change



4-momentum cons

$$p_1^\mu + p_2^\mu + \dots + p_n^\mu = p_{nt+1}^\mu + \dots + p_{nt+n}^\mu$$

Often useful to calculate the MP of LHS and RHS.

Threshold Energy: Assume just 2 incoming particles of rest-masses m_1, m_2 . M outgoing particles of rest masses $m_3, \dots, m_{nt+2}$.

Is the process kinematically allowed? Depends on relative velocity of incoming particles and all the masses m_i .

~~Let~~ $p_1^\mu + p_2^\mu = p_3^\mu + \dots + p_{nt+2}^\mu = Q_f^\mu$ (total 4-momentum of initial/final state).

square $(p_1 + p_2)^2 = p_1^2 + p_2^2 + 2p_1 \cdot p_2 = Q_f^2$

compute in Rest frame of particle 1: $p_1^\mu = (m_1 c, \vec{0})$

$$p_2^\mu = (m_2 c \gamma v, m_2 \gamma v \vec{v})$$

$$\therefore p_1 \cdot p_2 = m_1 m_2 c^2 \gamma(v)$$

$$\therefore Q_f^2 = m_1^2 c^2 + m_2^2 c^2 + 2 m_1 m_2 c^2 \gamma(v)$$

$$\therefore \gamma(v) = \frac{Q_f^2 - m_1^2 c^2 - m_2^2 c^2}{2 m_1 m_2 c^2} \quad v: \text{rel. vel. of particles}^{\text{inc.}}$$

The minimum Q_f^2 allowed by the final state particles is obtained if there exists a ~~PIF~~ ^{PIF} where all the final state particles are at rest. If such a frame exists, one would have

$$(Q_f^2)_{\min} = c^2 \left(\sum_{i=3}^{n+2} m_i \right)^2$$

Example:

$$PP \rightarrow PP\pi^0$$

$$m_p \simeq 1000 \text{ MeV}/c^2 \simeq 1 \text{ GeV}/c^2$$

$$m_{\pi} \simeq 140 \text{ MeV}/c^2 = 0.14 \text{ GeV}/c^2$$

$$\gamma(v)_{\min} = \frac{c^2 (2m_p + m_{\pi})^2 - 2m_p^2 c^2}{2c^2 m_p^2} = \frac{2m_p^2 + 4m_p m_{\pi} + m_{\pi}^2}{2m_p^2}$$

$$= 1 + 2 m_{\pi}/m_p + \frac{m_{\pi}^2}{2m_p^2} \simeq 1.3$$

$$\therefore v \geq 0.63 c.$$

Another example

Pair Creation and Photons.

Photons are the Quanta of electromagnetic radiation.

Particles with zero ^(rest) mass, travelling at speed of light.
 relativistic concept. Impossible in Newtonian mechanics.

$$p^\mu = (E/c, \vec{p}) = \left(\frac{hf}{c}, \frac{hf}{c} \vec{n} \right) = \frac{hf}{c} (1, \vec{n})$$

h : Planck's constant

f : frequency

$\vec{n}$: direction of travel, $|\vec{n}| = 1$.

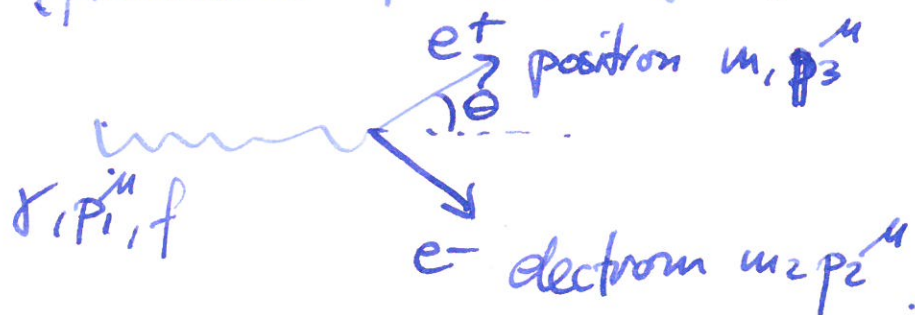
$$p^2 = p^\mu p_\mu = \frac{h^2 f^2}{c^2} - \frac{h^2 f^2}{c^2} |\vec{n}|^2 = 0!$$

a) Photons never stop

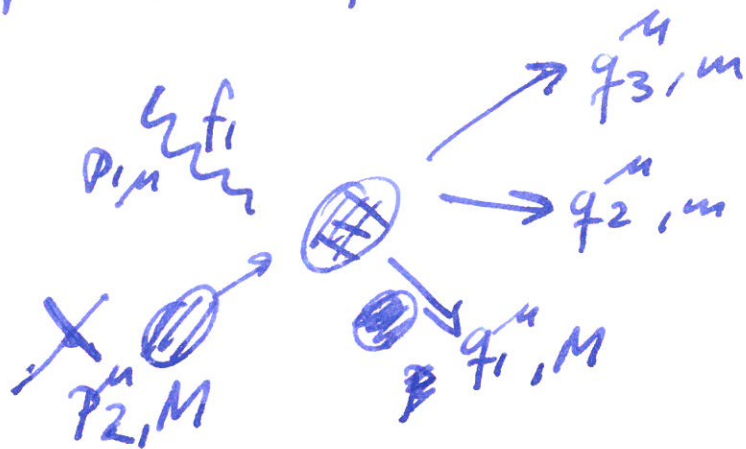
b) There is no rest frame for the photon

Often we use the notation γ for photons (NOT the gamma-factor).

Spontaneous "decay" or conversion of γ is impossible (production of e^+e^- pair)



In the presence of another particle, no spontaneous production is possible L19 p8



X is usually proton or nucleus.

In the rest frame of X :

$$p_1^\mu = \hbar f_1 / c (1, \vec{n})$$

$$p_2^\mu = (Mc, \vec{0})$$

Compute threshold frequency i.e. minimum frequency required.

$$(p_1 + p_2)^\mu = Q_f^\mu = (q_1 + q_2 + q_3)^\mu$$

squaring

$$p_1^2 + p_2^2 + 2p_1 \cdot p_2 = Q_f^2 \quad \left| \begin{array}{l} p_1^2 = 0 \\ p_2^2 = M^2 c^2 \end{array} \right.$$

$$\therefore p_1 \cdot p_2 = \frac{1}{2} (Q_f^2 - M^2 c^2)$$

$$\parallel \\ M \hbar f_1$$

Minimal $f_1 \rightarrow$ minimal Q_f^2 .

$$(Q_f^2)_{\min} = (2m + M)^2 c^2 \text{ so threshold frequency}$$

$$\text{for } \frac{2c^2 m^2 + Mm}{2Mm} \quad \left(\begin{array}{l} \rightarrow 2mc^2 \text{ for } M \gg m \\ \rightarrow \infty \text{ for } M \rightarrow 0 \end{array} \right)$$

Relativistic Electrodynamics

L11, p1

Summary of last lecture

Relativistic kinematics (constraints on 4-momentum)

4-momentum conservation

Closed system:

$$\sum P_i^{\mu} \text{ before} = \sum P_i^{\mu} \text{ after}$$

Elastic scattering: Number and types of particles unchanged

Inelastic scattering: ^{Number or types change} ~~change possible~~, eg. particle creation

Example $p p \rightarrow p p \pi^0$

Threshold energy: Minimum energy of inc. particles which allows the process to proceed. ~~all~~ $\exists S$: all $f \rightarrow$ particles at rest.

$$(p_1^{\mu} + p_2^{\mu}) = (p_3^{\mu} + p_4^{\mu} + p_5^{\mu}) = Q_f^{\mu} \quad [= \sum_i (m_i c, \vec{0})]$$

$$\therefore p_1^2 + p_2^2 + 2p_1 \cdot p_2 = Q_f^2$$

pair creation etc from f : impossible in vacuum, possible in presence of another particle

Today's lecture: Relativistic Dynamics: Movement under influence of forces.

First, discuss the relativistic definition of force, and acceleration

$\rightarrow$ L9 p6, L9 p7

Newton's laws and relativity:

1st law is built into rel. principle

3rd law (action-reaction) obviously problematic: If two objects are separated in space, $s_{ab} < 0$, so causal connection forbidden! Suppose in some frame S the force of A on B at instant t is $F(t)$. Force ~~one~~ of B on A is $-F(t)$ in this frame. But ~~there~~ a moving observer will not see these two events of force measurements at equal times, so in general 3rd law is violated.

2nd law: $\vec{F} = \frac{d\vec{p}}{dt}$ [Newton] holds in SR provided we use the relativistic momentum. (note: t , not τ !)

Motion under a constant force.

Q: Particle, mass m , constant force $\vec{F}$. Starts from rest at origin. Find $x(t)$.

Soln: $\frac{dp}{dt} = F \therefore p = Ft + \text{constant}$. $p(t=0) = 0 \therefore \text{constant} = 0$.

$$\therefore p = \frac{mu}{\sqrt{1-u^2/c^2}} = Ft.$$

Solve quadratic eqn for u leads to

$$u = \frac{(F/m)t}{\sqrt{1 + (Ft/mc)^2}}$$

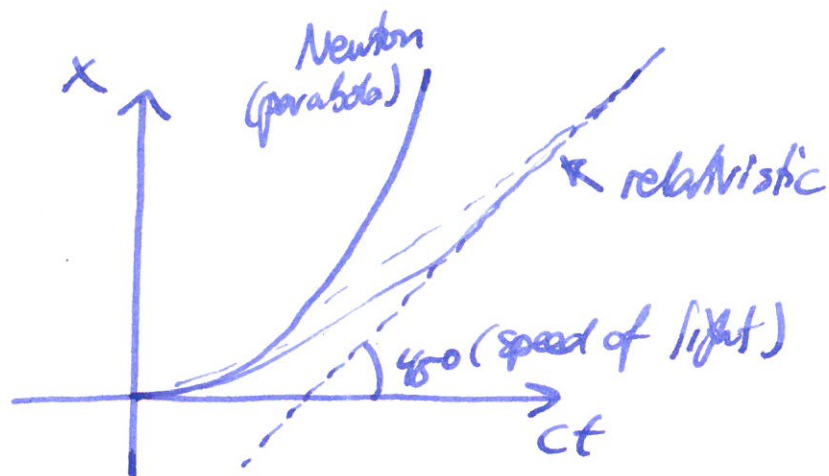
Numerator: Classical soln. Rel. denom ensures $u < c$.

$$x(t) = \frac{F}{m} \int_0^t \frac{t'}{\sqrt{1 + (Ft'/mc)^2}} dt'$$

$$= \frac{mc^2}{F} \left[\sqrt{1 + (Ft'/mc)^2} \right]_{t'=0}^{t'=t} = \frac{mc^2}{F} \left(\sqrt{1 + (Ft/mc)^2} - 1 \right).$$

Newton:
 Classically $x(t) = \left(\frac{F}{2m}\right)t^2$ parabola

constant force:
 hyperbola hyperbolic motion.



