Physics at Colliders

The Two-Loop QCD Matrix Element for $e^+e^- \to 3$ Jets

Lee Garland April 2002

In collaboration with T. Gehrmann,

E. W. N. Glover, A. Koukoutsakis, E. Remiddi

I. Motivation

II. Two-Loop Integrals and their reduction

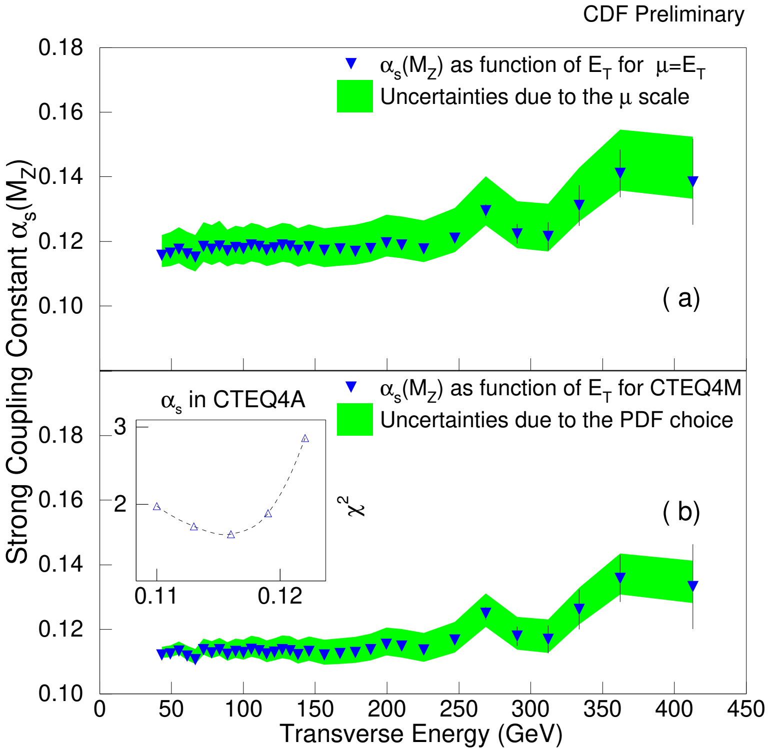
III. Matrix Element

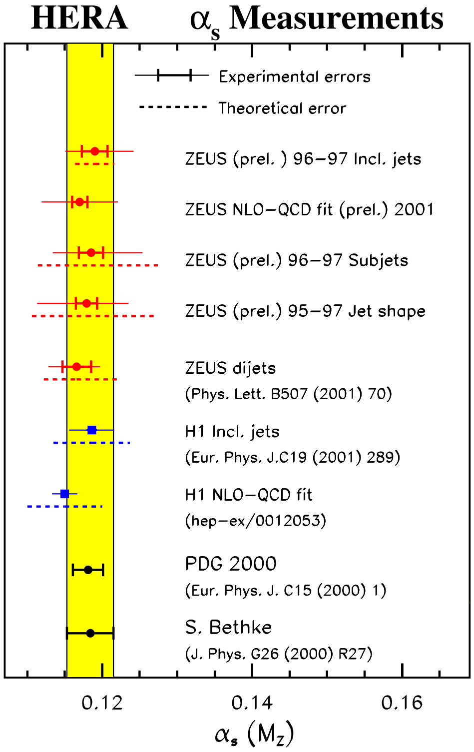
IV. Structure of Divergences

V. Future

Motivation

⋆ Experimental precision has reached a level whereby it is overtaking theoretical accuracy





