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On the Moduli Space for Strings on Group Manifolds

based on work together with Daniel Roggenkamp:
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Outline

- Introduction
- Boson on a Circle
- SU(2)
- semi simple, compact Group
- Conclusion

Introduction

Often a consistent string background (CFT) belongs to a "family" of consistent backgrounds which can be deformed into each other by contionously changing parameters

Example: Strings on a torus $T^n = U(1)^n$

Parameters which can be changed are in $n \times n$ Matrix $G + B$

equivalent backgrounds related by T-duality $O(n, n, \mathbb{Z})$

Subject of the talk: consider $U(1)^n \subset G$ (semi simple, compact group)

non trivial since no product geometry .

Boson on a Circle ($U(1)_R$)

$$S = \frac{1}{2\pi\alpha'} \int d^2z\; R^2\, \partial_z X \partial_{\bar{z}} X$$

equations of motion

$$\partial_{\bar{z}} J = \partial_z \bar{J} = 0$$

with

$$J = \partial_z X \quad , \quad \bar{J} = \partial_{\bar{z}} X.$$

primaries of dimension $(1,0)$, $(0,1)$, respectively

Response to $R \to R + \delta R$

$$\delta S = \frac{R\delta R}{\pi\alpha'} \int d^2z\; J\bar{J}$$

$\to$ everything can be obtained by marginally deforming the theory at $R = 1$

In less trivial cases $J, \bar{J}$ may depend on "deformation parameter" R.

T-Duality: $R \to \alpha'/R$

Some Trivialities

Coset Formulation:

start with U(1) $\times$ U(1)$_{\tilde{R}}$:

$$S = \frac{1}{2\pi\alpha'} \int d^2z \, \left(\partial_z X \partial_{\bar{z}} X + \tilde{R}^2 \partial_z Y \partial_{\bar{z}} Y \right)$$

Consider coset theory obtained by gauging $X \to X + c, Y \to Y + c$ i.e.

$$\partial_i X \to \partial_i X + A_i \ , \ \partial_i Y \to \partial_i Y + A_i$$

Gauge fix e.g. $Y = 0$ and eliminate A_i by solving its algebraic equation of motion

$$S = \frac{1}{2\pi\alpha'} \int d^2z \, R^2 \, \partial_z X \partial_{\bar{z}} X,$$

with $R^2 = \tilde{R}^2 / \left(1 + \tilde{R}^2\right)$ $\quad \to R \in (0, 1)$

Orbifold Formulation:

let $k \in \mathbb{N}$ and consider orbifold

$$\left(\frac{U(1)_k}{U(1)} \times U(1)_{kR} \right) / \mathbb{Z}_k = U(1)_{kR} / \mathbb{Z}_k = U(1)_R$$

Strings on SU(2)

Sigma-Model action (WZW Model):

$$S^{\mathrm{WZW}} = S^{\mathrm{kin}} + S^{\mathrm{WZ}} = \frac{k}{4\pi}\left[\int_\Sigma L^{\mathrm{kin}} + \int_B \omega^{\mathrm{WZ}}\right], \quad k \in \mathbb{N},\ \partial B = \Sigma$$

with

$$\omega^{\mathrm{WZ}} = \frac{1}{3}\mathrm{Tr}\left(g^{-1}dg\right)^3$$

and

$$L^{\mathrm{kin}} = \mathrm{Tr}\left(\partial_z g \partial_{\bar{z}} g^{-1}\right) \equiv -\left\langle g^{-1}\partial_z g, g^{-1}\partial_{\bar{z}} g\right\rangle$$

parameterize SU(2) group element

$$g = \cos x \cos\tilde{\theta} - i \sin x \sin\theta\, \sigma^1 + i \sin x \cos\theta\, \sigma^2 + i \cos x \sin\tilde{\theta}\, \sigma^3$$

$$\left(x = 0 \ldots \frac{\pi}{2}\,,\ \theta = -\pi \ldots \pi\,,\ \tilde{\theta} = -\pi \ldots \pi\right).$$