Work is defined as the line integral of the force

$$W \equiv \int \vec{F} \cdot d\vec{r}$$

Work-Energy Theorem holds (Net work equals increase in kinetic energy).

$$W = \int \frac{d\vec{p}}{dt} \cdot d\vec{r} = \int \frac{d\vec{p}}{dt} \frac{d\vec{r}}{dt} dt = \int \frac{d\vec{p}}{dt} \cdot \vec{u} dt$$

$$\text{But } \frac{d\vec{p}}{dt} \cdot \vec{u} = \frac{d}{dt} \left(\frac{m\vec{u}}{\sqrt{1-u^2/c^2}} \right) \cdot \vec{u}$$

$$= \frac{m\vec{u}}{(1-u^2/c^2)^{3/2}} \frac{d\vec{u}}{dt}$$

$$\left[\frac{d}{dt} \left(\frac{f}{g} \right) = \frac{f'g - g'f}{g^2} \right]$$

$$= \frac{d}{dt} \left(\frac{mc^2}{\sqrt{1-u^2/c^2}} \right) = \frac{dE_{rel}}{dt}$$

$$\text{so } W = \int \frac{dE_{rel}}{dt} dt = E_{final} - E_{initial}$$

[rest energy cancels out].

$\bar{F}$ derivative wrt. time, not proper-time. $\bar{F} = \frac{d\bar{p}}{d\bar{t}}$

L11, p6
L12 p2

$\bar{F}$ not 4-vector!

In S' , ^{std. conf.} have

$$\bar{F}_y = \frac{d\bar{p}_y}{d\bar{t}} = \frac{dp_y}{\gamma dt - \frac{\beta}{c} dx} = \frac{dp_y/dt}{\gamma(1 - \frac{\beta}{c} \frac{dx}{dt})} = \frac{F_y}{\gamma(1 - \beta u_x/c)}$$

sim. for z.

x comp. worst:

$$\bar{F}_x = \frac{d\bar{p}_x}{d\bar{t}} = \frac{\gamma p_x - \gamma \beta dp^0}{\gamma dt - \frac{\beta}{c} dx} = \frac{F_x - \frac{\beta}{c} (dE/dt)}{1 - \beta u_x/c}$$

$$\bar{F}_x = \frac{F_x - \beta (\bar{u} \cdot \bar{F})/c}{1 - \beta u_x/c}$$

If particle at rest in S , with $\bar{u}=0$, then

$$\bar{F}_\perp = \frac{1}{\gamma} F_\perp, \bar{F}_\parallel = F_\parallel \quad (\perp, \parallel \text{ wrt. boost between } S' \text{ and } S).$$

F parallel to boost unchanged!

F orthogonal to boost are divided by γ .

Define Minkowski Force $K^\mu = \frac{dp^\mu}{d\bar{t}}$.

4-vector

$$\bar{K} = \left(\frac{d\bar{t}}{d\bar{t}} \right) \frac{d\bar{p}}{d\bar{t}} = \gamma \bar{F}.$$

$$K^0 = \frac{dp^0}{d\bar{t}} = \frac{1}{c} \frac{dE_{rel}}{d\bar{t}}$$

Minkowski force: neat formulae.

Ordinary force: useful, since we are often interested in trajectory vs. time (of observer, not particle itself).

$$p_1^\mu = (hf/c, hf/c \vec{n}) \quad \text{choose cs. such that } \vec{n} = \hat{x}$$

$$= hf/c (1, 1, 0, 0)$$

$$p_2^\mu = (E_2/c, \vec{p}_2) \quad \text{with } (m_2 c^2)^2 = E_2^2 - (\vec{p}_2 c)^2 \geq 0.$$

$$|\vec{p}_2| = |\vec{p}_2| (\cos \theta, \sin \theta, 0)$$

4-momentum cons.

$$p_1^\mu = p_2^\mu + p_3^\mu \quad \therefore p_1^\mu - p_2^\mu = p_3^\mu$$

$$\therefore p_1^2 - 2p_1 \cdot p_2 + p_2^2 = p_3^2$$

$$\therefore 0 + m^2 c^2 - 2p_1 \cdot p_2 = m_c^2$$

$$\therefore 2p_1 \cdot p_2 = 0.$$

$$\text{But } p_1 \cdot p_2 = \frac{hf_1}{c} \left(\frac{E_2}{c} - \vec{n} \cdot \vec{p}_2 \right)$$

$$= \frac{hf_1}{c} (E_2/c - |\vec{p}_2| \cos \theta)$$

But this is larger than 0, since $E_2/c > |\vec{p}_2| \geq |\vec{p}_2| \cos \theta$!

Relativistic Electrodynamics

U2p1

Summary of last lecture

4-acceleration $a^\mu = \frac{d^2 x^\mu}{d\tau^2} = \frac{d u^\mu}{d\tau} = \gamma \left(\frac{d\gamma}{dt} c, \frac{d\gamma}{dt} \vec{v} + \gamma \vec{a} \right)$

New: compute $u^\mu a_\mu$:

a) Take general form, do lots of algebra

b) Choose "best possible" frame: $v=0$, $u^\mu = (c, \vec{0})$, $a^\mu = (0, \vec{a})$

$$\therefore u^\mu a_\mu = 0$$

c) use $u^\mu u_\mu = u \cdot u = c^2$, so $0 = \frac{d}{d\tau} (u^\mu u_\mu) = 2 u^\mu a_\mu \therefore u^\mu a_\mu = 0$.

4 acceleration $\perp$ 4 velocity!

4 force $f^\mu = \frac{dp^\mu}{d\tau}$ $\vec{F} = \frac{d\vec{p}}{dt}$ with rel. momentum $\vec{p} = m\gamma\vec{v}$ [t, not τ].

Newton's laws: 1✓ 2✓ 3x.

→ ~~Relativistic~~ SR: Motion under constant force: hyperbolic motion.

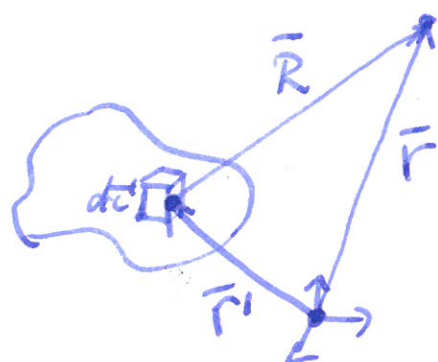
Work-Energy theorem holds. Net work equals increase in kinetic energy.

$$W \equiv \int \vec{F} \cdot d\vec{l}$$

Fields of Moving Point charge [10.3].

Return to electromagnetism. We paused there after deriving the scalar and vector potentials (in Lorenz-gauge) for non-static sources: Retarded potentials

$$V(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}', t_r)}{R} d\tau' \quad \vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}', t_r)}{R} d\tau'$$



Retarded time
 $t_r \equiv t - \frac{R}{c}$

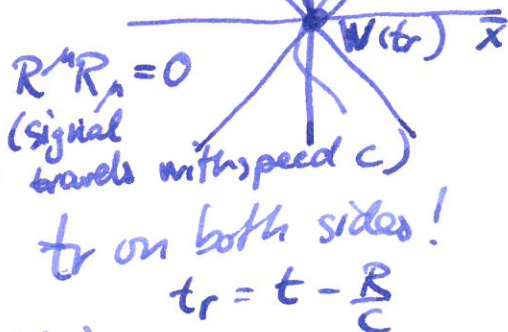
Now want to ~~derive~~ calculate the retarded potentials of a point charge moving on the trajectory

$\vec{w}(t)$ (position of q at time t).

Retarded time determined implicitly

$$|\vec{r} - \vec{w}(t_r)| = c(t - t_r)$$

$$t_r = t + \frac{|\vec{r} - \vec{w}(t_r)|}{c}$$



$\vec{w}(t_r)$: retarded position. $R = |\vec{r} - \vec{w}(t_r)|$.

At most one point is ~~in contact~~ at distance $|\vec{r} - \vec{w}(t_r)| = c(t - t_r)$ at any particular time t : Suppose two points with retarded times

$$t_1 \neq t_2 \quad R_1 = c(t - t_1) \quad \text{and} \quad R_2 = c(t - t_2).$$

$R_1 - R_2 = c(t_2 - t_1)$ so average speed ~~is~~ towards $\vec{r}$ is speed of light. $\Rightarrow$ only one unique point for a given time t .

Lienard-Wiechert potentials

L12, p4

a la A. Suger.

Compute the scalar and vector potential or the

4 potential $A^\mu = (\phi, \vec{A})$.

First work in a special IF (S'): The instantaneous rest frame of q @ time t_{ret} .

$$e \equiv \frac{q}{4\pi\epsilon_0}$$

$$S' : V = \frac{q}{4\pi\epsilon_0 |\vec{R}'|} = \frac{e}{R'} \quad \vec{A} = 0$$

Translate to arbitrary S : Trick: Find 4-vector that takes above form in S'

$$\text{claim: } A^\mu = \frac{e}{u_\nu R^\nu} u^\mu \quad u^\mu : \text{4-velocity of } q \text{ at time } t_{ret}.$$

is a 4-vector

check that in S' this is the correct 4-vector

$$u_\nu R^\nu = (c\gamma - \gamma \vec{v}) \cdot \begin{pmatrix} c(t - t_{ret}) = |\vec{R}| \\ \vec{R} \end{pmatrix} = \gamma c (|\vec{R}| - \frac{\vec{v} \cdot \vec{R}}{c})$$

$$\text{in } S' : u^\mu = (c, 0) ; \quad u_\nu R^\nu = c |\vec{R}|$$

$$\therefore A^\mu = \frac{e}{u_\nu R^\nu} u^\mu = \frac{e}{c |\vec{R}|} (c, \vec{0}) = \left(\frac{e}{R}, \vec{0} \right) \checkmark$$

True in S' . ~~Equation~~ ^{Relation} between 4-vectors.

$\therefore$ True in any S !

$$A^\mu = \frac{e u^\mu}{u_\nu R^\nu} \Big|_{ret} \quad V = \frac{e}{R - \frac{\vec{v} \cdot \vec{R}}{c}} \Big|_{ret} \quad \vec{A} = \frac{e \vec{v}}{c^2 (R - \frac{\vec{v} \cdot \vec{R}}{c})} \Big|_{ret}.$$

The Fields of a Moving Point Charge

L12p5

$$V(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{qc}{(Rc - \vec{R} \cdot \vec{v})}, \quad \vec{A}(\vec{r}, t) = \frac{\vec{v}}{c} V(\vec{r}, t)$$

$$\vec{E} = -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t}, \quad \vec{B} = \vec{\nabla} \times \vec{A}.$$

Difficulty: Differentiation tricky since

$$\vec{R} = \vec{r} - \vec{w}(tr), \quad v = \dot{\vec{w}}(tr)$$

both evaluated at retarded time tr defined implicitly through

$$|\vec{r} - \vec{w}(tr)| = c(t - tr)$$

so tr is itself a function of $\vec{r}$ and t .

$\therefore$ 2 pages of algebra 436-437.

$$\vec{u} \equiv c\hat{R} - \vec{v}$$

$$\vec{E}(\vec{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{R}{(\vec{R} \cdot \vec{u})^3} \left[(c^2 - v^2) \vec{u} + \vec{R} \times (\vec{u} \times \vec{a}) \right]$$

$$\vec{B}(\vec{r}, t) = \frac{1}{c} \hat{R} \times \vec{E}(\vec{r}, t).$$

Magnetic field from point charge is always perp. to the electric field from the point charge, and to the vector ~~to~~ from the retarded point to the observer in $\vec{R}$.

