Recursive actions for scalar theories [1]

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Abstract

We introduce a class of self-interacting scalar theories in which the various coupling contants obey a recursive relation. These imply a particularly simple form for the generating function of the Feynman amplitudes with vanishing external momenta, as well as for the effective potential. In addition we discuss an interesting duality inherent in these models. Specializing to the case of zero spacetime dimensions we find intriguing nullification properties for the amplitudes.

1 Introduction

In this paper we discuss a special class of Euclidean theories of self-interacting scalar fields. More in particular, we study amplitudes with vanishing external momentum, as for instance implied in the definition of an effective potential; but also the special case of theories in zero spacetime dimensions is subsumed

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(the zero-dimensional case is of course of paradigmatic interest because the path integral is a simple integral, amenable to straightforward solution and manipulation). Most of our results will, in fact, be derived for zero dimensions. Zero-dimensional field theories have been amply discussed: apart from the useful introductory treatment in [1] we may refer to [2, 3, 4, 5] as recent applications. The aim of this paper is to study properties of essentially non-polynomial theories in which the coupling constants obey a simple algebraic relation. In section 2, we define *recursive* theories, and show how the various zero-momentum Green's functions are related to one another in a surprisingly simple manner, which allows us to express the complete set n-particle amplitudes in terms of the tadpole alone. In section 3, we study a duality inherent in our models, that relates the elementary field in one theory with certain composite field in its dual. In section 4 we discuss several explicitly solvable zero-dimensional recursive theories. In section 5, we tackle the structure of loop corrections for general zero-dimensional self-interacting scalar theories, with special emphasis on the occurrence of 'nullification', that is, a special choice of parameters for which *all* loop corrections of a given loop order vanish. Section 6 is devoted to nullification in recursive thories, where an intriguing pattern is exhibited. In section 7, we address the application of renormalization in zero dimensions, in the spirit of [1].

2 Feynman amplitudes for recursive actions

We consider self-interacting theories of a field φ with mass m in d Euclidean spacetime dimensions, with potentials given by

$$V(\varphi) = \sum_{n\geq 3} \frac{\lambda_n}{n!}\varphi^n \quad . \tag{1}$$

We also introduce $\lambda_2 \equiv \mu = m^2$. Note that in the sum n runs, in principle, all the way up to infinity. Our class of models is characterized by the following property: there exist (dimensionful) constants α and β such that

$$\lambda_{n+1} = \lambda_n(\alpha n + \beta) \ \ , \ \ n \geq 2 \ \ . \tag{2}$$

Of course, we can determine α, β from

$$\alpha = \frac{\lambda_4}{\lambda_3} - \frac{\lambda_3}{\mu} \quad , \quad \beta = 3\frac{\lambda_3}{\mu} - 2\frac{\lambda_4}{\lambda_3} \quad , \tag{3}$$

and the combination $\alpha\varphi$ is dimensionless. The Lagrangian density of these models is given by

$$\mathcal{L} = \frac{1}{2}(\vec{\nabla}\varphi)^2 + \frac{\mu}{p\beta}\left((1-\alpha\varphi)^{-\beta/\alpha} - 1 - \beta\varphi\right) \quad , \quad p = \alpha + \beta \quad , \tag{4}$$

and in zero dimensions the action itself is simply

$$S(\varphi) = \frac{\mu}{p\beta}\left((1-\alpha\varphi)^{-\beta/\alpha} - 1 - \beta\varphi\right) \quad . \tag{5}$$

We call these models *recursive*. When $-\beta/\alpha$ is an integer $K \geq 2$, the potential is a finite polynomial of order K: φ^3 theory is recursive, but neither pure φ^4 theory nor spontaneously broken φ^4 theory are recursive.

Consider a connected Feynman diagram $D(n)$, entering into the $1 \to n$ amplitude. Let this graph have I internal lines, $E = 1 + n$ external lines, V_k vertices of type φ^k and L loops. We have

$$D(n) \propto \; \hbar^L \mu^{-I-E+dL/2} \lambda_3^{V_3} \lambda_4^{V_4} \cdots \quad . \tag{6}$$

Here we have assumed that the regularization of the loop integrals is performed in a manner that does not introduce another physical mass scale (as would be the case in say, Pauli-Villars regularization), so that a d-dimensional loop integral contributes a factor $m^d = \mu^{d/2}$; an example is dimensional regularization. In that case there enters, of course, an 'engineering dimension', which we include in the (possibly very complicated) proportionality constant. Note that this is consistent also for $d = 0$ since then loop integrals are simply absent.

The two topological relations

$$\sum_{k\geq 3} kV_k = 2I + E \quad , \quad \sum_{k\geq 3} V_k = I + 1 - L \quad , \tag{7}$$

can be written as follows for this diagram:

$$\begin{aligned} \sum_{k\geq 2} k\lambda_k \frac{\partial}{\partial\lambda_k} D(n) &= d\hbar\frac{\partial}{\partial\hbar} D(n) - (n+1)D(n) \quad , \\ \sum_{k\geq 2} \lambda_k \frac{\partial}{\partial\lambda_k} D(n) &= (\frac{d}{2} - 1)\hbar\frac{\partial}{\partial\hbar} D(n) - nD(n) \quad . \end{aligned} \tag{8}$$

Since this holds for any $D(n)$, it holds *a fortiori* also for the full $1 \to n$ amplitude $a(n)$.

Let us now consider what happens if we add one external leg to the amplitudes. This may be done in several ways. In the first place, we may simply attach the external line to any φ^k vertex, thereby turning it into a φ^{k+1} vertex, giving a factor $V_k\lambda_{k+1}/(\lambda_k\mu)$. In other words, attaching a line to any vertex in the diagram is equivalent to the operation

$$D(n) \to \sum_{k\geq 3} \frac{\lambda_{k+1}}{\mu}\frac{\partial}{\partial\lambda_k} D(n) \quad .$$

In the second place, we may attach the external line to any line by a three-point vertex. If the momentum flowing in the original line is q, the attachment turns $(q^2+\mu)^{-1}$ into $-\lambda_3\mu^{-1}(q^2+\mu)^{-2}$, so that we can write this procedure as

$$D(n) \to \frac{\lambda_3}{\mu}\frac{\partial}{\partial\mu} D(n) \quad .$$

Note that this also works for internal lines in loop diagrams, owing to the fact that the external momenta all vanish. In this way, we can form all amplitudes from the vacuum bubbles of the theory, with the single exception of the bare propagator. We also want to stress that, in this procedure, the symmetry factors of all diagrams will come out correct automatically: in a sense, our procedure is how the symmetry factors are defined in the first place anyway. We therefore have the following recursion between amplitudes:

$$\mu\, a(n+1) = \delta_{n,0} + \sum_{k\geq 2} \lambda_{k+1}\frac{\partial}{\partial\lambda_k} a(n) \quad , \quad n \geq 0 \quad . \tag{9}$$

So far this is general for zero-momentum amplitudes. In the case of recursive actions, we can use the relation between λ_{k+1} and λ_k to good effect. Let us denote by $\phi(x)$ the generating function of all $1 \to n$ amplitudes:

$$\phi(x) = \sum_{n\geq 0} a(n)\frac{x^n}{n!} \quad . \tag{10}$$

For recursive actions, this then satisfies the differential equation

$$(\mu + px)\frac{\partial}{\partial x}\phi + \alpha\phi + ((1-d/2)\beta - d\alpha)\,\hbar\frac{\partial}{\partial\hbar}\phi = 1 \quad . \tag{11}$$

The solution can be written as

$$\begin{aligned}
\phi(x) &= \frac{1}{\alpha}\left(1-\frac{1}{(1+px/\mu)^{\alpha/p}}\right)\\
&+ \frac{1}{(1+px/\mu)^{\alpha/p}}\sum_{L\geq 1} t_L(\mu;\alpha,\beta)\left(\frac{\hbar}{(1+px/\mu)^{(\beta(1-d/2)-d\alpha)/p}}\right)^L
\end{aligned} \quad (12)$$

we see that *for recursive actions, all connected amplitudes are completely determined by the tadpole* $\phi(0)=\sum_L t_L\hbar^L$. The above argument does not, however, allow us to determine the tadpole itself. This reflects the fact that, whereas the operation of adding an extra external *line* is, in the above, formulated as a fairly simple algorithm, the operation of adding an extra *loop* does not appear to follow any simple algorithm yielding the right symmetry factors.