This gives

$$\begin{aligned} S^{\mathrm{WZW}} &= \frac{k}{2\pi}\int d^2z \Big\{ \partial_z x \partial_{\bar{z}} x + \sin^2 x\, \partial_z\theta\partial_{\bar{z}}\theta + \\ &\quad + \cos^2 x\, \partial_z\tilde{\theta}\partial_{\bar{z}}\tilde{\theta} + \cos^2 x \left(\partial_z\theta\partial_{\bar{z}}\tilde{\theta} - \partial_z\tilde{\theta}\partial_{\bar{z}}\theta\right)\Big\} \end{aligned}$$

Deformed SU(2) Model

Hassan, Sen 1992; Giveon, Kiritsis 1993; Sfetsos, Tseytlin 1993.

$$S^R = \frac{k}{2\pi}\int d^2z \left\{ \partial_+ x \partial_- x + \frac{\sin^2 x}{\cos^2 x + R^2 \sin^2 x}\partial_+\theta\partial_-\theta + \frac{R^2\cos^2 x}{\cos^2 x + R^2\sin^2 x}\partial_+\tilde{\theta}\partial_-\tilde{\theta} + \frac{\cos^2 x}{\cos^2 x + R^2\sin^2 x}\left(\partial_+\theta\partial_-\tilde{\theta} - \partial_+\tilde{\theta}\partial_-\theta\right)\right\}.$$

(Non-trivial Dilaton to be discussed later) Chiral and Antichiral Currents from EOM
(R dependent)

$$\partial_{\bar{z}} J = \partial_z \bar{J} = 0$$

with

$$J = k\left(\sin^2 x \partial_z\theta - \cos^2 x \partial_z\tilde{\theta}\right) / \left(\cos^2 x + R^2\sin^2 x\right),$$

$$\bar{J} = k\left(\sin^2 x \partial_{\bar{z}}\theta + \cos^2 x \partial_{\bar{z}}\tilde{\theta}\right) / \left(\cos^2 x + R^2\sin^2 x\right).$$

One easily finds response to infinitesimal change in deformation parameter R:

$$S^{R+\delta R} = S^R - \frac{k}{2\pi}\delta R^2 \int d^2z J\bar{J}.$$

Orbifold Picture

From CFT (algebraic) considerations Gepner, Qiu 1987 Yang 1988 Giveon, Kiritsis 1993

$$\text{Deformed Model} = \left(\frac{\mathsf{SU}(2)_k}{\mathsf{U}(1)} \times \mathsf{U}(1)_{kR}\right)/\mathbb{Z}_k$$

Easy to see in sigma model by T-dualizing direction diagonal in Coset times U(1) SF, Roggenkamp 03

Details:

$$S^{\text{Coset}} = \frac{k}{2\pi}\int d^2z \left\{\partial_z x \partial_{\bar z} x + \tan^2 x \partial_z \theta \partial_{\bar z}\theta\right\}$$

$$S^{\mathsf{U}(1)} = \frac{k}{2\pi}\int d^2z \, \frac{1}{R^2}\partial_z y \partial_{\bar z} y$$

T-dualize $\theta + y$ direction:

1st gauge constant shifts in $\theta + y$, i.e.

$$\partial_i\theta \to \partial_i\theta + A_i/2 \ , \ \ \partial_i y \to \partial_i y + A_i/2$$

2nd constrain field strength $F = dA$ to vanish via Lagrange mutliplier, i.e. add

$$\frac{1}{2\pi}\int d^2z \, \lambda F_{z\bar z}$$

$$\left(\begin{array}{l} F = 0 \to A_i = \partial_i\phi \ \ \& \ \ y + \theta \to y + \theta - \phi \\ \to \text{ungauged model}\end{array}\right)$$

3rd Implement Orbifold by 'global twist' on A

Roček, Verlinde 1992

add topological term: $S^{\text{top}} = -\frac{1}{2\pi}\int d\,(\lambda A)$

specify compactification radius of λ:

$$\lambda \equiv \lambda + 2\pi k$$

on torus-worldsheet topological term gives:

$$S^{\text{top}} = k\, n_a \oint_b A + k\, n_b \oint_a A$$

For $A = d\phi$ sum over winding modes yields

$$\phi \equiv \phi + 2\pi/k \longrightarrow \theta + y \equiv \theta + y + 2\pi/k$$