First term in $\vec{E}$ falls off as $\frac{1}{R^2}$: Generalised Coulomb field.
(if $\vec{v}$ and $\vec{a} = 0$, reproduces Coulomb).

Second term falls off as $\frac{1}{R}$, and dominates at large distances
Radiation field, or acceleration field.

Lorentz Force on particle of charge Q from
accelerated charge q :
and velocity $\vec{V}$

L2, p6

$$\vec{F} = Q(\vec{E} + \vec{v} \times \vec{B})$$

$$= \frac{qQ}{4\pi\epsilon_0} \frac{\vec{R}}{(\vec{R} \cdot \vec{u})^3} \left\{ \left[(c^2 - v^2) \vec{u} + \vec{R} \times (\vec{u} \times \vec{a}) \right] \right. \\ \left. + \frac{\vec{V}}{c} \times \left[\hat{R} \times \left[(c^2 - v^2) \vec{u} + \vec{R} \times (\vec{u} \times \vec{a}) \right] \right] \right\}.$$

$\vec{R}, \vec{u}, \vec{v}, \vec{a}$ evaluated at retarded time.

Relativistic Electrodynamics

Dr Andersen

18 lectures + 3 examples classes in Michaelmas Term

Textbooks:

Required: Introduction to Electrodynamics, D. J. Griffiths (Pearson, 3rd Edition)

The course is defined by material contained in this book and in particular the material defined in the syllabus below where the numbers refer to the sections in the book.

Syllabus:

- 1. Einsteins postulates [12.1]
- 2. The geometry of relativity [12.1]
- 3. Lorentz transformations [12.1]
- 4. Structure of space-time [12.1]
- 5. Proper time and proper velocity [12.2]
- 6. Relativistic energy and momentum [12.2]
- 7. Relativistic Kinematics [12.2]
- 8. Relativistic Dynamics [12.2]
- 9. Magnetism as a relativistic phenomena [12.3]
- 10. How the Fields transform [12.3]
- 11. The Field Tensor [12.3]
- 12. Electrodynamics in Tensor notation [12.3]
- 13. Relativistic potentials [12.3]
- 14. Scalar and Vector potentials [10.1]
- 15. Gauge transformations [10.1]
- 16. Coulomb gauge [10.1]
- 17. Retarded potentials [10.2]
- 18. Fields of a moving point charge [10.3]
- 19. Dipole radiation [11.1]
- 20. Radiation from point charges [11.2]

10.1, 10.2, 11.1, 11.2, 12.1, 12.2, 12.3, 10.3

7 sections, 18 Lectures

10.1: 416 - 422	7p
10.2: 422 - 429	7p
11.1: 443 - 459	17p
11.2: 460 - 472	13p
12.1: 477 - 506	29p
12.2: 507 - 520	14p
12.3: 522 - 542	21p

+ 10.3: 10p!

~~98~~ 108p ~ 6p per lecture.

108p / 18 lectures ~ 6p per lecture → ~ 8p / lecture.

Relativistic Electrodynamics

113, p1

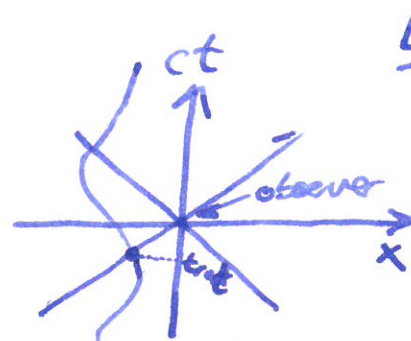
Summary of last lecture

Liendard - Wieckert potentials

4 potential $A^\mu = (V, \vec{A})$

$A^\mu = \frac{e}{4\pi\epsilon_0 R^\nu} u^\mu \Big|_{\text{ret}}$ u^μ : 4-velocity of q at the time

$$V = \frac{e}{R - \frac{\vec{v} \cdot \vec{R}}{c}} \Big|_{\text{ret}} \quad \vec{A} = \frac{e\vec{v}}{c^2 (R - \frac{\vec{v} \cdot \vec{R}}{c})} \Big|_{\text{ret}}$$



Fields of moving point charge. Take derivatives of above

$$\vec{E}(\vec{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{R}{(\vec{R} \cdot \vec{u})^3} \left[\underbrace{(c^2 - v^2)}_{\text{Generalized Coulomb field / velocity field}} \vec{u} + \underbrace{\vec{R} \times (\vec{u} \times \vec{a})}_{\text{Radiation or acceleration field}} \right] \Big|_{\text{ret}}$$

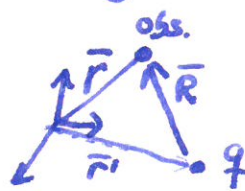
$$\vec{B}(\vec{r}, t) = \frac{1}{c} \hat{R}_{\text{ret}} \times \vec{E}(\vec{r}, t) \quad \vec{u} \equiv c \hat{R} - \vec{v}$$

Very important results. Learn how to evaluate the terms.

Let us ~~then~~ now evaluate the power radiated by the accelerated point charge

The Poynting vector is

If we choose coordinate system such that $\vec{r}_i = 0$ at $t_{\text{ret}} = 0$, then $\vec{r} = \vec{R}$ at $t_{\text{ret}} = t - R/c$



$$\vec{S} = \frac{1}{\mu_0} (\vec{E} \times \vec{B}) = \frac{1}{\mu_0 c} [\vec{E} \times (\hat{R} \times \vec{E})] = \frac{1}{\mu_0 c} [E^2 \hat{R} - (\hat{R} \cdot \vec{E}) \vec{E}]$$

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

Not all of this radiation will be radiated to infinity.
[example class: $\vec{E}, \vec{B}$ of particle moving with const. velocity].

The radiated energy propagates to infinity. L13, p2

To calculate the total power radiated at time t_{ret} , we should ^(large) integrate the Poynting vector over a sphere of radius R , centered at the position of the particle at time t_{ret} . (All points of sphere reached simultaneously by radiation emitted at time t_{ret}).

Like in dipole-discussion: Will concentrate on terms in $\bar{S}$ that fall off like $1/R^2$. Those that fall off faster will not contribute for sufficiently large R . (and there are none that fall off slower).

The acceleration fields represent the true radiation.

$$\bar{E}_{rad} = \frac{q}{4\pi\epsilon_0} \frac{R}{(\bar{R} \cdot \bar{u})^3} [\bar{R} \times (\bar{u} \times \bar{a})].$$

[The velocity field carries energy too - dragged along with the charge, but it does not contribute at large R].

$\bar{E}_{rad} \perp \hat{R}$, so second term in $\bar{E} \cdot \bar{S}$ vanishes.

$$\bar{S}_{rad} = \frac{1}{\mu_0 c} E_{rad}^2 \hat{R}$$

Calculate a few examples of increasing difficulty

1) Charge is instantaneously at rest (at time t_r): $\bar{u} = c \hat{R}$, so

$$c^2 = \frac{1}{\epsilon_0 \mu_0} \quad \bar{E}_{rad} = \frac{q}{4\pi\epsilon_0 c^2 R} [\hat{R} \times (\hat{R} \times \bar{a})] = \frac{\mu_0 q}{4\pi R} [(\hat{R} \cdot \bar{a}) \hat{R} - \bar{a}]$$

Then, since $|\hat{\mathbf{R}} \cdot \mathbf{a} \hat{\mathbf{R}} - \bar{a}|^2 = ((\hat{\mathbf{R}} \cdot \bar{\mathbf{a}}) \hat{\mathbf{R}} - \bar{a}) \cdot ((\hat{\mathbf{R}} \cdot \bar{\mathbf{a}}) \hat{\mathbf{R}} - \bar{a}) \quad \frac{L(3p3)}{4\pi R^2}$
 $= (\hat{\mathbf{R}} \cdot \mathbf{a})^2 - (\hat{\mathbf{R}} \cdot \bar{\mathbf{a}})(\hat{\mathbf{R}} \cdot \mathbf{a}) - (\bar{\mathbf{a}} \cdot \hat{\mathbf{R}})(\hat{\mathbf{R}} \cdot \bar{\mathbf{a}}) - \bar{a}^2 = (a^2 - (\hat{\mathbf{R}} \cdot \bar{\mathbf{a}})^2)$
 ~~$\bar{E} = \frac{1}{4\pi R^2}$~~

$$\bar{S} = \frac{1}{\mu_0 c} \left(\frac{\mu_0 q}{4\pi R} \right)^2 [a^2 - (\hat{\mathbf{R}} \cdot \bar{\mathbf{a}})^2] \hat{\mathbf{R}} = \frac{\mu_0 q^2 a^2}{16\pi^2 c} \left(\frac{\sin^2 \theta}{R^2} \right) \hat{\mathbf{R}}$$

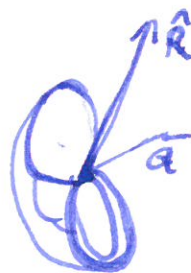
θ : angle between $\bar{\mathbf{a}}$ and $\hat{\mathbf{R}}$.

No power radiated in direction of acceleration!

Donut about the direction of instantaneous acceleration.

Total power radiated $\hat{\mathbf{n}}: \text{outward} = \hat{\mathbf{R}}$
 $d\bar{a} = \hat{\mathbf{R}} R^2 \sin \theta d\theta d\phi$

$$P = \oint \bar{S}_{\text{rad}} \cdot d\bar{a}$$



$$= \frac{\mu_0 q^2 a^2}{16\pi^2 c} \int \frac{\sin^2 \theta}{R^2} R^2 \sin \theta d\theta d\phi$$

$$P = \frac{\mu_0 q^2 a^2}{16\pi^2 c} \int_0^\pi \sin^3 \theta d\theta \int_0^{2\pi} d\phi$$

No ϕ dep: $\int_0^{2\pi} d\phi \rightarrow 2\pi$.

$x = \cos \theta$

$$\int_0^\pi \sin^3 \theta d\theta = - \int_1^{-1} \sin^2 \theta d(\cos \theta) = \int_{-1}^1 (1-x^2) dx$$

$$= \left[x - \frac{1}{3} x^3 \right]_{-1}^1 = \frac{4}{3}$$

$$P = \frac{\mu_0 q^2 a^2}{6\pi c} \quad \text{Larmor Formula.}$$

$\bar{v} = 0$

Complications for $\bar{v} \neq 0$.