In zero dimensions, simple dimensional analysis shows that $t_L(\mu;\alpha,\beta)$ can be written as μ^{-L} times an expression in α and β that is homogeneous of degree $2L-1$. Moreover, since the tadpole contains α and β only through the coupling constants λ_k, which only enter in the numerator of any Feynman diagram, we conclude that, for $p\neq 0$,

$$t_L = \frac{p^{2L-1}}{\mu^L}R_{2L-1}(u) \ \ , \quad (13)$$

where R_q is a polynomial of degree q and the ratio between α and β is encoded in

$$u = \frac{2\alpha+\beta}{\alpha+\beta} = 1+\alpha/p \ \ . \quad (14)$$

It is clear that we may put $\mu = p = 1$ without loss of generality (except for the special case $\alpha+\beta=0$), since they can always be put back into any expression.

Finally, by judicious choice of α and β we can single out theories with interesting properties. For instance, when α/p is a negative integer $-n$, the tree-level amplitude generating function $\phi_0(x)$ is a finite polynomial of degree n. This implies that the tree level amplitudes with $n+2$ or more external legs all vanish. In higher orders, though, these amplitudes may be nonzero since $\phi_L(x)$ has exponent $n-(n+1)L$. In the same vein, when $p=0$ the amplitudes do not necessarily vanish, but the generating function goes with a power of $\exp(-x)$ at every loop order, which implies that the amplitudes go as $1/n!$ rather than as $n!$, a situation that has been discussed (at the kinematic threshold rather than at zero momentum) elsewhere [6].

Similarly, if $\beta/\alpha = -n/(n+1)$, with n a nonnegative integer, the L-loop contribution $\phi_L(x)$ is a finite polynnomial in x of degree $n(L-1)-1$ for $L \geq 2$, which implies that L-loop corrections vanish for Green's functions with more that $n(L-1)$ external legs.

In the one-dimensional case, an interesting situation is that where $\beta = 2\alpha$. In that case,

$$\phi(x) = \frac{1}{\alpha} - \frac{1}{(1+px/\mu)^{\alpha/p}}\left(\frac{1}{\alpha} - \sum_{L\geq 1} \hbar^L t_L\right) \quad , \tag{15}$$

so that in the zero-momentum amplitudes *all* loop corrections can be completely absorbed into finite tadpole and mass renormalization (see also the discussion in a later section of this paper).

For theories with $\beta/\alpha = -2n/(1+2n)$ we find, in this case, that $\phi_L(x)$ contains the exponent $(1+3n)(L-1)+n$, so that again loop corrections vanish for sufficiently large numbers of legs.

In higher dimensions, the tadpole factors t_L will themselves also depend on d, and in fact for $d \geq 2$ they contain divergences. As long as we use dimensional regularization it is therefore tempting, but erroneous, to choose attractive-looking values for d. For instance, choosing $d = 2$ and $\alpha = 0$ would at first sight seem to eliminate the x dependence in the loop corrections, but the more careful treatment $d = 2 - 2\epsilon$, $\epsilon \to 0$ reveals that t_L will, in general contain poles in ϵ up to ϵ^{-L} so that, in fact all amplitudes have loop corrections.

We may use a similar argument for the one-particle irreducible (1PI) diagrams of the theory. The only difference is that the new line may *not* be attached to an existing external line, so that the relevant operation reads

$$D(n) \to \sum_{k\geq 2} \frac{\lambda_{k+1}}{\mu}\frac{\partial}{\partial\lambda_k} D(n) + (n+1)\frac{\lambda_3}{\mu^2} D(n) \quad .$$

Denoting the generating function of the 1PI $1 \to n$ amplitudes by ϕ^{1PI}, we now find the differential equation

$$(\mu - \alpha x)\frac{\partial}{\partial x}\phi^{\mathrm{1PI}} - p\phi^{\mathrm{1PI}} + ((1-d/2)\beta - d\alpha)\,\hbar\frac{\partial}{\partial\hbar}\phi^{\mathrm{1PI}} = 1 \quad , \tag{16}$$

which is quite similar to the equation for the connected amplitudes. Its solution reads

$$\begin{aligned}\phi^{\text{1PI}}(x) &= -\frac{1}{p}+\frac{1}{p}(1-\alpha x/\mu)^{-p/\alpha}\\ &+ (1-\alpha x/\mu)^{-p/\alpha}\sum_{L\geq 1} t_L^{\text{1PI}}(\mu;\alpha,\beta)\left(\hbar(1-\alpha x/\mu)^{(\beta(1-d/2)-d\alpha)/\alpha}\right)^L\end{aligned} \tag{17}$$

and again everything is determined by the (1PI) tadpole. The effective action of the theory, $\Gamma(\varphi)$, can simply be found from

$$\Gamma'(\varphi) = \mu\phi^{\text{1PI}}(\mu\varphi) \ . \tag{18}$$

A final note on the divergence structure of the amplitudes is in order. On first sight it might be thought that, since any L-loop amplitude is proportional to t_L, these amplitudes must all have the same divergence structure. That this is not necessarily the case can be seen from the following simple example. Consider the 1PI one-loop amplitudes for the pure φ^3 theory in d dimensions, *i.e.* $\beta=-3\alpha$ and $L=1$. We find

$$\begin{aligned}\phi_1^{\text{1PI}}(x) &= \hbar t_1\left(1-\frac{\alpha x}{\mu}\right)^{d/2-1}\\ &= \hbar t_1\left(1-\left(\frac{\alpha x}{\mu}\right)\frac{(d-2)}{2}+\left(\frac{\alpha x}{\mu}\right)^2\frac{(d-2)(d-4)}{8}+\cdots\right)\end{aligned} \tag{19}$$

The tadpole contribution,

$$\hbar t_1 = -\frac{\alpha\hbar}{2(4\pi)^{d/2}}\mu^{d/2-1}\Gamma\left(1-\frac{d}{2}\right) \ , \tag{20}$$

is divergent for $d=2,4,6,\ldots$. It can be seen that in two dimensions the 1PI propagator is then finite. In four dimensions, the 1PI propagator is divergent (but less so than the tadpole) and the 1PI three-point function is finite, and so on; precisely in accordance with what is expected on the basis of the Feyman diagrams, that all consist of a single loop beaded with three-point vertices.

Before finishing this section we wish to point out the following. In the definition of the recursive action it is very important that the one-point coupling λ_1 is *not* included. Consider a theory with an explicit tadpole term, given by

$$V_{\text{tad}}(\varphi) = \lambda_1\varphi + V(\varphi) \ , \tag{21}$$

with a recursive potential $V(\varphi)$ as given in Eq.(1). It is easily seen that the generating function $\phi_{\text{tad}}(x)$ of this theory is related to the one without the tadpole term as

$$\phi_{\text{tad}}(x) = \phi(x-\lambda_1) = \frac{1}{\alpha} - \frac{1}{\alpha\left(1 - \frac{\lambda_1 p}{\mu} + \frac{xp}{\mu}\right)^{\alpha/p}} + (\text{loop corrections}) \ . \tag{22}$$

If we let λ_1 approach its recursive value, $\lambda_1 = \mu/p$, the generating function obtains its singularity at $x = 0$, and perturbation theory breaks down even at the tree level. This is caused by the fact that, with a nonzero bare tadpole term, every amplitude at every loop order contains an infinite number of Feynman diagrams, the sum of which, order by order, is no longer convergent at precisely the recursive value of the tadpole. For larger absolute values of the tadpole coupling, ϕ can only be obtained by analytic continuation.