4th Integrating out A instead of λ gives deformed action S^R wit $\lambda = k\tilde{\theta}$

<u>Coset Picture</u>

$$\text{Deformed Model} = \left(\text{SU}(2)_k \times \text{U}(1)_{\tilde{R}}\right)/U(1)$$

Sfetsos, Tseytlin 1993

Strings on G

$G =$ semisimple, compact group

recall

$$S^{\text{WZW}} = S^{\text{kin}} + S^{\text{WZ}} = \frac{k}{4\pi}\left[\int_\Sigma L^{\text{kin}} + \int_B \omega^{\text{WZ}}\right], \quad \begin{matrix} k \in \mathbb{N}, \\ \partial B = \Sigma \end{matrix}$$

with

$$\omega^{\text{WZ}} = \frac{1}{3}\text{Tr}\left(g^{-1}dg\right)^3$$

and

$$L^{\text{kin}} = \text{Tr}\left(\partial_z g \partial_{\bar{z}} g^{-1}\right) \equiv -\left\langle g^{-1}\partial_z g, g^{-1}\partial_{\bar{z}} g\right\rangle$$

<u>Orbifold Construction of Deformed Models</u>

$$\left(G/H \times U(1)^r_E\right)/\Gamma_k \qquad \text{(Gepner 1987)}$$

with

$$H = \text{CSG} \;\; , \;\; r = \text{Rank(G)} \;\; , \;\; E = \text{“}G + B\text{''}$$

$\Gamma_k =$ (Weight Lattice)$/k$(Lattice of Long Roots)

e.g. $\Gamma_k : g \to \exp\left(i\vec{\varphi}\vec{H}/k\right) g \qquad \vec{H} = \begin{pmatrix} \vec{\mu}_1 & & \\ & \ddots & \\ & & \vec{\mu}_{d-r} \end{pmatrix}$

similar to: $\mathbb{Z}_k = \mathbb{Z} \bmod k \cong \left\{e^{2\pi i/k}, \ldots, e^{2\pi i(k-1)/k}, 1\right\}$

Construction of Orbifold $\rightarrow$ Deformed

1st Start with vector gauged model

(see e.g. Gawędzki, Kupiainen 1989)

$$S^{(1)} = S^{WZW} + S\,(g, A)$$

Local symmetries:

$$g \rightarrow hgh^{-1} \quad , \quad A_i \rightarrow A_i + h\partial_i h^{-1} \qquad h \in \text{CSG}$$

2nd: Add $\text{U}(1)^r$ factor (convenient to use y taking values in copy of CSG)

$$S^{(2)} = S^{(1)} - \frac{k}{4\pi}\int d^2z \left\langle y^{-1}\partial_z y, E\, y^{-1}\partial_{\bar{z}} y \right\rangle ,$$

with $E = r{\times}r$-Matrix acting on Cartan generators

3rd: Identify global symmetry with respect to which we could try to "T-Dualize"

$$g \rightarrow fgf \quad , \quad y \rightarrow f^2 y \quad , \qquad f \in \text{GSG}$$

4th: gauge that symmetry

$$S^{(3)} = S^{(2)} + S\,(g, B) + S\,(y, B) + S\,(A, B)$$

Gauge Field: $B_i \rightarrow B_i + f^{-1}\partial_i f$.

5^{th}: Check whether original vector Symmetry ($g \to hgh^{-1}$) still there

Answer: No!

In asymmetrically gauged models usually constraint between the two gauge fields is imposed (Bars, Sfetsos 1992)

6^{th}: Kill two birds with one stone, i.e. save gauge invariance by introducing Lagrange multiplier term which is needed for T-duality, anyway.