L13, p4

Obs: E rad more complicated

Subtle: Power radiated does not equal power observed!
Simple doppler effect. passing through the sphere R/c later.

let $\frac{dW}{dt}$ be the rate at which energy passes through

an infinitesimal area of the sphere at radius R , then
the rate at which the energy left the particle at time t_r

$$\text{was } \frac{dW}{dt_r} = \frac{dW}{dt} \times \frac{dt}{dt_r}.$$

Find $\frac{dt}{dt_r}$.

$$c(t - t_r) = R \therefore c^2(t - t_r)^2 = R^2 = \bar{R} \cdot \bar{R}$$

Diff. w.r.t. t

$$2c^2(t - t_r) \left(1 - \frac{\partial t_r}{\partial t}\right) = 2\bar{R} \cdot \frac{\partial \bar{R}}{\partial t}$$

$$\therefore CR \left(1 - \frac{\partial t_r}{\partial t}\right) = \bar{R} \cdot \frac{\partial \bar{R}}{\partial t}$$

$$\bar{R} = \bar{r} - \bar{u}(t_r), \text{ so } (\bar{r} \text{ fixed})$$

$$\frac{\partial \bar{R}}{\partial t} = -\frac{\partial u(t_r)}{\partial t} = -\frac{\partial u(t_r)}{\partial t_r} \frac{\partial t_r}{\partial t} = -\bar{v} \frac{\partial t_r}{\partial t}$$

$$\therefore CR \left(1 - \frac{\partial t_r}{\partial t}\right) = -\bar{R}\bar{v} \frac{\partial t_r}{\partial t}$$

$$\therefore CR = \frac{\partial t_r}{\partial t} (CR - \bar{R}\bar{v}) = \frac{\partial t_r}{\partial t} (\bar{R} \cdot \bar{u})$$

$$\therefore \frac{\partial t_r}{\partial t} = \frac{CR}{\bar{R} \cdot \bar{u}}.$$

$$\text{So } \frac{dW}{dt} = \left(\frac{\bar{\mathbf{R}} \cdot \bar{\mathbf{u}}}{Rc} \right) \frac{dW}{dt}.$$

L13p5

$$\frac{\bar{\mathbf{R}} \cdot \bar{\mathbf{u}}}{Rc} = 1 - \frac{\hat{\mathbf{R}} \cdot \bar{\mathbf{v}}}{c}$$

$$\bar{\mathbf{u}} = c\hat{\mathbf{R}} - \bar{\mathbf{v}} \Rightarrow \bar{\mathbf{u}} = \hat{\mathbf{R}} - \bar{\boldsymbol{\beta}}$$

$$\text{So } \frac{dP}{d\Omega} = \left(\frac{\bar{\mathbf{R}} \cdot \bar{\mathbf{u}}}{Rc} \right) \frac{1}{\mu_0 c} E_{\text{rad}}^2 R^2 = \frac{q^2}{16\pi^2 \epsilon_0} \frac{|\hat{\mathbf{R}} \times (\bar{\mathbf{u}} \times \bar{\mathbf{a}})|^2}{(\bar{\mathbf{R}} \cdot \bar{\mathbf{u}})^5} \\ = \frac{q^2}{16\pi^2 \epsilon_0 c} \frac{|\hat{\mathbf{R}} \times (\bar{\mathbf{R}} - \bar{\boldsymbol{\beta}}) \times \bar{\mathbf{a}}|^2}{(1 + \bar{\mathbf{R}} \cdot \bar{\boldsymbol{\beta}})^5}.$$

Difficult to integrate in full generality - you will get to do a few examples.

$$P = \frac{\mu_0 q^2}{6\pi c} \gamma^6 \left(a^2 - \frac{|\bar{\mathbf{v}} \times \bar{\mathbf{a}}|^2}{c^2} \right).$$

$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$. Power increases enormously for relativistic particles.

Radiation Reaction

Accelerated charges radiate $\therefore$ For given force, charged particles accelerate less than a neutral particle of same mass.

"Recoil force". Radiation Reaction force.

Encouraged to read sec. 11.2.2

$$\bar{\mathbf{F}}_{\text{rad}} = \frac{\mu_0 q^2}{6\pi c} \bar{\mathbf{a}} \quad (\text{non-rel}).$$

Summary of previous lectures

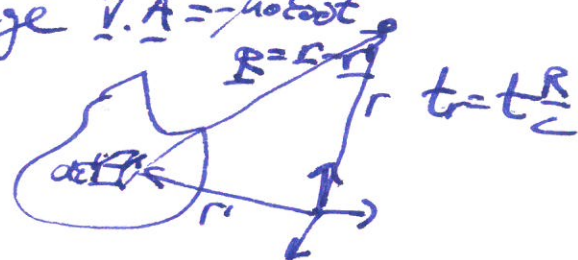
2013, LS

~~Electric~~

Scalar and vector potentials of time-varying, continuous charge and current distributions. Use Lorenz-gauge $\nabla \cdot \underline{A} = -\mu_0 \epsilon_0 \frac{\partial V}{\partial t}$

$$V(\underline{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\underline{r}', t_r)}{R} d\tau'$$

$$\underline{A}(\underline{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\underline{J}(\underline{r}', t_r)}{R} d\tau'$$



Applied to Electric (and Magnetic) dipole.

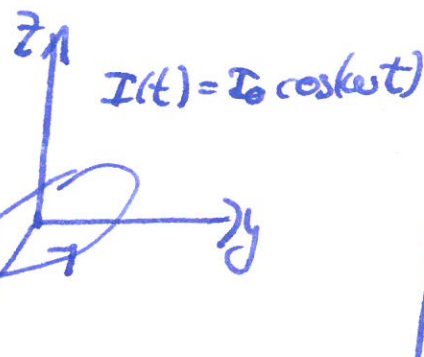
Get Electric and Magnetic fields by taking derivatives of these

$$\underline{E} = -\nabla V - \frac{\partial \underline{A}}{\partial t} \quad \underline{B} = \nabla \times \underline{A} \quad (\text{rather than from Jefimenko's Eqn}).$$

Question of integration in magnetic dipole

$$\underline{A}(\underline{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\underline{J}(\underline{r}', t_r)}{R} d\tau' \quad \leftarrow \text{Volume}$$

$$= \frac{\mu_0}{4\pi} \oint \frac{I_0 \cos[\omega(t - R/c)]}{R} d\vec{l} \quad \leftarrow \text{line element.}$$



This lecture:

Fields of a moving point charge

Recap relativity:
 E1: Laws of physics the same in all inertial frames.
 Express as tensor equations.

Power radiated to infinity $\underline{S} = \frac{1}{\mu_0} \underline{E} \times \underline{B}$.

$\oint \underline{S} \cdot d\vec{a}$ keep terms in $\underline{S}$ only that fall off no faster than $1/r^2$.

$\sim \frac{1}{r}$ in $\underline{E}$ and $\underline{B}$.

Relativistic Electrodynamics

Summary of last lecture.

Fields from the Liénard-Wiechert potentials

$$\vec{E}(\vec{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{\vec{R}}{(\vec{R} \cdot \vec{u})^3} \left[\underbrace{(c^2 - v^2) \vec{u}}_{\text{Vel. or Coulomb field}} + \underbrace{\vec{R} \times (\vec{u} \times \vec{a})}_{\text{acc. or Rad. field}} \right]$$

$$\vec{u} = c\hat{R} - \vec{v}$$

$$\vec{B}(\vec{r}, t) = \frac{1}{c} \hat{R} \times \vec{E}(\vec{r}, t)$$

 $\vec{B} \perp \vec{E}$. Special property.

$$\vec{S}(\vec{r}, t) = \frac{1}{\mu_0} \vec{E} \times \vec{B} = \frac{1}{\mu_0 c} [\vec{E}^2 \hat{R} - (\hat{R} \cdot \vec{E}) \vec{E}]$$

Radiation fields only:

$$\vec{S}_{\text{rad}} = \frac{1}{\mu_0 c} E_{\text{rad}}^2 \hat{R}$$

Power radiated

$$\vec{v} = 0 : \text{Larmor's formula } P = \frac{\mu_0 q^2 a^2}{6\pi c}$$

$\vec{v} \neq 0$: Power emitted $\neq$ Power observed. Photons ~~observed~~ emitted at equal ~~same~~ time intervals from source arrive with a different time spacing at observer, depending on $\vec{v}$ and $\hat{R}$.

$$\frac{dw}{d\Omega} = \frac{dw}{dt} \frac{dt}{d\Omega} \quad , \quad \frac{dt}{d\Omega} = \frac{\vec{R} \cdot \vec{u}}{Rc}$$

$$\vec{u} = c\hat{R} - \vec{v}$$

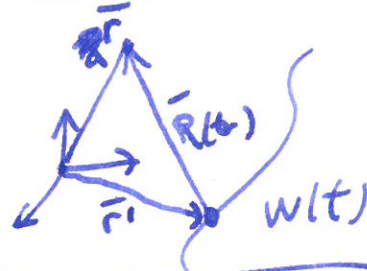
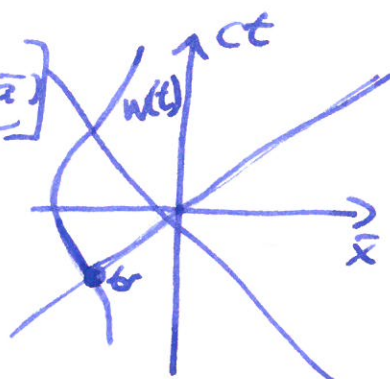
$$\frac{dP}{d\Omega} = \left(\frac{\vec{R} \cdot \vec{u}}{Rc} \right) \frac{1}{\mu_0 c} E_{\text{rad}}^2 R^2$$

General solⁿ

$$P = \frac{\mu_0 q^2}{6\pi c} \left(a^2 - \frac{|\vec{v} \times \vec{a}|^2}{c^2} \right)$$

Radiation-Reaction
back

For a given external force, charged particles accelerate less than uncharged particles
Liénard's generalisation of Larmor's formula.



Sec. 12.2 L14, p2

Today, we will discuss how magnetism arise as a relativistic phenomenon. ~~Electric fields~~ + relativity $\Rightarrow$ magnetism.

Maxwells eqn and Lorentz force fully consistent with spec. relativity. In the following we will re-express EM to expose this.
 $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$. $\vec{F}$ nonzero for $\vec{B}$ and $\vec{v} \neq 0$. But what about frame where $\vec{v} = 0$?
 First return to relativity to learn how to "add" two velocities.

Einstein's velocity addition rule.

Particle moves distance dx in time dt as measured in S .

velocity $u = \frac{dx}{dt}$

$\bar{S}$ moving with vel. v in x -dir.

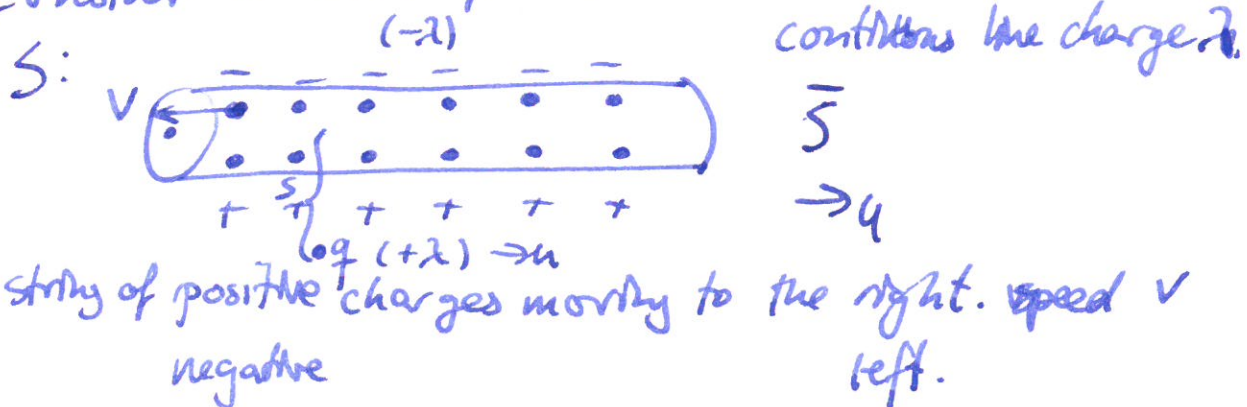
In $\bar{S}$, it has moved $d\bar{x} = \gamma(dx - vdt)$

In an interval $d\bar{t} = \gamma(dt - \frac{v}{c^2}dx)$.