3 Duality in zero dimensions

Throughout this section, we shall assume $d = 0$. The Euclidean path integral in the presence of a source x is then given by

$$Z(x) = \int\limits_C d\varphi \ \exp\left(-\frac{1}{\hbar}(S_{\alpha,\beta}(\varphi) - x\varphi)\right) \ , \tag{23}$$

where we have explicitly indicated the parameters entering in $S(\varphi)$. The integration contour C is preferably such that the integrand vanishes sufficicently fast at the endpoints. However, if we restrict ourselves to perturbation theory it is sufficient that the endpoints do not approach the perturbative extremum $\varphi = 0$. This is particularly important when β/α is not integer so that the action displays branch cuts starting at $\varphi = 1/\alpha$. The generating funciton of the connected amplitudes is given by

$$\phi(x) = \frac{\hbar}{Z(x)}\frac{\partial}{\partial x}Z(x) \ . \tag{24}$$

The tadpole for this theory is therefore

$$T_{\alpha,\beta} \equiv \phi(0) = \frac{\langle\varphi\rangle_{\alpha,\beta}}{\langle 1\rangle_{\alpha,\beta}} \ \ , \tag{25}$$

where

$$\langle A(\varphi)\rangle_{\alpha,\beta} \equiv \int d\varphi\ A(\varphi)\exp\left(-\frac{S_{\alpha,\beta}(\varphi)}{\hbar}\right) \tag{26}$$

for any function $A(\varphi)$ of φ. By partial integration we may prove the following useful lemma:

$$\left\langle A(\varphi)S'_{\alpha,\beta}(\varphi)\right\rangle_{\alpha,\beta} = \hbar\left\langle A'(\varphi)\right\rangle_{\alpha,\beta} \ \ . \tag{27}$$

Now, consider the object ψ dual to φ, defined by

$$(1-\beta\psi)^{\alpha}(1-\alpha\varphi)^{\beta} = 1 \quad \Rightarrow \quad \psi = \frac{1}{\beta}\left(1-(1-\alpha\varphi)^{-\beta/\alpha}\right) \ \ . \tag{28}$$

In terms of φ, ψ is a 'composite' object, and we have

$$S_{\alpha,\beta}(\varphi) = -\frac{\mu}{p}(\varphi+\psi) = S_{\beta,\alpha}(\psi) \ \ . \tag{29}$$

We may replace φ as a dummy integration variable by ψ. For instance,

$$\langle 1\rangle_{\alpha,\beta} = \left\langle\frac{d\varphi}{d\psi}\right\rangle_{\beta,\alpha} = \left\langle -1-\frac{p}{\mu}S'_{\beta,\alpha}(\psi)\right\rangle_{\beta,\alpha} = -\langle 1\rangle_{\beta,\alpha} \ \ . \tag{30}$$

The zero-dimensional action satisfies a linear differential equation:

$$(1-\alpha\varphi)S'_{\alpha,\beta}(\varphi) - \beta S_{\alpha,\beta}(\varphi) = \mu\varphi \ \ , \tag{31}$$

so that S can be expressed in terms of S'. Together with the lemma, this allows us to compute also

$$\begin{aligned}\alpha\langle\varphi\rangle_{\alpha,\beta} &= \alpha\left\langle\left(1+\frac{p}{\mu}S'_{\beta,\alpha}(\psi)\right)\left(\psi+\frac{p}{\mu}S_{\beta,\alpha}(\psi)\right)\right\rangle_{\beta,\alpha} \\ &= \frac{p\hbar}{\mu}(\alpha-\beta)\langle 1\rangle_{\beta,\alpha} - \beta\langle\psi\rangle_{\beta,\alpha} \ \ .\end{aligned} \tag{32}$$

We therefore find a simple relation between the tadpoles of the two dual theories with $S_{\alpha,\beta}$ and $S_{\beta,\alpha}$:

$$\alpha\left(T_{\alpha,\beta}(\hbar)+\frac{p\hbar}{\mu}\right) = \beta\left(T_{\beta,\alpha}(\hbar)+\frac{p\hbar}{\mu}\right) \ \ . \tag{33}$$

This duality allows for some immediate conclusions. In the first place, the free theory is recursive, with $\beta = -2\alpha$ so that $\lambda_3 = 0$, and it has a vanishing tadpole. Its dual is the action given by

$$S_{-2\alpha,\alpha}(\varphi) = \frac{\mu}{\alpha^2}\left(1 + \alpha\varphi - \sqrt{1+2\alpha\varphi}\right) \quad , \tag{34}$$

and its tadpole is therefore immediately seen to be

$$T_{-2\alpha,\alpha} = \frac{3\alpha\hbar}{2\mu} \quad ; \tag{35}$$

in this theory, *all* loop corrections beyond the one-loop level vanish identically! Similarly, the action with $\beta \to 0$ and hence $u = 2$, given by

$$S_{\alpha,0}(\varphi) = \frac{\mu}{\alpha^2}\left(-\alpha\varphi - \log(1-\alpha\varphi)\right) \quad , \tag{36}$$

has for its tadpole

$$T_{\alpha,0} = -\frac{\alpha\hbar}{\mu} \quad , \tag{37}$$

and again all higher orders vanish identically. The results (35) and (37) are confirmed by explicit computation of the zero-dimensional path integral.

The zero-dimensional duality has another interesting consequence. Putting $\mu = p = 1$ and writing $\alpha = u - 1$, $\beta = 2 - u$ we can write the tadpole duality as

$$(u-1)R_{2L-1}(u) = (2-u)R_{2L-1}(3-u) \quad , \quad L \geq 2 \quad , \tag{38}$$

where $R_{2L-1}(u)$ is the polynomial entering in t_L as discussed above. This means that $R_{2L-1}(u)$ must have a root at $u = 2$ (as indeed we have seen). Moreover, since $u = 0$ corresponds to the free action, $R_{2L-1}(u)$ must vanish for $u = 0$, and by duality also for $u = 3$. We can therefore write

$$R_{2L-1}(u) = u(2-u)(3-u)P_{L-2}(\omega) \quad , \quad \omega = u(3-u) \quad , \quad L \geq 2, \tag{39}$$

where $P_{L-2}(\omega)$ is a polynomial of degree $L - 2$ only.

4 Explicit solutions in zero dimensions

In this section we discuss a few explicitly solvable models with $d = 0$. For simplicity, we shall take $\mu = 0$ and $p = 1$. The models are, then, completely specified by the parameter u. We may therefore write $S_{\alpha,\beta}(\varphi) = S_u(\varphi)$, and the duality operation is the interchange $u \leftrightarrow 3 - u$. The partition function $Z(x)$ is in general determined from the Schwinger-Dyson (SD) equation

$$S'\left(\hbar\frac{\partial}{\partial x}\right) Z(x) = xZ(x) \quad , \tag{40}$$

which leaves the overall normalization of $Z(x)$ undetermined. For recursive actions, the SD equation may be rewritten as

$$\left(1 - \alpha\hbar\frac{\partial}{\partial x}\right)^{-1/\alpha} Z(x) = (1 + x)Z(x) \quad . \tag{41}$$

The simplest case is the free theory, $u = 0$, leading to $Z(x) = \exp(x^2/2)$ and $\phi = x$; and the effective action is $\Gamma_0(\varphi) = S_0(\varphi)$. The next simplest case is $u = 2$:

$$S_2(\varphi) = -\varphi - \log(1 - \varphi) \quad , \tag{42}$$

leading to the SD equation

$$\frac{1}{1 - \hbar\partial} Z(x) = (1 + x)Z(x) \quad , \quad \partial \equiv \frac{\partial}{\partial x} \quad . \tag{43}$$

Multiplying from the left by $Z(x)^{-1}(1 - \hbar\partial)$ on both sides gives immediately the form of ϕ:

$$\phi = \frac{x - \hbar}{(1 + x)} \quad , \tag{44}$$

in agreement with the result from duality. The effective action is most simply obtained from inverting this relation:

$$\phi(x) = F(x) \to x(\phi) = \Gamma'(\phi) \quad .$$

In this case, we find

$$\Gamma_2(\varphi) = -\varphi - (1 + \hbar)\log(1 - \varphi) \quad , \tag{45}$$

so that also the effective action is free of $L \geq 2$ corrections.