$$S^{(4)} = S^{(3)} + \frac{ik}{\pi} \int d^2 z \left\{ - \langle \partial_z \lambda, \bar{B} \rangle + \langle B, \partial_{\bar{z}} \lambda \rangle \right\}$$

$$\lambda \to \lambda + \kappa \ \text{ for } \ h = \exp\left[i\kappa \right]$$

7^{th}: specify compactification torus for λ and integrate out λ:

$$\lambda \equiv \lambda + 2\pi\ell \ \text{ with } \ell \in \text{Long Root Lattice}$$

integration by parts $\to$ term constraining B_i to be locally pure gauge plus topological term; sum over winding modes $\to$ locally pure gauge is globally pure gauge (absorbable by field redefinition) if starting model (before gauging $g \to fgf$, $y \to f^2 y$) is

$$\left(G/H \times U(1)^r_E \right) / \Gamma_k$$

(Orbifold group element acts on y in the same way as on g on slide 9)

Now construct 'T-dual' by integrating out gauge field B instead of Lagrange multiplier λ
(Gauge fix e.g. $y = 1$)

In a next step integrate out gauge field A
(Gauge fix e.g. $\lambda = 0$)

Result:

$$S = S^{\mathsf{WZW}} +$$

$$+\frac{k}{2\pi}\int d^2z \left\langle \left(\mathsf{PAd}_g - R^{-1}\right)^{-1} \mathsf{P}\partial_z g g^{-1}, \mathsf{P} g^{-1}\partial_{\bar{z}} g \right\rangle$$

with

P projects on Cartan subalgebra,

$$\mathsf{Ad}_g X = g X g^{-1},$$

$$R = \frac{E^T - \mathsf{P}}{E^T + \mathsf{P}}.$$

Coset Construction: $\left(G \times H_{\tilde{E}}\right)/H$
(Tseytlin 1993 (for $\tilde{E} = \tilde{E}^T$)
S.F., Roggenkamp 03)

Symmetries

undeformed model $E = 1$ has chiral times anti-chiral symmetry: $G \times G$

$\rightarrow$ conserved currents:

$$J = -k\partial_z g g^{-1} \ , \ \ \bar{J} = k g^{-1}\partial_{\bar{z}} g.$$

For generic moduli $E \neq 1$ $(R \neq 0)$ this is broken to $U(1)^r \times U(1)^r$ with conserved currents:

$$\begin{aligned} J &= kR^{-1}\left(1 - RR^T\right)\left(\mathsf{PAd}_g - R^{-1}\right)^{-1}\mathsf{P}\partial_z g g^{-1}, \\ \bar{J} &= -kR^{-T}\left(1 - R^T R\right)\left(\mathsf{PAd}_{g^{-1}} - R^{-T}\right)^{-1}\mathsf{P} g^{-1}\partial_{\bar{z}} g. \end{aligned}$$

Response of sigma model action to infinitesimal change in deformation parameters:

$$\delta S = \frac{1}{2\pi k}\int d^2 z \, \langle \delta Q \, J, \bar{J} \rangle$$

with constant matrix:

$$\delta Q = R\left(R^T R - 1\right)^{-1}\left(\delta R^{-1}\right)\left(1 - RR^T\right)^{-1} R.$$

The non-trivial dilaton

Integrating out gauge fields means solving Gaussian integral $\rightarrow$ determinants

Or: Integrating up current-current perturbation and changing to covariant measure with respect to new metric $\rightarrow$ Jacobian

results in non-trivial dilaton
(special cases: Hassan, Sen 1992, Giveon, Kiritsis 1993, Sfetsos, Tseytlin 1993, Tseytlin 1993)

Pragmatic way of finding it: solve β-equations for dilaton

More elegant (Tseytlin 1993)

Compare eigenstate equation

$$(L_0 + \bar{L}_0 - a) \, |\text{physical}\rangle = 0$$

with eigenfunction equation

$$\Delta_\Phi \Psi = \frac{e^{2\Phi}}{\sqrt{G}} \partial_\mu \left(e^{-2\Phi} \sqrt{G} G^{\mu\nu} \partial_\nu \Psi \right) = -m^2 \Psi.$$

ground state corresponds to minimal m^2

compute Hamiltonian $(L_0+\bar{L}_0)$:(Bowcock 1989)
Go to Minkowski worldsheet signature $(J,\bar{J}\to J_+,J_-)$ and perform Legendre transform

with respect to the Poisson bracket $J_\pm$ form two copies of the Kac-Moody algebra
(Lie Algebra for zero modes)