The velocity in $\bar{S}$ is therefore

$$\bar{u} = \frac{d\bar{x}}{d\bar{t}} = \frac{\gamma(dx - vdt)}{\gamma(dt - \frac{v}{c^2}dx)} = \frac{(dx/dt - v)}{1 - v/c^2 dx/dt} = \frac{u - v}{1 - uv/c^2}$$

Consider a model of a wire:



Net current to the right $I = 2\lambda v$.

A distance S away there is a point charge q travelling L14, p3
to the right at speed $u < v$.

No electrical force (since no electrical field - the charges in the wire cancel) [But magnetic field]

Let us examine the situation from $\bar{S}$, moving to the right with speed $\bar{u}$. q is at rest $\therefore$ no magnetic force

Velocities of the positive and negative charges are

$$V_{\pm} = \frac{v \mp u}{1 \mp vu/c^2}$$

$V_- > V_+$ $\therefore$ Lorentz contractions of spacing between negative charges is more severe than that between pos. charges $\therefore$ more negative charges than positive for a given length of wire (in this frame).

$$\lambda_{\pm} = \pm (\gamma_{\pm}) \lambda_0$$

where $\gamma_{\pm} = \frac{1}{\sqrt{1 - V_{\pm}^2/c^2}}$

λ_0 : charge density in the rest frame of the charges.
($\neq \lambda$, since in S the charges move with speed v).

$$\lambda = \gamma \lambda_0, \gamma$$

concentrate on γ_+ :

L14p4

$$\gamma_{\pm} = \frac{1}{\sqrt{1 - \frac{1}{c^2} (v-u)^2}} = \frac{c^2 - uv}{\sqrt{(c^2 - uv)^2 - c^2 (v-u)^2}}$$

$$= \frac{c^2 - uv}{\sqrt{(c^2 - v^2)(c^2 - u^2)}} = \gamma(v) \frac{1 - uv/c^2}{\sqrt{1 - u^2/c^2}}$$

Sim. for γ_-

$$\lambda_{\text{tot}} = \lambda_+ - \lambda_- = \lambda_0 (\gamma_+ - \gamma_-) = \frac{-2\lambda_0 uv}{c^2 \sqrt{1 - u^2/c^2}}$$

Concl: Unequal Lorentz contractions between positive and negative charges (because of direction of current), a ~~wire~~ current-carrying wire that is neutral in one frame will be charged in another!

The charge sets up an electric field at the pos. of particle

$$E = \frac{\lambda_{\text{tot}}}{2\pi\epsilon_0 s}$$

Electric force on q !

$$\vec{F} = q\vec{E} = -\frac{\lambda_0}{\pi\epsilon_0 c^2 s} \frac{qu}{\sqrt{1 - u^2/c^2}}$$

But if there is a force on q in $\bar{S}$, then there must also be a force on q in S . (Both are IF).

Remember how forces transform [when particle at rest in S]

$$\vec{F}_{\perp} = \frac{1}{\gamma} \vec{F}_{\perp}, \quad \vec{F}_{\parallel} = \vec{F}_{\parallel}$$

Since q is at rest in $\bar{S}$, and $\bar{F} \perp u$, the force in S is L14p5

$$F = \gamma u \bar{F} = \sqrt{1 - u^2/c^2} \bar{F} = -\frac{2V}{\pi \epsilon_0 c^2} \frac{qu}{S}$$

The charge is attracted towards the wire by a force that is purely electrical in $\bar{S}$, but non-electrical in S (where the wire is neutral).

Electrostatics + relativity imply another force:
magnetic force.

Use $c^2 = \frac{1}{\epsilon_0 \mu_0}$, and $I = 2IV$, we find indeed

$$F = -qu \left(\frac{\mu_0 I}{2\pi S} \right)$$

Equivalent to $\bar{F} = q \bar{v} \times \bar{B}$ used in system S !

Relativistic Electrodynamics

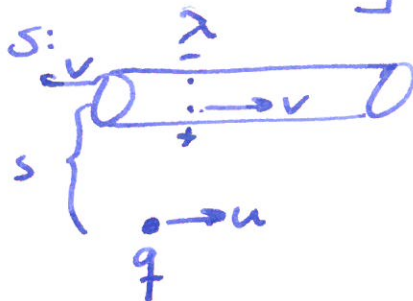
Summary of last lecture.

Electrostatics + relativity $\rightarrow$ magnetic forces.

Let us pretend we only know first part of Lorentz force

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

Point charge and wire
uncharged, carrying
current. $I = 2\lambda v$

 $\vec{S} \rightarrow u$

q at rest. No magnetic force.

diff. Lorentz contraction of positive and negative charge density.
 $\Rightarrow$ wire is charged in S' ! Electric field

$$\vec{F} = q\vec{E} = -\frac{\lambda v}{\pi \epsilon_0 c^2 s} \frac{qu}{\sqrt{1 - u^2/c^2}}$$

We know how to translate forces between IF.

When Particle is at rest in S' : $\vec{F}_\perp = \frac{1}{\gamma} F_\perp$, $\vec{F}_\parallel = F_\parallel$.In this case, particle at rest in S .

$$F_\perp = \frac{1}{\gamma} \vec{F}_\perp, F_\parallel = \vec{F}_\parallel.$$

$$= -\frac{\lambda v}{\pi \epsilon_0 c^2} \frac{qu}{s} = -qu \left(\frac{\mu_0 I}{2\pi s} \right)$$

ie result of using $\vec{F} = q\vec{v} \times \vec{B}$!
with the $\vec{B}$ field of Biot-Savart law!

12.3.2. Lorentz-transformations of the E-M fields.

L5, p2

Given $\vec{E}, \vec{B}$ in S , what are $\vec{E}', \vec{B}'$ in S' ?

Input from physics: Charge is Lorentz-invariant.

Assumption: Transformation laws for the fields are independent of how the fields were produced (the sources).

Will use this to setup sources that we understand how transform.

Also, assume the transformation is linear: Investigate each component independently. Consider ^{an arbitrary} a general $\vec{E}$ at a point in

space-time. We want to know $\vec{E}'$ in a different IF S' .

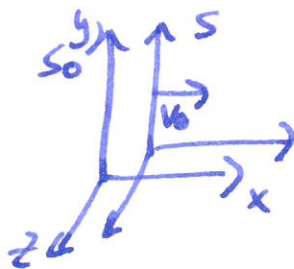
IF S_0 . Let us pretend each component of $\vec{E}$ was generated by simple ~~cap~~ parallel plate capacitors.

Start by considering the situation where the capacitor is at rest in S_0 and plates are ⁱⁿ $x-z$ plane. i.e. y -comp. of $\vec{E}$.

$$\vec{E}_0 = \frac{\sigma_0}{\epsilon_0} \hat{y} \quad \sigma_0: \text{surface charge in } S_0.$$

In S , plates are moving to the left, but the field still takes the form

$$\vec{E} = \frac{\sigma}{\epsilon_0} \hat{y}$$



Surface charge is increased due to Lorentz contraction of the moving plates $\sigma = \gamma(v_0) \sigma_0$.

So in this case $\vec{E}'^\perp = \gamma(v_0) \vec{E}_0^\perp$.

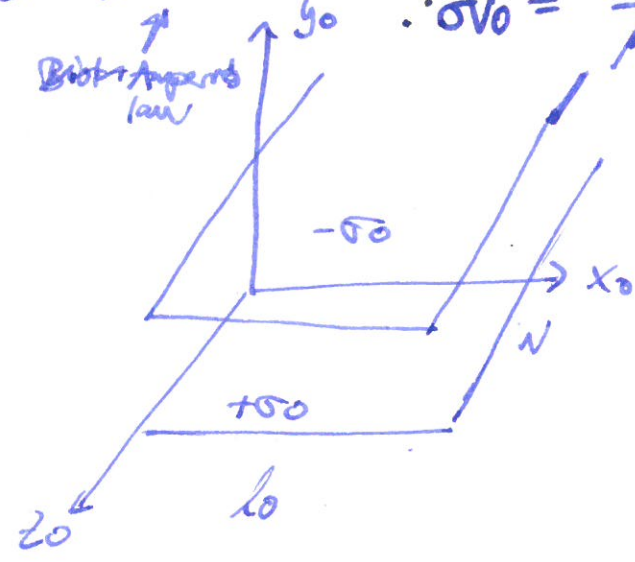
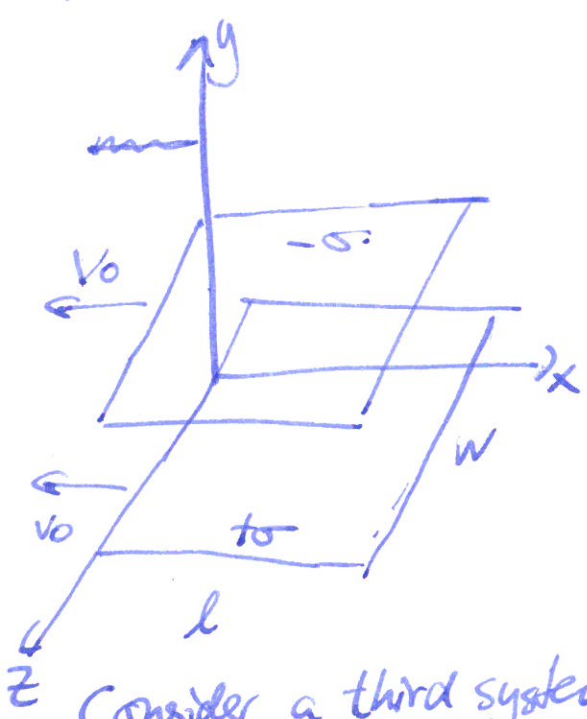
[one would have got same result by having plates parallel to $x-y$ plane and consider field in z -direction].

Not the most general transformation rules, since we constructed a setup with no magnetic field. Will now generalise.

We note that in S , there will be both $\vec{E}$ and $\vec{B}$ - fields!
 Magnetic field generated due to the surface currents from the moving capacitors $\vec{K}_{\pm} = \mp \sigma v_0 \hat{x}$.

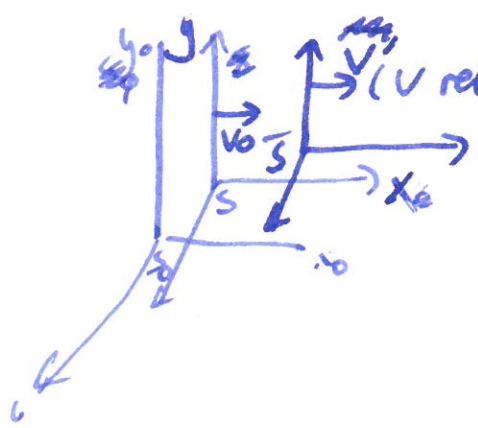
Magnetic field generated by this surface current will lie in the plane of the capacitor $B_z = -\frac{\mu_0 \sigma v_0}{2}$ [right-hand rule]

$\therefore \sigma v_0 = -\frac{B_z}{\mu_0}$



Consider a third system S' , travelling to the right with speed v relative to S :

in S' we would have



$E'_y = \frac{\sigma'}{\epsilon_0}$, $B'_z = -\mu_0 \sigma' V$

$V = \frac{v + v_0}{1 + vv_0/c^2}$, $\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$

$\sigma' = \gamma \sigma$

Express $\bar{E}', \bar{B}'$ in terms of $\bar{E}, \bar{B}$.

$$E'_y = \left(\frac{r'}{r_0}\right) \frac{\sigma}{\epsilon_0} \quad B'_z = -\left(\frac{r'}{r_0}\right) \mu_0 \sigma v'$$

One finds $\frac{r'}{r_0} = \frac{\sqrt{1 - v_0^2/c^2}}{\sqrt{1 - v'^2/c^2}} = \gamma(v) \left(1 + \frac{v v_0}{c^2}\right)$.