The case $u = 1$ corresponds to the action dual to $S_2(\varphi)$:

$$S_1(\varphi) = e^{\varphi} - 1 - \varphi \ \ , \tag{46}$$

leading to a *functional* form for the SD equation:

$$e^{\hbar\partial} Z(x) = Z(x+\hbar) = (1+x)Z(x) \ \ , \tag{47}$$

and a functional equation for $\phi(x)$:

$$\phi(x+\hbar) = \phi(x) + \frac{\hbar}{1+x} \ \ . \tag{48}$$

Together with the requirement $\lim_{\hbar\to 0}\phi(0) = 0$ this implies

$$\begin{aligned} \phi(x) &= \log\hbar + \psi\left(\frac{1+x)}{\hbar}\right) \\ &= \log(1+x) - \frac{\hbar}{2(1+x)} - \sum_{L\geq 2}\frac{B_L}{L}\left(\frac{\hbar}{1+x}\right)^L \ \ , \end{aligned} \tag{49}$$

where $\psi()$ denotes the digamma function, and we have indicated the asymptotic expansion, where B_L are the Bernoulli numbers. The behaviour with x of this result is, of course, already given from the recursivity of the model, but as stated before the tadpole itself can only be obtained from the SD equation. Since $B_L = 0$ for odd $L \geq 3$, we conclude that for this model *all odd-loop amplitudes beyond the one-loop level vanish completely.* This result is significant since, in this model, every coupling constant λ_n is unity, and the value of every Feynman diagram is given by only its symmetry factor times a factor (-1) for every vertex; we therefore have, here, a strictly graph-theoretic result. The one-loop tadpole consists, of course, of only a single diagram and can never vanish. For the effective action we find a similar result, at least empirically. Writing x as a function of ϕ gives the effective action:

$$\Gamma_1'(\varphi) = -1 + e^{\varphi} + \sum_{L\geq 1} a_L \hbar^L e^{(1-L)\varphi} \ \ , \tag{50}$$

where $a_1 = 1/2$ and the first few even coefficients a_L read

$$\begin{aligned} a_2 &= -1/24 \ \ , \\ a_4 &= 3/640 \ \ , \end{aligned}$$

$$
\begin{aligned}
a_6 &= -1525/580608 \ , \\
a_8 &= 615881/199065600 \ , \\
a_{10} &= -3058641/504627200 \ , \\
a_{12} &= 38800188510523/2191186722816000 \ , \\
a_{14} &= -3213747182969063/44497945755648000 \ , \\
a_{16} &= 100462329712125/255806104666112 \ .
\end{aligned} \tag{51}
$$

The only one-loop 1PI amplitude is the tadpole. All coefficients for odd $L \geq 3$ appear to vanish again: we have checked this up to 60 loops, but we have not been able to prove it rigorously. The fact that the pattern of zeroes in both ϕ and ϕ^{1PI} is the same is intimately tied up with the occurrence of the Bernoulli numbers: if we assume the given x dependence in ϕ and insist that the 3,5,7,... loop corrections vanish in both ϕ and ϕ^{1PI} we recover the above result for ϕ as the unique solution.

For $u = 3$ we have the action dual to the free one:

$$
S_3(\varphi) = 1 - \varphi - \sqrt{1 - 2\varphi} \ . \tag{52}
$$

Its SD equation reads

$$
DZ = yZ \quad , \quad y = 1 + x \quad , \quad D \equiv (1 - 2\hbar\frac{\partial}{\partial y})^{-1/2} \ . \tag{53}
$$

Although this is an infinite-order differential equation, it is solvable by using the fact that D and y obey a commutation relation

$$
[D, y] = \hbar D^3 \ . \tag{54}
$$

This allows us to write

$$
D^2 Z = D(yZ) = yDZ + \hbar D^3 Z = y^2 Z + \hbar D^2(yZ) \ . \tag{55}
$$

Multiplying from the left by $D^{-2} = 1 - 2\hbar\partial/\partial y$ then gives a linear SD equation for Z, from which ϕ follows algebraically:

$$
Z = y^2 Z - 3\hbar y Z - 2\hbar y^2 \partial Z \quad , \quad \phi = \frac{1}{2}\left(1 - \frac{1}{(1+x)^2}\right) - \frac{3\hbar}{2}\frac{1}{(1+x)} \ , \tag{56}
$$

which is precisely the result obtained earlier by duality. The effective action is given by

$$\Gamma_3'(\varphi) = -1 + \frac{1}{1-2\varphi}\left[\frac{3\hbar}{2} + \sqrt{1 - 2\varphi + \left(\frac{3\hbar}{2}\right)^2}\,\right] \quad . \tag{57}$$

For this theory, the 1PI amplitudes for odd $L \geq 3$ vanish identically.

For polynomial recursive actions with highest interaction term φ^{K+1} (integer K) and $p = \mu = 1$ we have $\alpha = -1/K$, $\beta = (K+1)/K$, and hence the SD equation has finite order:

$$\left(1 + \frac{\hbar}{K}\frac{\partial}{\partial y}\right)^K Z = yZ \quad . \tag{58}$$

We can show that, even though the dual theories of these actions are not of finite polynomial form, nevertheless their SD equations can also be cast in the form of differential equations of order K, as follows. The dual actions have $\alpha = (K+1)/K$, and the SD equation is

$$\left(1 - \frac{\hbar(K+1)}{K}\frac{\partial}{\partial y}\right)^{-K/(K+1)} Z = D^K Z = yZ \quad , \tag{59}$$

where

$$D = \left(1 - \frac{\hbar(K+1)}{K}\frac{\partial}{\partial y}\right)^{-1/(K+1)} \quad . \tag{60}$$

The differential operator D has the following commutation relation with y:

$$D^s y = yD^s + \frac{s\hbar}{K}D^{s+K+1} \quad , \tag{61}$$

for general s, and hence

$$D^s Z = D^{s-K}(yZ) = yD^{s-K}Z + \frac{\hbar(s-K)}{K}D^{s+1}Z \quad . \tag{62}$$

By repeating this operation, it is easily seen that

$$\begin{aligned} A_0 &\equiv D^s Z = A_1 = A_2 = A_3 = \ldots, \\ A_m &= \sum_{n=0}^{m} \gamma_n^{(m)} y^{m-n}\hbar^n D^{s-mK+n(K+1)} Z \quad , \\ \gamma_n^{(m)} &= \gamma_n^{(m-1)} + \gamma_{n-1}^{(m-1)}\frac{s-(m+1)K+n(K+1)-1}{K} \quad , \end{aligned} \tag{63}$$

and the recursion relation for the γ's starts at $\gamma_n^{(0)} = \delta_{n,0}$. Choosing $s = K(K+1)$ and $m = K$ we are then left with

$$D^{K(K+1)}Z = \sum_{n=0}^{K} \gamma_n^{(K)} y^{K-n} \hbar^n D^{n(K+1)+K} Z \quad , \tag{64}$$

or, in other words,

$$\left(1 - \frac{\hbar(K+1)}{K}\frac{\partial}{\partial y}\right)^{-K} Z = \sum_{n=0}^{K} \gamma_n^{(K)} y^{K-n} \hbar^n \left(1 - \frac{\hbar(K+1)}{K}\frac{\partial}{\partial y}\right)^{-n} (yZ) \quad . \tag{65}$$

By multiplying from the left by $\left(1 - \frac{\hbar(K+1)}{K}\frac{\partial}{\partial y}\right)^K$, we obtain the differential equation of finite order K mentioned above, with coefficients containing powers of y up to y^{K+1}. We give here the results for the first few K values:

$$\begin{aligned}
K = 1 \quad &: \quad Z_0 = y^2 Z_1 - 3\hbar y Z_0 \quad , \\
K = 2 \quad &; \quad Z_0 = y^3 Z_2 - 6\hbar y^2 Z_1 + 5\hbar^2 y Z_0 \quad , \\
K = 3 \quad &: \quad Z_0 = y^4 Z_3 - 10\hbar y^3 Z_2 + \frac{65}{3}\hbar^2 y^2 Z_1 - \frac{70}{9}\hbar^3 y Z_0 \quad , \\
K = 4 \quad &: \quad Z_0 = y^5 Z_4 - 15\hbar y^4 Z_3 + 60\hbar^2 y^3 Z_2 \\
&\qquad - \frac{525}{8}\hbar^3 y^2 Z_1 + \frac{189}{16}\hbar^4 y Z_0 \quad ,
\end{aligned} \tag{66}$$

where $Z_n \equiv \left(1 - \frac{\hbar(K+1)}{K}\frac{\partial}{\partial y}\right)^n Z$.