$\rightarrow$ useful to express Hamiltonian in terms of these currents, e.g. in the undeformed case

$$H_{E=1}=-\frac{1}{4k\pi}\int d\sigma\left(\Big\langle J_+,J_+\Big\rangle+\langle J_-,J_-\rangle\right).$$

for deformed case identify $J_\pm$ such that they are identical as phase space functions to the quantities in the undeformed case $\rightarrow$ Poisson algebra is not altered by deformation

Hamiltonian in terms of these $J_\pm$:

$$H_E= -\frac{1}{4\pi k}\int d\sigma\Bigg(\bigg\langle\Big(1+\mathsf{P}(E-1)\left(E+E^T\right)^{-1}\left(E^T-1\right)\mathsf{P}\Big)J_+,J_+\bigg\rangle + \bigg\langle\Big(1+\mathsf{P}\left(E^T-1\right)\left(E+E^T\right)^{-1}(E-1)\,\mathsf{P}\Big)J_-,J_-\bigg\rangle +2\bigg\langle\left(E^T-1\right)\left(E+E^T\right)^{-1}(E+1)\,\mathsf{P}J_+,\mathsf{P}J_-\bigg\rangle\,,$$

Laplacian:

$$\Delta_\Phi \equiv \frac{e^{2\Phi}}{\sqrt{G}} \partial_\mu e^{-2\Phi} \sqrt{G} G^{\mu\nu} \partial_\nu$$

introduce coordinates and vielbeins

$$g^{-1}dg = \tilde{e}_{A\mu} T^A dx^\mu \ , \ -dgg^{-1} = e_{A\mu} T^A dx^\mu.$$

$\rightarrow$two sets of differential operators (vector fields) satisfying Lie algebra: $e^\mu_A \partial_\mu$, $\tilde{e}^\mu_a \partial_\mu$.
for notational convenience introduce
(Scalar product = Trace)

$$j_+ = T^A e^\mu_A \partial_\mu \ , \ j_- = T^A \tilde{e}^\mu_A \partial_\mu$$

one gets

$$2k\alpha' \Delta_\Phi =$$

$$\left\langle f^{-1} j_+, f \Big(1 + \mathsf{P} \left(E^T - 1\right) \left(E + E^T\right)^{-1} (E - 1) \mathsf{P}\Big) j_+ \right\rangle$$

$$+ \left\langle f^{-1} j_-, f \left(1 + \mathsf{P} (E - 1) \left(E + E^T\right)^{-1} \left(E^T - 1\right) \mathsf{P}\right) j_- \right\rangle$$

$$+ \left\langle f^{-1} \mathsf{P} j_+, f \left(E^T + 1\right) \left(E + E^T\right)^{-1} (E - 1) \mathsf{P} j_- \right\rangle$$

$$+ \left\langle \mathsf{P} f^{-1} j_-, f (E + 1) \left(E + E^T\right)^{-1} \left(E^T - 1\right) \mathsf{P} j_- \right\rangle,$$

with: $f = e^{-2\Phi} \sqrt{G/G_0}$.
'matches' with Hamiltonian if $f =$ const, i.e.

$$e^{-2\Phi} \sqrt{G} = \text{independent of deformation}$$

.

Conclusions

Summary:

Deformations of WZW models parameterized by $r \times r$ matrix E $\rightarrow$ deformed metric, B-field, Dilaton

Moduli Space: Duality $\backslash\, O(r,r)/O(r) \times O(r)$

Duality Group: $O(r,r,\mathbb{Z}) \cap$ Duality of WZW

$$\text{Duality of WZW} = \widehat{\mathrm{W}} \times \widehat{\mathrm{W}} \ltimes \text{outer autom.}$$

$\widehat{\mathrm{W}}$ is affine Weyl group (Kiritsis 1993)

Outlook:

add D-branes work in progress with D. Roggenkamp

Orientifolds:
symmetry: $z \leftrightarrow \bar{z},\ \ g \leftrightarrow \pm g^{-1},\ \ E \leftrightarrow E^T$
$\rightarrow$ $E = E^T$ mod large gauge trafo
$\rightarrow$ B-field quantization

. . .