Therefore $E'_y = \gamma \left(1 + \frac{v v_0}{c^2}\right) \frac{\sigma}{\epsilon_0} = \gamma \left(E_y - \frac{v}{c^2 \epsilon_0 \mu_0} B_z\right)$

[using $\sigma v_0 = -\frac{B_z}{\mu_0}$]

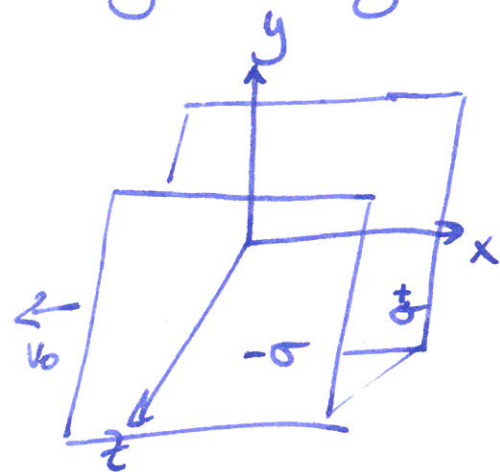
$$B'_z = -\gamma \left(1 + \frac{v v_0}{c^2}\right) \mu_0 \sigma \left(\frac{v + v_0}{1 + v v_0/c^2}\right) = \gamma (B_z - \mu_0 \epsilon_0 v E_y).$$

Using $\mu_0 \epsilon_0 = 1/c^2$ we find

$$E'_y = \gamma (E_y - v B_z)$$

$$B'_z = \gamma \left(B_z - \frac{v}{c^2} E_y\right).$$

We now know how E_y, B_z transform. Get E_z and B_y similarly by considering them generated by a capacitor || to xy-plane.



$$E_z = \frac{\sigma}{\epsilon_0} \quad B_y = \mu_0 \sigma v_0$$

right-hand rule: diff. sign to earlier

$$E'_z = \gamma (E_z + v B_y)$$

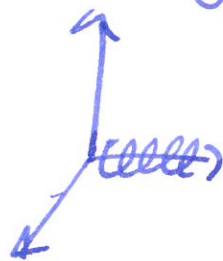
$$B'_y = \gamma \left(B_y + \frac{v}{c^2} E_z\right)$$

x-comp: If capacitor $\parallel$ to yz-plane, no change in surface charge when boosting in x-direction:

$$\vec{E}_x' = E_x$$

But this setup ~~carries no $\vec{B}$ field~~ does not tell us how to transform B_x -~~field~~ component. (since no B_x -comp. in S).

Consider instead the B_x -component generated by solenoid pointing in x-direction



$$B_x = \mu_0 n I$$

winding # per unit length

$n' = \gamma n$, but also time dilates.

$$I' = \frac{1}{\gamma} I \quad (I \text{ is charge per unit time})$$

time in S' runs slower by factor $\gamma(v)$, $\frac{d}{dt'} = \frac{1}{\gamma} \frac{d}{dt}$

Two factors of γ cancel. so $B_x' = B_x$!

$\vec{E}_{\parallel} \vec{B}_{\parallel}$ unchanged!

$$\vec{E}_x' = E_x \quad E_y' = \gamma(E_y - v B_z) \quad E_z' = \gamma(E_z + v B_y)$$

$$B_x' = B_x \quad B_y' = \gamma(B_y + \frac{v}{c^2} E_z) \quad B_z' = \gamma(B_z - \frac{v}{c^2} E_y)$$

Special cases! If $\vec{B} = 0$ in S : $\vec{B}' = \gamma \frac{v}{c^2} (E_z \hat{y} - E_y \hat{z}) = \frac{v}{c^2} (E_z \hat{y} - E_y \hat{z})$

$$\Rightarrow \vec{B}' = -\frac{1}{c^2} (\vec{v} \times \vec{E}')$$

2! If $\vec{E} = 0$ in S : $\vec{E}' = \vec{v} \times \vec{B}'$.

Relativistic Electrodynamics

L16, p1

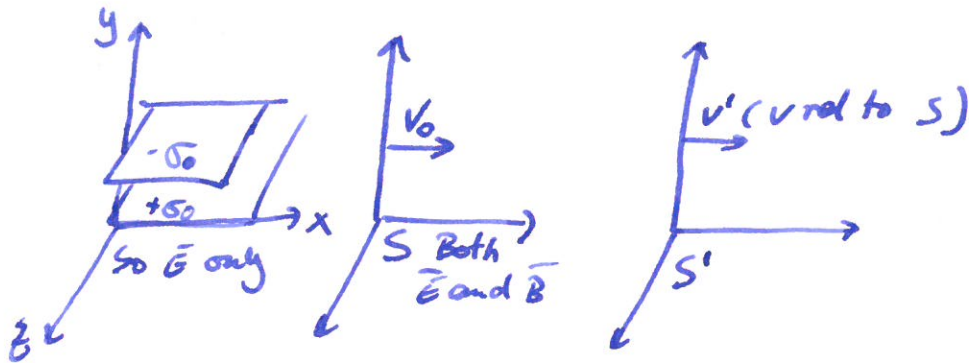
Summary of last lecture.

Derived the rules for Lorentz-transforming the $\vec{E}, \vec{B}$ fields.

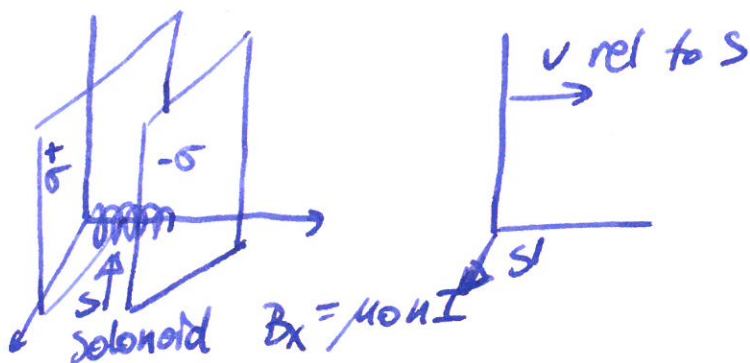
Derived setups for generating $\vec{E}, \vec{B}$ fields.

Studied the modification under Lorentz boosts.

For components $\perp$ to boost:



For Comp. $\parallel$ to boost



$$n' = \gamma n$$

$I' = \frac{1}{\gamma} I$ (I is charge per unit time)
When moving solenoid is analysed in S' , then
the time of the comoving (solenoid's)
clock is running slower.

$$E'_x = E_x \quad E'_y = \gamma(E_y - v B_z) \quad E'_z = \gamma(E_z + v B_y)$$

$$B'_x = B_x \quad B'_y = \gamma(B_y + \frac{v}{c^2} E_z) \quad B'_z = \gamma(B_z - \frac{v}{c^2} E_y)$$

Special cases: If $\vec{B} = 0$ in S : $\vec{B}' = -\frac{1}{c^2} (\vec{v} \times \vec{E})$

If $\vec{E} = 0$ in S : $\vec{E}' = \vec{v} \times \vec{B}'$

L16, p2

$\vec{E}$ and $\vec{B}$ do not transform like the spatial components of a 4-vector - they even mix under Lorentz-transformations. Can we find a Lorentz-covariant object with 6 components, which transforms like $\vec{E}\vec{B}$? A: Antisymmetric, rank-2 tensor!

Remember 4-vector transforms like

$$x^\mu \rightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

For boost along x-axis summed

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cosh y & -\sinh y & 0 & 0 \\ -\sinh y & \cosh y & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

↑
1st index: row

2nd index: column

$y = \text{rapidity}$

A rank-2 tensor is an object with two free indices, where the elements transform as

$$t^{\mu\nu} \rightarrow t'^{\mu\nu} = \Lambda^\mu{}_\sigma \Lambda^\nu{}_\lambda t^{\sigma\lambda}$$

can display the tensor as a matrix
represent

$$t^{\mu\nu} = \begin{pmatrix} t^{00} & t^{01} & t^{02} & t^{03} \\ t^{10} & t^{11} & t^{12} & t^{13} \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & t^{33} \end{pmatrix}$$

Symmetric tensor: $t^{\mu\nu} = t^{\nu\mu}$ 10 distinct components

Anti-symmetric tensor: $t^{\mu\nu} = -t^{\nu\mu}$ 6 distinct components.

L16, p3

Anti-symmetric $t^{\mu\nu} = \begin{pmatrix} 0 & t^{01} & t^{02} & t^{03} \\ -t^{01} & 0 & t^{12} & t^{13} \\ -t^{02} & -t^{12} & 0 & t^{23} \\ -t^{03} & -t^{13} & -t^{23} & 0 \end{pmatrix}$

Check out transformation rule for a few components

$$t'^{01} = \Lambda^0_{\lambda} \Lambda'^1_{\sigma} t^{\lambda\sigma}$$

Consider boost along x-axis (like usual). Then $\Lambda^0_{\lambda} = 0$ unless $\lambda \in \{0, 1\}$. $\Lambda'^1_{\sigma} = 0$ unless $\sigma \in \{0, 1\}$. Four terms only.

$$t'^{01} = \Lambda^0_0 \Lambda'^1_0 t^{00} + \Lambda^0_0 \Lambda'^1_1 t^{01} + \Lambda^0_1 \Lambda'^1_0 t^{10} + \Lambda^0_1 \Lambda'^1_1 t^{11}$$

But for the anti-symmetric tensor $t^{00} = t^{11} = 0$ and $t^{10} = -t^{01}$.

$$\text{so } t'^{01} = (\Lambda^0_0 \Lambda'^1_1 - \Lambda^0_1 \Lambda'^1_0) t^{01} = (\cosh^2 \eta - \sinh^2 \eta) t^{01} = t^{01}.$$

Can infer the components by comparison and construct the electromagnetic field tensor $F^{\mu\nu}$

$$\left[\begin{array}{lll} t'^{01} = t^{01} & t'^{02} = \gamma(t^{02} - \beta t^{12}) & t'^{03} = \gamma(t^{03} + \beta t^{31}) \\ t'^{23} = t^{23} & t'^{31} = \gamma(t^{31} + \beta t^{03}) & t'^{12} = \gamma(t^{12} - \beta t^{02}) \end{array} \right]$$

Sign conventions etc.