Another class of models is that for which $\alpha = 1/K$, with K a positive integer, hence $u = 1 + 1/K$. Their SD equation reads

$$Z = \left(1 - \frac{\hbar}{K}\frac{\partial}{\partial y}\right)^K (yZ) \quad , \tag{67}$$

again a linear equation of finite order. Among these models is the 'self-dual' action with $K = 2$ and $u = 3/2$, with solution

$$Z = y^{-1/2} \exp\left(-\frac{2y}{\hbar}\right) I_1\left(\frac{4y^{1/2}}{\hbar}\right) \quad , \tag{68}$$

where I is the modified Bessel function of the first kind. The resulting tadpole reads

$$T(\hbar) = 2\frac{I_0(4/\hbar)}{I_1(4/\hbar)} - 2 - \hbar \quad . \tag{69}$$

The other solution, which has the modified Bessel function of the second kind, K_1 instead of I_1 in $Z(x)$, has a nonvanishing tree-level tadpole and hence does not correspond to a perturbative solution. For $K \to \infty$ we return to the case $u = 1$ discussed above.

A final case of interest is the 'almost-free' theory, with $u = \epsilon \ll 1$. The action reads, in this case

$$S_\epsilon(\varphi) = \frac{1}{2}\varphi^2 + \frac{\epsilon}{4}\left(-2\varphi - 3\varphi^2 + 2(1+\varphi)^2\log(1+\varphi)\right) + \mathcal{O}(\epsilon^2) \ , \tag{70}$$

and the SD equation,

$$\left((1-\epsilon)\hbar\frac{\partial}{\partial x} + \epsilon\left(1 + \hbar\frac{\partial}{\partial x}\right)\log\left(1 + \hbar\frac{\partial}{\partial x}\right) - x\right) Z(x) = \mathcal{O}(\epsilon^2) \ , \tag{71}$$

looks quite hopeless. However, we may solve it by realizing that, for small ϵ, the coupling constants are also very small:

$$\lambda_k = (-)^{(k-3)}(k-3)!\epsilon + \mathcal{O}(\epsilon^2) \ , \ \ k \geq 3 \ . \tag{72}$$

Therefore, the tadpole is dominated by diagrams with only one vertex. The L-loop tadpole therefore contains only λ_{2L+1}, and

$$T(\hbar) = -\sum_{L\geq 1}\frac{(2L-2)!}{2^L L!}\hbar^L\epsilon = -\frac{\epsilon}{2}\int_0^\infty dz\ \frac{\exp(-z)}{z}\left(1 - \sqrt{1-2z\hbar}\right) \ . \tag{73}$$

In the integral representation, the ambiguity arising from the branch cut shows up as a nonpertrubative effect only. Note that this result allows us to conclude that

$$P_{L-2}(0) = -\frac{(2L-2)!}{2^{L+1}\cdot 3\cdot L!} \ , \ \ L \geq 2 \ . \tag{74}$$

5 Higher loops and nullification patterns

The observed patterns of vanishing higher loop corrections leads naturally to the question of whether there are more such patterns, maybe for non-recursive actions. To answer this it is necessary to determine the general structure of higher-loop corrections in general zero-dimensional theories. To this end, we may write the L-loop term in the SD equation as

$$\sum_{\{c_{l,n}\}} S^{(k)}\left(\phi_0(x)\right) \prod_{l,n\geq 0} \left[\frac{1}{c_{l,n}!}\left(\phi_l^{(n)}(x)\frac{\hbar^n}{(n+1)!}\right)^{c_{l,n}}\right] = 0 \quad , \tag{75}$$

where the sum extends over all nonnegative integer values of $c_{l,n}$ with the condition that $\sum(l+n)c_{l,n} = L$, and $k = 1 + \sum(n+1)c_{l,n}$: the upper indices in brackets denote derivatives. This equation is valid for general zero-dimensional actions: all specifics of the action are encoded in $\phi_0(x)$, or rather $f(x) = \phi_0'(x)$: since by definition $S'(\phi_0(x)) = x$, we can find the higher derivative terms using

$$S^{(k)}(\phi_0(x)) = \frac{1}{f(x)}\frac{\partial}{\partial x}S^{(k-1)}(\phi_0(x)) \quad . \tag{76}$$

In the above representation of the SD equation, the L-loop correction $\phi_L(x)$ occurs only in the combination $\phi_L(x)S^{(2)}(\phi_0(x))$, and therefore $\phi_L(x)$ is simply expressed in terms of the lower ones and their derivatives, and hence eventually in terms of $f(x)$ and its derivatives. The first few loop corrections are

$$\begin{aligned}
\phi_1 &= \frac{1}{2}f_1 \quad , \\
\phi_2 &= \frac{1}{24f}\left(12f_1^3 - 14f_2f_1 + 3f_3\right) \quad , \\
\phi_3 &= \frac{-1}{48f^2}\left(-144f_1^5 - 68f_1^2f_3 + 11f_4f_1 - f_5 - 96f_1f_2^2\right. \\
&\quad \left. +276f_1^3f_2 + 20f_3f_2\right) \quad , \\
\phi_4 &= \frac{1}{5760f^3}\left(138480f_1^4f_3 - 1212f_4f_3 - 136800f_1^2f_2f_3 + 13260f_2^2f_3\right. \\
&\quad +9360f_1f_3^2 + 204480f_1^7 - 300f_6f_1 + 15f_7 - 780f_5f_2 + 3320f_5f_1^2 \\
&\quad +390960f_1^3f_2^2 - 545280f_1^5f_2 + 14620f_1f_4f_2 \\
&\quad \left. -25200f_1^3f_4 - 64440f_1f_2^3\right) \quad , \\
\phi_5 &= \frac{1}{11520f^4}\left(8400f_3^3 + 3f_9 + 7292160f_1^9 - 152520f_1^2f_5f_2 + 9760f_6f_1f_2\right. \\
&\quad +17872f_1f_5f_3 + 1552320f_1^3f_2f_4 - 237560f_1^2f_3f_4 + 39400f_2f_3f_4
\end{aligned}$$

$$-335820 f_1 f_2^2 f_4 - 430000 f_1 f_2 f_3^2 + 4212960 f_1^2 f_2^2 f_3 - 11004960 f_1^4 f_2 f_3$$
$$-202800 f_2^3 f_3 - 320 f_7 f_2 - 19640 f_1^3 f_6 + 1640 f_7 f_1^2 + 992160 f_2^4 f_1$$
$$-10357920 f_1^3 f_2^3 - 95 f_8 f_1 + 27145440 f_1^5 f_2^2 - 24914880 f_1^7 f_2$$
$$+994240 f_1^3 f_3^2 + 6411840 f_1^6 f_3 - 962 f_5 f_4 - 672 f_6 f_3 + 12660 f_2^2 f_5$$
$$-1217040 f_1^5 f_4 + 10870 f_1 f_4^2 + 176400 f_1^4 f_5 \Big) \quad , \tag{77}$$

where

$$f = f(x) \quad , \quad f_n = \frac{1}{f(x)} f^{(n)}(x) \quad . \tag{78}$$

Note the 'homogeneity' in the number of derivatives in each term: this also follows from simple dimensional arguments. The highest derivative occurring in ϕ_L is f_{2L-1}.

The requirement of one-loop nullification, $\phi_1 = 0$, gives immediately that $f'(x) = 0$, so that the free action is the only possibility, as we know. Let us therefore study nullification at two loops, that is, $\phi_2 = 0$. This implies the following differential equation for $g(x) = f_1$:

$$3g'' - 5gg' + g^3 = 0 \quad . \tag{79}$$

Writing $g' = g^2 s(g)$ we can rewrite this as

$$3gss' + 6s^2 - 5s + 1 = 0 \quad . \tag{80}$$

Two obvious solutions are $s = 1/2$ and $s = 1/3$: the corresponding actions are $S_2(\varphi)$ and $S_3(\varphi)$, discused above. Otherwise, we can integrate the equation to get

$$\log(g) = \log(3s - 1) - \frac{3}{2}\log(2s - 1) + c \quad , \tag{81}$$

where c is the constant of integration. This in turn tells us that s is the solution of a third-order algebraic equation involving g and c, so that there are in principle 3 solutions for $s(g) = -(1/g)'$. Working back to $\phi_0(x)$ we pick up 2 additional constants of integration, which correspond to a trivial scaling transform $\phi_0(x) \to a_1\phi_0(a2x)$. Disregarding this, we conclude that there are 3 different one-parameter classes of actions that show two-loop nullification. Note that these are not recursive, since any recursive action implies $s =$constant.