⋆ Further improvement in the future, e.g. HERA high luminosity programme

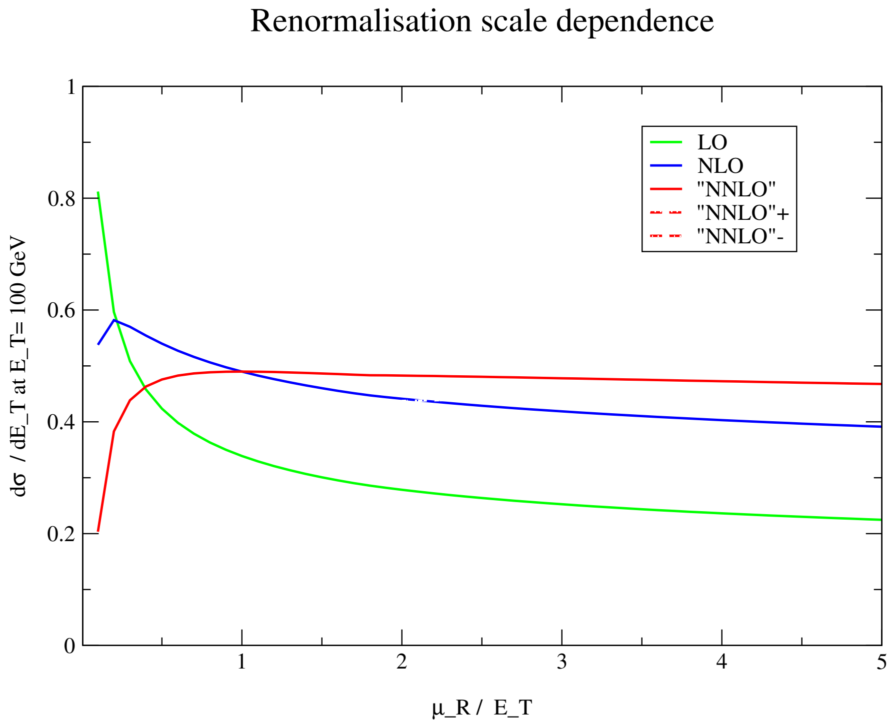
⋆ Large uncertainties from renormalisation scale dependence and pdf's

⋆ $\Rightarrow$ Higher order corrections

⋆ Renormalisation scale dependence in $p\bar{p} \to$ jet;

$$\begin{aligned}\frac{d\sigma}{dE_T} &= \alpha_s(\mu_R)^2 A \\ &+ \alpha_s(\mu_R)^3(B + 2b_0 L A) \\ &+ \alpha_s(\mu_R)^4(C + 3b_0 L B + (3b_0^2 L^2 + 2b_1 L) A)\end{aligned}$$

with $L = \log(\mu_R/E_T)$ and b_i the beta function coefficients



Single jet inclusive distribution at $E_T = 100$Gev and

$0.1 < |\eta| < 0.7$ at $\sqrt{s} = 1800$Gev

NNLO coefficient C unknown. Guess $C = 0$ solid and $C = \pm B^2/A$ dashed

⋆ Overall decrease in μ_R dependence at NNLO

- ⋆ An NLO calculation decreases scale dependence
- ⋆ An NNLO calculation provides an estimate of the uncertainty
- ⋆ Much work has already been done on $2 \to 2$ scattering at two-loops, e.g.,
  - ▷ $e^+e^- \to e^+e^-$ Bern, Dixon, Ghinculov
  - ▷ $\gamma\gamma \to \gamma\gamma$ Bern, De Freitas, Dixon, Ghinculov, Wong
  - ▷ $gg \to \gamma\gamma$ Bern, De Freitas, Dixon, Ghinculov
  - ▷ QCD scattering amplitudes, e.g., $q\bar{q} \to q\bar{q}$ Anastasiou, Glover, Oleari, Tedjeda-Yeomans
  - ▷ $q\bar{q} \to \gamma\gamma$, $q\bar{q} \to g\gamma$ Anastasiou, Glover, Tedjeda-Yeomans
- ⋆ Need to extend this work to include $1 \to 3$ scattering at two-loops, e.g.,
  - ▷ Three Jet events, e.g., $e^+e^- \to q\bar{q}g$

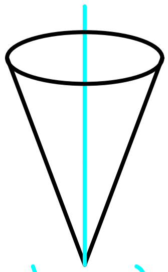
And related processes,

- ▷ DIS $(2+1)$-Jet production
- ▷ Hadron–Hadron 2-Jet and $V+1$-Jet production

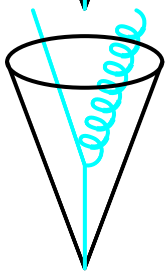
Jets at NNLO

- Jet observables important testing ground in QCD for e^+e^- annihilation
- 3–jet production rate allows determination of α_s
- Current error on α_s dominated by theoretical uncertainty
- Higher orders allow possibility to combine more partons into jets

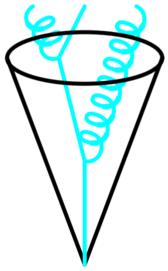
LO 1 parton per jet



NLO 2 partons per jet



NNLO 3 partons per jet



- An NNLO calculation of $e^+e^- \to 3$ jets should reduce uncertainty on α_s down to $\sim 1\%$