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -cB_z & cB_y \\ E_y & cB_z & 0 & -cB_x \\ E_z & -cB_y & cB_x & 0 \end{pmatrix}$$

Combines $\vec{E}$ and $\vec{B}$ into a single object. Clearly, $\vec{E}, \vec{B}$ are interrelated - two ~~same~~ faces of the same coin.

Actually, ~~to~~ we could have related the elements differently L16p4

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & cB_x & cB_y & cB_z \\ -cB_x & 0 & -E_z & E_y \\ -cB_y & E_z & 0 & -E_x \\ -cB_z & -E_y & E_x & 0 \end{pmatrix}.$$

Dual field strength tensor.

Still seems like a lot of signs to remember.

^{next week} Will introduce E/M in tensor notation; this will take care of signs etc.

Even though $\vec{E}, \vec{B}$ change under LT, some combinations are invariant

$F^{\mu\nu} F_{\mu\nu}$ is Lorentz scalar.

Note: This is not on the form of a matrix multiplication. to get it on that form:

$$F^{\mu\nu} F_{\mu\nu} = -F^{\rho\mu} F_{\mu\rho}$$

Now it is the trace of a matrix multiplication.

one finds $\frac{1}{2} F^{\mu\nu} F_{\mu\nu} = \vec{E}^2 - c^2 \vec{B}^2$ invariant!

Similarly $F_{\mu\nu} \tilde{F}^{\mu\nu} = -4c \vec{E} \cdot \vec{B}$ invariant!

No further LI quantities $\tilde{F}^{\mu\nu} \tilde{F}_{\mu\nu} = \vec{E}^2 - c^2 \vec{B}^2$

$$g_{\mu\nu} F^{\mu\nu} = 0 \text{ etc.}$$

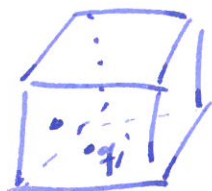
Two invariants: $\vec{E}^2 - c^2 \vec{B}^2$
 $\vec{E} \cdot \vec{B}.$

Consequences:
if $\vec{E} \perp \vec{B}$ in one IF, then in all IF
if $\vec{E} \parallel \vec{B}$ in one IF, then in all IF
if in one IF: $\vec{E} \neq 0, \vec{B} = 0$ then there is no IF with $\vec{E} = 0, \vec{B} \neq 0$ and vice versa

4-current density

q_i style point charge : $j_i^\mu \equiv q_i u^\mu = q_i \gamma (c, \vec{v})$ 4-vector.

Volume element with many point charges



$$\text{At rest } \rho_0 = \frac{Q_{\text{tot}}}{V} = \frac{\sum_i q_i}{V}$$

↑
volume

Moving



$$\rho = \frac{Q_{\text{tot}}}{V'} = \frac{\sum_i q_i}{V/\gamma} = \gamma \rho_0$$

$$j^\mu \equiv \underbrace{\rho_0}_{\text{scalar}} \underbrace{\gamma (c, \vec{v})}_{\text{4-vector}} = \rho (c, \vec{v}) = (\rho c, \vec{j})$$

4-current density

Continuity equations

$$\frac{\partial}{\partial t} (\vec{\nabla} \cdot \vec{E}) = \frac{1}{\epsilon_0} \frac{\partial}{\partial t} \rho$$

$$\vec{\nabla} \cdot \vec{\nabla} \times \vec{B} = 0 = \mu_0 \epsilon_0 \frac{\partial}{\partial t} \vec{\nabla} \cdot \vec{E} + \mu_0 \vec{\nabla} \cdot \vec{j} \quad \left. \vphantom{\frac{\partial}{\partial t} \vec{\nabla} \cdot \vec{E}} \right\} \Rightarrow \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0.$$

$$\frac{\partial}{\partial t} Q_{\text{tot}} = \frac{\partial}{\partial t} \int_{\text{Volume}} \rho \cdot dV = - \int_{\text{Surface}} (\vec{\nabla} \cdot \vec{j}) dV = - \int_{\text{Surface}} \vec{j} \cdot d\vec{a}$$

change of ^{total} charge inside surface happens only ^{surface} (Klein)
by flow through surface (charge does not disappear into nothing).

$$\partial_\mu j^\mu = 0 \quad \partial_\mu = \left(\frac{1}{c} \frac{\partial}{\partial t}, -\vec{\nabla} \right), \quad \partial_\mu = \left(\frac{1}{c} \frac{\partial}{\partial t}, +\vec{\nabla} \right)$$

The 4-potential $A^\mu = (\phi, c\vec{A})$.

Is it a 4-vector? Demonstrate that the Lorentz-transform of A^μ leads to the correct $\vec{E}, \vec{B}$ as derived in the previous lectures.

Gauge transformations: A^μ not unique for given $\vec{E}, \vec{B}$.

In 4-vector notation $A^\mu \rightarrow A^\mu - \partial^\mu \phi$.

Choose gauge freedom to fix $\vec{\nabla} \cdot \vec{A}$.

Lorenz gauge: $\vec{\nabla} \cdot \vec{A} = -\frac{1}{c^2} \frac{\partial \phi}{\partial t}$. Retarded potentials.

In 4-vector talk: $\boxed{\partial_\mu A^\mu = 0}$

Let us rewrite the ~~Maxwell~~ ^{Maxwell} equations in ~~4-vector notation~~ ^{covariant} 4-vector notation.

M1: $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$. Plug in $\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$:

$$\cancel{-\vec{\nabla} \cdot \vec{\nabla} \phi} - \vec{\nabla} \cdot \vec{\nabla} \phi - \frac{\partial}{\partial t} \underbrace{\vec{\nabla} \cdot \vec{A}}_{= -\frac{1}{c^2} \frac{\partial \phi}{\partial t} \text{ in Lorenz gauge}} = \frac{1}{\epsilon_0} \rho$$

$$\therefore -\nabla^2 \phi + \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = \rho/\epsilon_0$$

$$M4: \vec{\nabla} \times \vec{B} = \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{j} = \frac{1}{c^2} \frac{\partial}{\partial t} \left(-\vec{\nabla} \phi - \frac{\partial \vec{A}}{\partial t} \right) + \mu_0 \vec{j}$$

$$\text{Use now } \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} = -\frac{1}{c^2} \vec{\nabla} \frac{\partial \phi}{\partial t} - \nabla^2 \vec{A}$$

$$\therefore -\nabla^2 \vec{A} + \frac{1}{c^2} \frac{\partial \vec{A}}{\partial t} = \mu_0 \vec{j} = \frac{1}{c^2 \epsilon_0} \vec{j}$$

But the operator $-\nabla^2 + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} = \partial_\mu \partial^\mu$

2 Maxwell

Wave eqⁿ take the form $\underbrace{\partial_\mu \partial^\mu}_{4\text{-vector}} \underbrace{A^\nu}_{\text{4-vector}} = \frac{1}{c\epsilon_0} \underbrace{j^\nu}_{\text{4-vector}}$

$\nu = 0, 1, 2, 3$:
4 eqⁿ.

The Field-Strength Tensor

We showed that the construct $F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -cB_z & cB_y \\ E_y & cB_z & 0 & -cB_x \\ E_z & -cB_y & cB_x & 0 \end{pmatrix}$ transforms as a rank-2 tensor. L17p3

We now note that $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$!

No more signs to remember. Only the def of $\vec{E}$ & $\vec{B}$ in terms of potentials.

$F^{\mu\nu}$ obviously anti-symmetric: $F^{\mu\nu} = -F^{\nu\mu}$

Consider $F^{\mu\nu}$ in Lorenz gauge gauge choice.
↓ (Lorenz condition)

$$\partial_\mu F^{\mu\nu} = \partial_\mu \partial^\mu A^\nu - \underbrace{\partial_\mu \partial^\nu A^\mu}_{= \partial^\nu \partial_\mu A^\mu} = \frac{1}{c\epsilon_0} j^\nu = 0$$

$\therefore$ The inhomogeneous Maxwell equations can be written as

$$\partial_\mu F^{\mu\nu} = \frac{1}{c\epsilon_0} j^\nu$$

$$\begin{aligned} \text{Find elements: } F^{01} &= \partial^0 A^1 - \partial^1 A^0 = \frac{1}{c} \frac{\partial}{\partial t} (cA_x) - \left(-\frac{\partial}{\partial x}\right) \phi \\ &= \left(\nabla \phi + \frac{\partial \vec{A}}{\partial t}\right)_x = -E_x. \end{aligned}$$

$$F^{12} = \partial^1 A^2 - \partial^2 A^1 = \frac{\partial}{\partial x} cA_y - \left(-\frac{\partial}{\partial y} cA_x\right) = \frac{1}{c} (\nabla \times \vec{A})_z = -cB_z.$$

Def. Dual field strength tensor as

$$\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \quad (\text{or } \tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma})$$

$\epsilon^{\mu\nu\rho\sigma}$ the 4-dim Levi-Civita tensor: Totally anti-symmetric

$\epsilon_{\mu\nu\rho\sigma} = -\epsilon_{\nu\mu\rho\sigma} = +\epsilon_{\nu\rho\mu\sigma} \dots$. Two labels identical: $\epsilon = 0$

$\epsilon_{0123} = -\epsilon^{0123} = +1$ by convention.

$$\tilde{F}^{01} = \frac{1}{2} (\epsilon^{0123} F_{23} + \epsilon^{0132} F_{32}) = \frac{1}{2} (-F_{23} + F_{32}) = -F_{23}$$

$$= -F^{23} = cB_x$$

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & cB_x & cB_y & cB_z \\ -cB_x & 0 & -E_z & E_y \\ -cB_y & E_z & 0 & -E_x \\ -cB_z & -E_y & E_x & 0 \end{pmatrix}$$

anti-symmetric μ, ν

Consider $\partial_\mu \tilde{F}^{\mu\nu} = \partial_\mu \underbrace{\frac{1}{2} \epsilon^{\mu\nu\rho\sigma}}_{\substack{\text{symmetric } \mu, \rho \\ \text{symmetric } \mu, \sigma}} (\partial_\rho A_\sigma - \partial_\sigma A_\rho) = 0!$

$$\partial_\mu F^{\mu\nu} = \frac{1}{c\epsilon_0} j^\nu \quad \text{inhomogeneous Maxwell eqn.}$$

$$\partial_\mu \tilde{F}^{\mu\nu} = 0 \quad \text{homogeneous Maxwell eqn.}$$

$$\partial_\mu F^{\mu\nu} = \frac{j^\nu}{c\epsilon_0} \begin{cases} \nu=0: & \vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0 \\ \nu=1,2,3: & \vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{j} \end{cases}$$

$$\partial_\mu \tilde{F}^{\mu\nu} = 0 \begin{cases} \nu=0: & \vec{\nabla} \cdot \vec{B} = 0 \\ \nu=1,2,3: & \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \end{cases} \Rightarrow \text{We can use potentials } A^\mu \text{ to get the fields.}$$

Using covariant notation

$$\frac{d\vec{p}}{dt} = q(\vec{E} + \frac{\vec{v}}{c} \times \vec{B})$$

$$f^{\mu} = \frac{dp^{\mu}}{d\tau} = \frac{q}{c} F^{\mu\nu} u_{\nu}$$

$$p^* = (p_0, \bar{p}) = \arg \min (E/C, \bar{p})$$

$$\frac{d\mathbf{p}'}{d\tau} = \left(\frac{d\mathbf{E}}{dt}, \frac{d\mathbf{F}}{dt} \right).$$

This equation reduces to Lorentz-force law $u^\mu = j(c, \vec{v})$

$$\vec{F} = \frac{d\vec{p}}{dt} = q(\vec{E} + \vec{v} \times \vec{B}) \quad \frac{d\vec{E}}{dt} = q \underbrace{\vec{E} \cdot \vec{v}}_{\text{electric field}} \text{ velocity.}$$

Simple special cases:

① $\vec{E} = 0$, uniform $\vec{B}$ -field. Energy conserved! No work from external field on particle.