If, in addition to two-loop nullification, there is also nullification at some higher L, we may employ $\phi_2 = 0$ to express $f_{3,4,5,\ldots}$ in terms of f_1 and f_2. Nullification at L loops therefore implies, because of the homogeneity, that

$$\phi_L = 0 \Rightarrow P\left(\frac{f_2}{f_1^2}\right) = 0 \quad , \tag{82}$$

where P is some finite polynomial. This in turn means that g'/g^2 is a constant, so that the action is necessarily recursive, and $S_{2,3}(\varphi)$ appear as the only possibilities. We conclude that *if the L-loop amplitudes vanish for $L = 2$ and one higher value of L, all loop corrections are identically zero beyond the one-loop level.*

For the actions $S_{0,1,2,3}(\varphi)$ we see that both ϕ and ϕ^{1PI} have interesting nullification patterns. It is natural to wonder whether there are other theories in which the effective action has no corrections beyond the one-loop level, that is: is there a theory in which

$$\Gamma'(\phi) = S'(\phi) + \hbar\Gamma_1'(\phi) = S'(\phi_0) \quad . \tag{83}$$

This question can be answered by inserting the loop expansion for ϕ involving $f(x)$ and its derivatives, and making a Taylor expansion in $\hbar$ in the above equation. We immediately find, from the term linear in $\hbar$:

$$\Gamma_1'(\phi_0) = -\frac{f'(x)}{2f(x)^2} \quad , \tag{84}$$

so that not only the derivatives of S but also those of Γ_1 are completely expressed in terms of f and f_j. The $\hbar^2$ term then results in

$$3g''(x) = -4g(x)^3 + 11g(x)g'(x) \quad , \quad g(x) = f'(x)/f(x) \quad , \tag{85}$$

so that all terms $\hbar^L$, $L \geq 3$ are completely expressed in terms of g'/g^2: again we are naturally led to recursive actions. From the $\hbar^3$ and $\hbar^4$ terms we find the conditions

$$\begin{aligned} 0 &= -2g^5(6v+1)(2v-1) \quad , \\ 0 &= -8g^7(2v-1)(1395v^2 + 690v - 646)) \quad , \end{aligned} \tag{86}$$

where $g' = vg^2$ so that $v = 1/u$. The only common solutions are $g = 0$, corresponding to the free action S_0, and $u = 2$, corresponding to the action S_2, and we know that for these theories the effective action indeed stops at one loop. Thus we have proven that these are, in fact, the only theories with this property.

6 Nullification for recursive actions

Computing the higher-order amplitudes from Eq(75) is in principle straightforward, but for large L it becomes impractical. The number of terms in the expression (75) for given L is easily seen to be equal to the coefficient of x^L in the function

$$\prod_{m\geq 1}\left(\frac{1}{1-x^m}\right)^{m+1} \quad ,$$

and hence grows much faster than the number of partitions of m which is known to grow as $\sim \exp(\pi\sqrt{2n/3})$. For recursive actions, however, we can simplify the treatment, as follows. The SD equation, written in terms of $\phi(x)$, reads

$$S'\left(\phi(x)+\hbar\frac{\partial}{\partial x}\right)e(x) = x \quad , \tag{87}$$

where $e(x) = 1$ is the unit function. This means that the SD equation is built up from

$$R_n = \left(\phi(x)+\hbar\frac{\partial}{\partial x}\right)^n e(x) \quad , \quad n = 1,2,3,\ldots \tag{88}$$

and we must evaluate these objects efficiently. As before, we put $\mu = p = 1$. Now, the R_n are all of the form

$$\begin{aligned} R_{n+1} &= g_{0,n\alpha}\sum_{L\geq 0}\hbar^L g_{0,L\beta}P_{n,L}(z) \quad , \\ g_{m,n} &\equiv \phi_0(x)^m\frac{1}{(1+x)^n} \quad , \\ z &\equiv g_{1,-\alpha} \quad . \end{aligned} \tag{89}$$

This hinges on the fact that

$$\frac{\partial}{\partial x}g_{m,n} = (m-nz)g_{m-1,n+1+\alpha} \quad , \quad \frac{\partial}{\partial x}z = (1+\alpha z)g_{0,1} \quad . \tag{90}$$

By inspection of the form of R_2 we obtain the following recursive definition[7] of the polynomials $P_{n,L}$:

$$P_{n,L}(z) \quad = \quad \theta(n=1)\left\{\theta(L=0)z+\theta(L\geq 1)t_L\right\}$$

[7]The logical step function $\theta(A)$ is one if A is true, else zero.

$$
\begin{aligned}
&+ \quad \theta(n \geq 2)\left\{\sum_{M=0}^{L} P_{1,M}(z)P_{n-1,L-M}(z)\right.\\
&+ \quad \theta(L \geq 1)((1-\alpha)(1-L)-(n-1)\alpha)P_{n-1,L-1}(z)\\
&+ \quad \left.\theta(L \geq 1)(1+\alpha z)\frac{\partial}{\partial x}P_{n-1,L-1}(z)\right\} \ . \qquad (91)
\end{aligned}
$$

Here we have written $t_L = t_L(u)$ for $\phi_L(0)$, with $u = 1+\alpha$. Using this recursion, we can compute the $P_{n,L}(z)$ to quite high order in L: notice that for given L, they have to be computed up to $n = 2L$. Since the SD equation holds for any x we may evaluate it at $x = 0$, where $z = 0$ and $g_{0,k} = 1$ for any k: it then becomes

$$
\sum_{n\geq 2} \frac{\lambda_{n+1}}{n!} \lfloor R_n \rfloor_{x=0} = 0 \ . \qquad (92)
$$

This allows us to successively determine the t_L. We have implemented this approach in a `FORM` program [7]. Note that the complexity of the algorithm in its most straightforward form is of order L^6; by various optimizations we managed to go up to $L = 50$ which takes about 24 hours of `FORM`. Note that for $L = 50$ Eqn(75) contains 213,927,397,257 terms!

The lowest-order polynomials $t_L(u)$ read

$$
\begin{aligned}
t_1(u) &= -\frac{u}{2} \ ,\\
t_2(u) &= -\frac{u}{24}(u-2)(u-3) \ ,\\
t_3(u) &= -\frac{u}{24}(u-1)(u-2)^2(u-3) \ ,\\
t_4(u) &= -\frac{u}{1920}(u-2)(u-3)(7u^2-21u+12)(23u^2-69u+50) \ ,\\
t_5(u) &= -\frac{u}{1440}(u-1)(u-2)^2(u-3)\\
&\quad (367u^4-2202u^3+4685u^2-4146u+1260) \ ,\\
t_6(u) &= -\frac{u}{580608}(u-2)(u-3)\\
&\quad (601285u^8-7215420u^7+37068226u^6-106328304u^5\\
&\quad +185954749u^4-202661124u^3+134127612u^2\\
&\quad -49166352u+7620480) \ ,\\
t_7(u) &= -\frac{u}{60480}(u-1)(u-2)^2(u-3)
\end{aligned}
$$

$$
\begin{aligned}
& (318344u^8 - 3820128u^7 + 19590653u^6 - 55981845u^5 \\
& +97298966u^4 - 105068217u^3 + 68637897u^2 \\
& -24716070u + 3742200) \ , \\
t_8(u) \ = \ & -\frac{u}{199065600}(u-2)(u-3) \\
& (6389072441u^{12} - 115003303938u^{11} + 935605664709u^{10} \\
& -4546312395750u^9 + 14686419780735u^8 - 33204111807078u^7 \\
& +53832598760431u^6 - 63007849676250u^5 + 52791473853204u^4 \\
& -30847914995544u^3 + 11919566344320u^2 \\
& -2731077648000u + 280215936000) \ , \qquad (93)
\end{aligned}
$$

and so on.

From duality we have seen that $t_L(u)$ is of the form

$$
t_L(u) = u(u-2)(u-3)Q_L(u) \ , \qquad (94)
$$

with $Q_L(u)$ a polynomial of degree $2L-4$ in u, which is symmetric under $u \leftrightarrow 3-u$: it is therefore a polynomial of degree $L-2$ in the variable $u(3-u)$, which makes finding the roots simpler. We therefore discuss only the 'lower half' of the roots of $Q_L(u)$. Surprisingly, all roots are real up to $L = 18$, where a pair of conjugate roots appear. Other pairs appear at $L = 27, 34, 41$ and 48. In figure 1 we give the distribution of the real values of the roots of $Q_L(u)$ that are smaller than $3/2$. It is suggestive to follow roots over trajectories as L increases. At $L = 18$ the second and third lowest lines appear to merge, leading to conjugate complex roots. For higher L, the fourth line does not merge with these two, but actually crosses them (this is borne out by inspecting which roots are real, and which ones complex). For $L = 27$ a similar phenomenon occurs, and so on. For large L, there is an apparent asymptotic upper limit $3/2$ which is just an artefact of our restriction to $\mathrm{Re}(u) < 3/2$. There is also a lower asymptotic bound $1/2$. This bound can, in fact, be understood: it is easy to see that, for $0 < u < 1/2$, the single vertex $-\lambda_{n+1}$ occurring in a diagram has *the same sign* as the product $\lambda_n\lambda_3$, corresponding to 'opening up' the vertex by insertion of a propagator. This imples that in that case all diagrams contributing to a given amplitude have precisely the same sign, so no cancellation is possible and no root of $Q_L(u)$ can be in $(0, 1/2)$ for any L: by duality, the same holds for the interval $(5/2, 3)$. On the other hand, it is not clear why there should be no roots with negative real part (or real part larger than 3), or why there should not be any complex roots with real part between 0 and 1/2. Figure 2 shows the distribution of the imaginary parts of the roots, where the branching structure becomes especially apparent.

7 Renormalization

Because of the fairly simple structure of $\phi(x)$ in zero dimensions, it is often possible to carry through a renormalization program for recursive actions, as we shall now show.

Reinserting generic values for α and μ, the action $S_2(\varphi)$ has for its solution

$$\phi(x) = \frac{x - \alpha\hbar}{\mu + \alpha x} \quad . \tag{95}$$

As a first step, we add a tadpole counterterm to the action, which has the effect of shifting the variable x by a constant. The tadpole renormalization condition is that $\phi(x)$ should have no tadpole left after renormalization. Denoting the renormalized generating function by $\phi_r(x)$, we therefore have

$$\phi_r(x) \equiv \phi(x + c) \quad , \quad \phi_r(0) \equiv 0 \quad \Rightarrow \quad c = \alpha\hbar \quad , \tag{96}$$

so that

$$\phi_r(x) = \frac{x}{\mu + \alpha^2\hbar + \alpha x} \quad . \tag{97}$$

The second renormalization condition is that of mass renormalization. Denoting the renormlized, physical mass by m, we therefore require

$$\phi_r'(0) = \frac{1}{m^2} \quad , \tag{98}$$

which fixes μ:

$$\mu = m^2 - \alpha^2\hbar \quad . \tag{99}$$

The resulting renormalized generating function and the renormalized effective action can therefore be written as

$$\phi_r(x) = \frac{x}{m^2 + \alpha x} \quad , \quad \Gamma(\varphi) = \frac{m^2}{\alpha^2}\left(-\alpha\varphi - \log(1 - \alpha\varphi)\right) \quad . \tag{100}$$

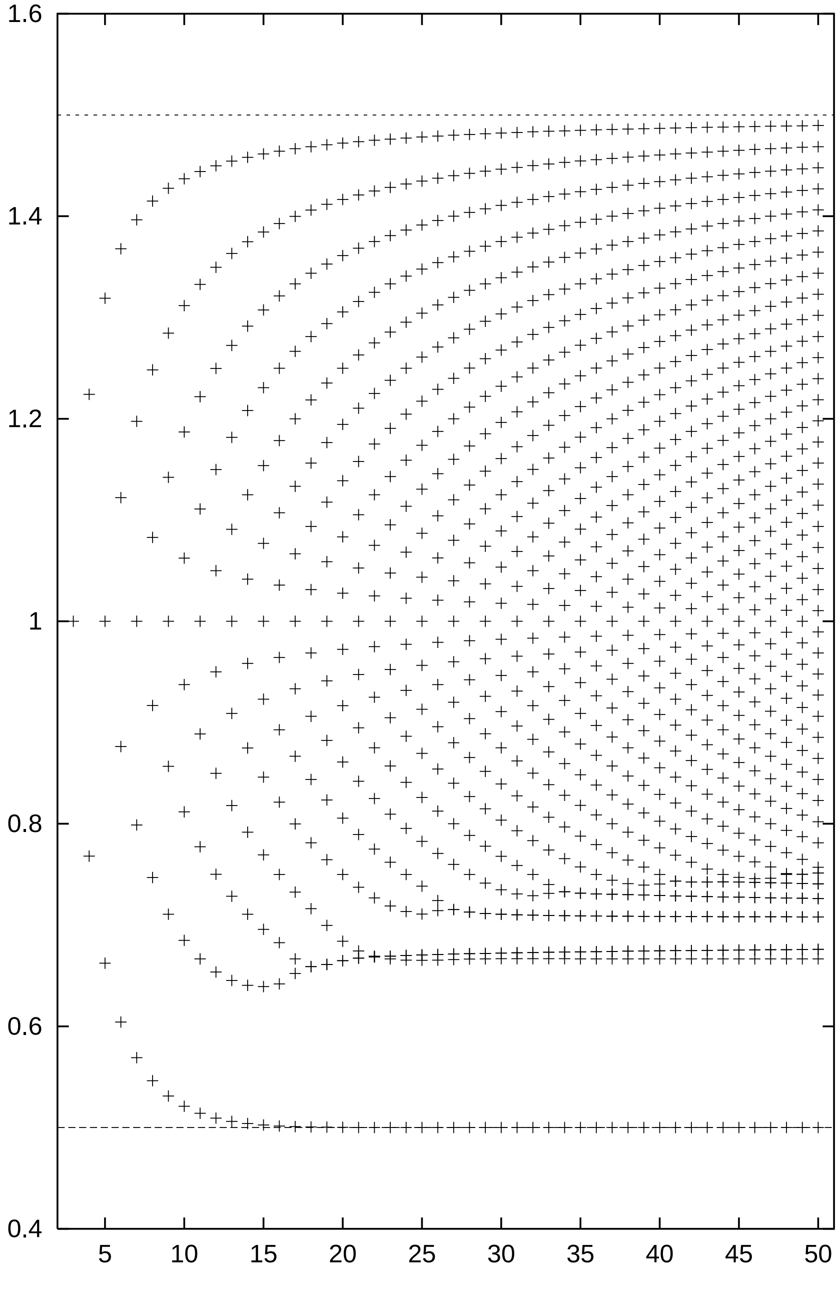

Figure 1: Distribution of the real values of u for which $Q_L(u)$ vanishes, for $3 \leq L \leq 50$. Horizontal: L, vertical: $\mathrm{Re}(u)$.