$\Rightarrow v \equiv |\vec{v}|$ constant. If $\vec{v} \perp \vec{B}$: circular motion.

$$\vec{F} = \frac{d\vec{p}}{dt} = q \vec{v} \times \vec{B} = \frac{d}{dt} (m_f \vec{v}) = m_f \frac{d\vec{v}}{dt}$$

$\frac{dI}{dt} = 0$ since W constant.

~~11/11/11~~

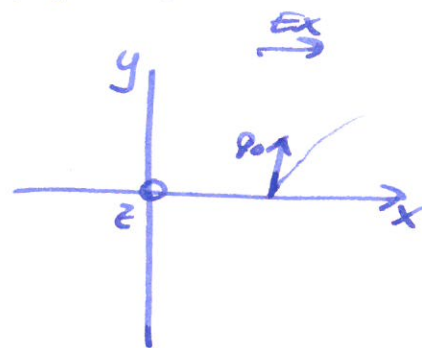
$$\frac{d\phi}{dt} = P \frac{d\theta}{dt} = P \frac{V}{R} \therefore qVB = P \frac{V}{R} \therefore P = qBR.$$

② $\vec{B}=0$, $\vec{E}$ -field.

For $t=0$, point charge of mass m is on x -axis.

with $p(t=0) = (0, p_0, 0)$

given $\vec{E} = (E_x, 0, 0)$, want $\vec{x}(t)$



$$\frac{d\vec{p}}{dt} = q\vec{E} \quad \therefore \dot{p}_x = qE_x, \quad \dot{p}_y = 0, \quad \dot{p}_z = 0$$

$$p_x(t) = qE_x t, \quad p_y(t) = p_0, \quad p_z(t) = 0$$

Next, determine $\vec{v}$ from $\vec{p}$

$$\vec{p} = m\gamma\vec{v}, \quad E = mc^2 \quad \therefore v = \frac{pc^2}{E} = \frac{pc^2}{\sqrt{m^2c^4 + c^2p_0^2 + (cqE_x t)^2}}$$

$$v_x = \frac{dx}{dt} = \frac{p_x c^2}{\sqrt{m^2c^4 + p^2c^2}} = \frac{qE_x t c^2}{\sqrt{m^2c^4 + c^2p_0^2 + (cqE_x t)^2}} \rightarrow c \text{ for } t \rightarrow \infty$$

$$v_y = \frac{dy}{dt} = \frac{p_y c^2}{\sqrt{m^2c^4 + p^2c^2}} = \frac{p_0 c^2}{\sqrt{m^2c^4 + c^2p_0^2 + (cqE_x t)^2}} \rightarrow 0 \text{ for } t \rightarrow \infty$$

Determine $\vec{x}$ from $\vec{v}$.

$$x(t) = \int v_x(t) dt = \frac{1}{qE_x} \sqrt{m^2c^4 + c^2p_0^2 + (cqE_x t)^2} + \text{const.}$$

$$y(t) = \int v_y(t) dt = \frac{p_0 c}{qE_x} \text{Arctanh} \frac{cqE_x t}{\sqrt{m^2c^4 + c^2p_0^2}} + 0$$

combine
$$x = \frac{1}{qE_x} \sqrt{m^2c^4 + c^2p_0^2} \cosh \frac{qE_x y}{p_0 c}$$

Hyperbolic motion!

Relativistic Electrodynamics

L18pt

Summary of last lecture

4-potential $A^\mu = (\phi, c\vec{A})$

Gauge-transformation: $A^\mu \rightarrow A^\mu - \partial^\mu \psi$

Lorenz gauge $\vec{\nabla} \cdot \vec{A} = -\frac{1}{c^2} \frac{\partial \phi}{\partial t} \therefore \partial_\mu A^\mu = 0$.

M1 & M4: $\partial_\mu \partial^\mu A^\nu = \frac{1}{c\epsilon_0} j^\nu$ [Lorenz gauge - Lorentz inv.]

Field strength tensor $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

$\partial_\mu F^{\mu\nu} = \frac{1}{c\epsilon_0} j^\nu$ [M1 & M4]

Dual Field strength tensor $\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$

$\epsilon^{\mu\nu\rho\sigma}$: 4-dim Levi-Civita tensor. Totally antisymmetric.

Two labels identical value: $\epsilon = 0$, $\epsilon_{0123} = -\epsilon^{0123} = 1$

$\partial_\mu \tilde{F}^{\mu\nu} = 0$: Homogeneous Maxwell eqn.

Maxwell's eqn done - now just need Lorentz Force in cov. notation.

Lorentz Force in covariant notation.

$$f^\mu = \frac{dp^\mu}{d\tau} = \frac{q}{c} F^{\mu\nu} u_\nu$$

$$p^\mu = (p_0, \vec{p}) = (E/c, \vec{p})$$

$$\frac{dp^\mu}{d\tau} = \left(\frac{1}{c} \frac{dE}{dt}, \gamma \vec{F} \right)$$

$$u^\mu = \gamma(c, \vec{v})$$

$$F^{\mu\nu} u_\nu = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -cB_z & cB_y \\ E_y & cB_z & 0 & -cB_x \\ E_z & -cB_y & cB_x & 0 \end{pmatrix} \begin{pmatrix} \gamma c \\ -\gamma v_x \\ -\gamma v_y \\ -\gamma v_z \end{pmatrix}$$

L18p2

$$= \left(\gamma \bar{E} \cdot \bar{v}, \gamma c [\bar{E} - \bar{B} \times \bar{v}] \right)$$

$\bar{E} + \bar{v} \times \bar{B}$

$$\text{So } \left(\frac{1}{c} \frac{dE}{dt}, \gamma \bar{F} \right) = \left(\frac{q}{c} \gamma \bar{E} \cdot \bar{v}, q \gamma [\bar{E} + \bar{v} \times \bar{B}] \right)$$

0-component $\frac{dE}{dt} = q \bar{E} \cdot \bar{v}$

Rel. Energy of point particle

electric field.

1,2,3-component: Lorentz Force.

Special cases) $\bar{E} = 0$, (uniform $\bar{B}$):

$$\bar{E} = 0 \therefore \frac{dE}{dt} = 0 \text{ i.e. kin energy conserved.}$$

$$\therefore v = |\bar{v}| \text{ is constant.}$$

A little taste of things to come

L1B, p3

Lagrangian and Hamiltonian for a Rel. particle

Recall: Lagrangian $L(\vec{x}, \dot{\vec{x}})$ ~~is a function of~~ is a function of
(generalised) ~~Cartesian~~ coord.

$$\text{Action } S = \int_{t_1}^{t_2} L(\vec{x}, \dot{\vec{x}}) dt.$$

The true path (in conf. space) is the one which minimises the action.

$$\text{Generalised momentum: } p = \frac{\partial L}{\partial \dot{x}}$$

$$\text{Lagrange eqn: } \frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = 0.$$

Energy function (Hamiltonian)

$$H = \dot{x} \frac{\partial L}{\partial \dot{x}} - L.$$

Modifications for relativistic systems:

Action for a point mass. Must be Lorentz invariant

$$S = k \int_a^b ds$$

$$ds = \sqrt{c^2 dt^2 - d\vec{x}^2} = \sqrt{c^2 d\tau^2} = c d\tau$$

k constant to be fixed by NR limit. Recall $d\tau = \frac{dt}{\gamma}$

$$S = k \int_{t_1}^{t_2} \frac{c}{\gamma} dt \quad \text{we can evaluate this in any IF.}$$

↑ time of event as seen in one IF.

$$\therefore L = kc\sqrt{1-v^2/c^2} \stackrel{\text{NR limit}}{\approx} kc - \underbrace{k\frac{v^2}{2c}}_{\text{set } k = -mc} + \mathcal{O}(v^4/c^4) \quad \underline{L18, p1}$$

Constant term does not affect equation of motion
Drop it.

So for a free point mass

$$L = -mc^2\sqrt{1-v^2/c^2} = -mc^2/\gamma.$$

$$\& S = -mc \int_a^b ds.$$

Momentum

$$\vec{p} = \frac{\partial L}{\partial \dot{\vec{x}}} = \frac{\partial L}{\partial \vec{v}} = \frac{\partial}{\partial \vec{v}} (-mc^2\sqrt{1-v^2/c^2}) = m\gamma\vec{v}.$$

Lagrange eqn $\frac{\partial L}{\partial \vec{x}} = \frac{d}{dt} \frac{\partial L}{\partial \dot{\vec{x}}} \therefore \dot{\vec{p}} = 0$. [free particle].

Now must add potential. Must be Lorentz invariant.
(EM)

$$S = \int_a^b \left(-mc ds - \underbrace{\frac{q}{c} A_\mu dx^\mu}_{\text{constant fixed by NR limit.}} \right)$$

$$= \int_a^b \left(-mc ds - \frac{q}{c} \phi c dt + \frac{q}{c} c \vec{A} \cdot d\vec{x} \right)$$

$$d\vec{x} = \frac{d\vec{x}}{dt} dt = \vec{v} dt.$$

$$= \int_{t_1}^{t_2} \underbrace{(-mc^2 \sqrt{1-v^2/c^2} + q \bar{\mathbf{A}} \cdot \bar{\mathbf{v}} - q \bar{\Phi})} dt$$

L18, p5

L for charged point mass in EM potential.

Generalised momentum is

$$\bar{\mathbf{p}}_{\text{gen}} = \frac{\partial L}{\partial \dot{\bar{\mathbf{x}}}} = \frac{\partial L}{\partial \bar{\mathbf{v}}} = m \bar{\mathbf{v}} + q \bar{\mathbf{A}} = \bar{\mathbf{p}} + q \bar{\mathbf{A}}$$

[Follow through ... End up with the Lorentz force.]

Lagrange eqⁿ $\frac{d}{dt} \frac{\partial L}{\partial \bar{\mathbf{v}}} = \frac{\partial L}{\partial \bar{\mathbf{x}}}$

Can even define a Lagrangian for the EM field itself!

$$S = \int d^4x \mathcal{L}(\bar{A}^\mu, \partial^\mu \bar{A}^\nu)$$

$$\mathcal{L} = -\frac{\epsilon_0}{4} F^{\mu\nu} F_{\mu\nu} \quad ! \quad [\text{in vacuum}]$$

With sources:

$$\mathcal{L} = -\frac{\epsilon_0}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{c} j^\nu A_\nu$$

The Euler-Lagrange eqⁿ then give the Maxwell eqⁿ!

$$\boxed{\partial_\mu F^{\mu\nu} = \frac{1}{c\epsilon_0} j^\nu}$$

This is how Modern Theoretical physics operates:

Symmetry

[Here: Lorentz invariance]

↓
Get Lagrangian

↓
E-L eqⁿ.

↓
solve!