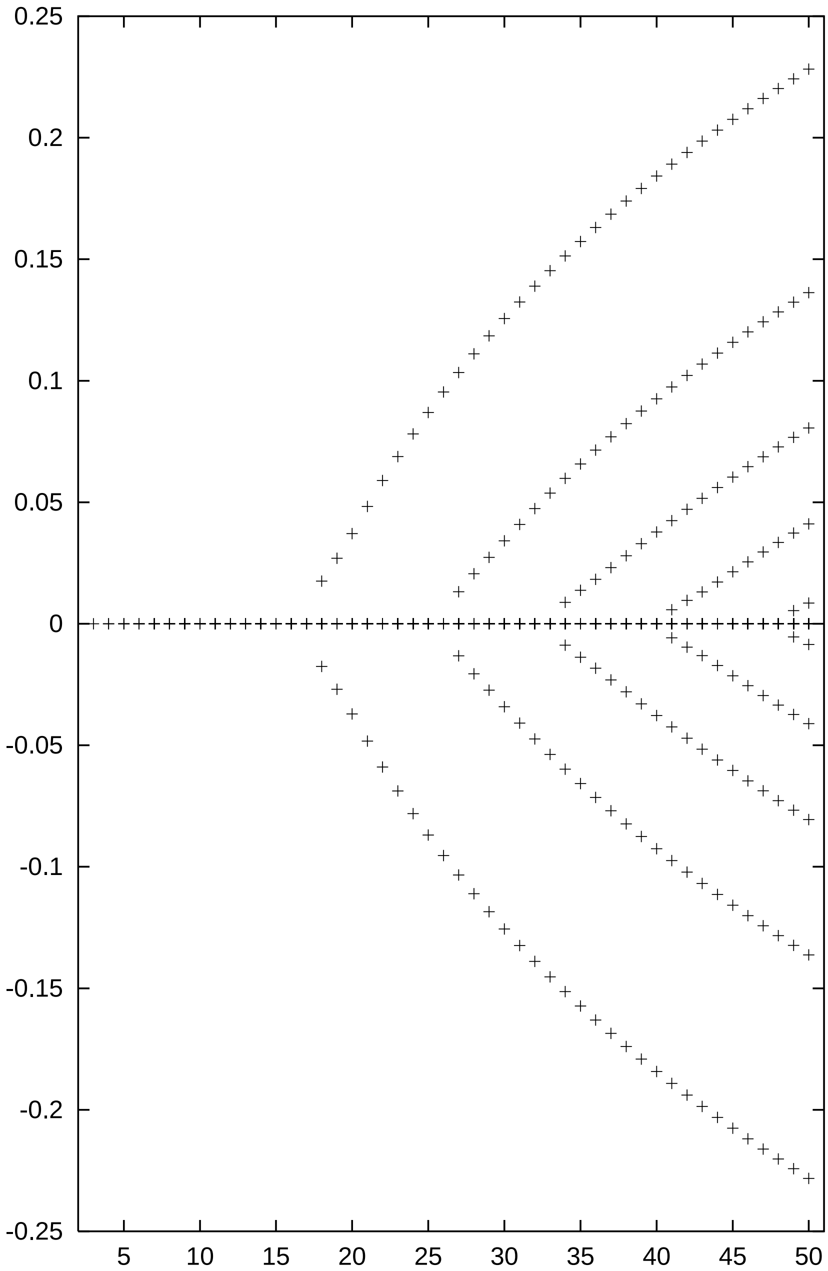

Figure 2: Distribution of the imaginary values of u for which $Q_L(u)$ vanishes, for $3 \leq L \leq 50$. Horizontal: L, vertical: $\mathrm{Im}(u)$.

Hence, *all* loop corrections have been completely absorbed. As mentioned before, a similar finding occurs for $d = 0, u = 4/3$.

The action $S_3(\varphi)$ also leads to fairly simple results. We have

$$\phi(x) = \frac{1}{\alpha} - \frac{4\mu^2}{\alpha}\frac{1}{(2\mu+\alpha x)^2} - \frac{3\alpha\hbar}{2}\frac{1}{2\mu+\alpha x} \quad . \tag{101}$$

Writing the tadpole counterterm as $(c-2)\mu/\alpha$ we have for the renormalized ϕ:

$$\phi_r(x) = \frac{1}{\alpha} - \frac{4\mu^2}{\alpha}\frac{1}{(c\mu+\alpha x)^2} - \frac{3vm^2}{2\alpha}\frac{1}{c\mu+\alpha x} \quad , \tag{102}$$

where the renormalization conditions are, as before,

$$\phi_r(0) = 0 \quad , \quad \phi_r'(0) = \frac{1}{m^2} \quad , \tag{103}$$

and we have introduced the dimensionless parameter $v = \alpha^2\hbar/m^2$. The two renormalization conditions imply two coupled equations for c and μ:

$$\begin{aligned} 0 &= -2\mu^2c^3 + 3vm^4c + 16\mu m^2 \quad , \\ 0 &= 16\mu m^2 + (-8\mu^2 + 3vm^4)c - 3\mu c^2 vm^2 \quad . \end{aligned} \tag{104}$$

These can be combined to give c as a function of μ, and a quadratic equation for μ^2:

$$\begin{aligned} c &= \frac{8\mu m^2(4-3v)}{16\mu^2 - 6vm^4 + 9v^2m^4} \quad , \\ 0 &= -128\mu^4 - 192m^4v\mu^2 + 128m^4\mu^2 + 18v^2m^8 - 27v^3m^8 \quad . \end{aligned} \tag{105}$$

The perturbative solution for μ is

$$\mu = \frac{m^2}{4}\sqrt{8 - 12v + \sqrt{64 - 192v + 180v^2 - 54v^3}} \quad , \tag{106}$$

so that the perturbative expansion of μ and c contain an infinite number of terms. we conclude that, although the unrenormalized amplitudes vanish for two or more loops, renormalization reintroduces nonzero amplitudes at all loops.

For the action $S_1(\varphi)$ we have

$$\phi(x) = \frac{1}{p}\left[\psi\left(\frac{\mu+px}{\hbar p^2}\right) + \log\left(\frac{\hbar p^2}{\mu}\right)\right] \quad . \tag{107}$$

The conditions for the renormalized function $\phi_r(x) = \phi(x+c)$ now read

$$\begin{aligned} \phi_r(0) = 0 \quad &\Rightarrow \quad \psi(w) = \log\left(\frac{\mu}{p^2\hbar}\right) \quad , \\ \phi_r'(0) = \frac{1}{m^2} \quad &\Rightarrow \quad \psi'(w) = \frac{p^2\hbar}{m^2} \quad , \end{aligned} \tag{108}$$

with $w = (\mu + pc)/(p^2\hbar)$. Since ψ' is monotonic for $w \geq -1$ and takes all real values, the second equation gives w uniquely for given m; and the first one then gives μ, and hence also c. Following through this program in perturbation theory gives the following interesting result. Using the dimensionless number $v = p^2\hbar/m^2$ and the asymptotic expansions for ψ and $\psi'(w)$, we obtain the following results:

$$\begin{aligned} w &= \frac{1}{v} + \frac{1}{2} - \frac{1}{12}v + \frac{11}{720}v^3 - \frac{379}{30240}v^5 + \frac{24369}{1209600}v^7 \cdots \quad , \\ \frac{\mu}{m^2} &= 1 - \frac{1}{24}v^2 + \frac{71}{5760}v^4 - \frac{31741}{2903040}v^6 + \frac{25265783}{1393459200}v^8 + \ldots \quad , \\ c\frac{p}{m^2} &= \frac{1}{2}v - \frac{1}{24}v^2 + \frac{17}{5760}v^4 - \frac{4643}{2903040}v^6 + \frac{559157}{278691840}v^8 + \ldots \quad , \end{aligned} \tag{109}$$

and, putting $m = p = 1$ for simplicity, we find for the renormalized generating function

$$\begin{aligned} \phi_r(x) = \; & \log(1+x) + \frac{x^2v^2}{24}\frac{1}{(1+x)^2} \\ & - \frac{x^2v^4}{2880}\frac{76+88x+33x^2}{(1+x)^4} \\ & + \frac{x^2v^6}{362880}\frac{3790x^4+18192x^3+34572x^2+31636x+12861}{(1+x)^6} \\ & + \cdots \end{aligned} \tag{110}$$

The higher powers of v in the results for (vw),μ,c and ϕ all appear to be even. We have checked this up through order v^{30}. The conclusion is that for the action S_1, *all* odd-loop corrections vanish after tadpole and mass renormalization, thereby even improving on the unrenormalized pattern. This is in accordance with our conjecture that for this theory the only odd-loop contribution to the effective action is the one-loop tadpole, which is removed by renormalization.

8 Conclusions

We have identified recursive theories, an essentially one-parameter class of self-interacting scalar theories in which zero-momentum amplitudes are related in a simple and systematic manner. In the case of zero-dimensional theories, several of these theories can be solved exactly and display an interesting pattern of vanishing higher-loop amplitudes and 1PI amplitudes. We have identified a duality property in which composite objects in a given theory are the elementary fields in its dual. A study of the dependence of the higher-loop amplitudes on the parameter, u, of the theory reveals a remarkable pattern of roots.

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