▷ $e^+e^- \to 3$ jets at NNLO

Sub-process	Partonic Final State
$\gamma^* \to 3$ partons, 2 loop	3 partons
$\gamma^* \to 4$ partons, 1 loop	4 partons 3+1 partons
$\gamma^* \to 5$ partons, tree	5 partons 4+1 partons 3+2 partons

Red indicates unresolved (soft or collinear)

▷ $\gamma^* \to 5$ partons

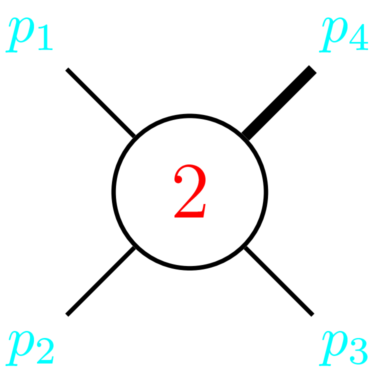
K. Hagiwara, D. Zeppenfeld
F. A. Berends, W. T. Giele, H. Kuijf
N. K. Falck, D. Graudenz, G. Kramer

▷ $\gamma^* \to 4$ partons

Z. Bern, L. J. Dixon, D. A. Kosower
Z. Bern, L. J. Dixon, D. A. Kosower, S. Weinzierl
E. W. N. Glover, D. J. Miller
J. .M. Campbell, E. W. N. Glover, D. J. Miller

▷ $\gamma^* \to 3$ partons requires calculation of 2-loop 4-point functions with 1 off-shell leg

Two-Loop Four-Point Functions



$$p_1^2 = p_2^2 = p_3^2 = 0$$
$$p_4^2 = (-p_1 - p_2 - p_3)^2 = p_{123}^2$$

⋆ General structure of 2 loop integral;

$$I_{t,r,s}(p_1, p_2, p_3) = \int d^D k_1 d^D k_2 \frac{S_1^{n_1} \cdots S_q^{n_q}}{D_1^{m_1} \cdots D_t^{m_t}}$$

- ▷ D_i massless scalar propagators
- ▷ S_i scalar products constructed from momenta
- ▷ t number of propagators
- ▷ $r = \sum_i m_i$ dimension of denominator
- ▷ $s = \sum_i n_i$ dimension of numerator

⋆ Topology of integral defined by the propagators;

$$\{D_1, \ldots, D_t\}$$

Reduction of Integrals

▷ Integration-by-parts (IBP) G. 't Hooft, M. Veltman
K. G. Chetrkin, F. V. Tkachov

$$\int d^D k_1 d^D k_2 \frac{\partial}{\partial v_1^\mu} \left[v_2^\mu f(k_1, k_2, p_i) \right] = 0$$

$v_1 = k_1^\mu, k_2^\mu$ and $v_2 = k_1^\mu, k_2^\mu, p_i$

▷ Lorentz Invariance (LI) T. Gehrmann, E. Remiddi

$$\int d^D k_1 d^D k_2 \epsilon_\nu^\mu \left(\sum_i p_i^\nu \frac{\partial}{\partial p_i^\mu} \right) f(k_1, k_2, p_i) = 0$$

Two-loop four-point functions have 3 independent external momemta ⇒ 10 IBP and 3 LI identities

⋆ Repeated application allows reduction of all integrals to a small set of Master Integrals (MI)

Laporta Algorithm

⋆ Algorithm which allows the systematic solution of the IBP and LI identities

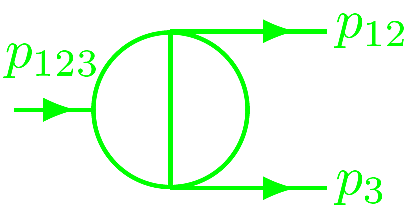
- ▷ Generate all sub-topologies i.e. $3, 4, \ldots$ propagator topologies up to the dimension of required integral
- ▷ Starting with smallest topology as 'seeds' generate system of all IBP (and LI) equations
- ▷ Using some priority (e.g. highest tensor power etc.) solve each IBP equation for highest priority integral and eliminate from system
- ▷ Move on to next level of topology again eliminating highest priority integrals and substituting all previous known integrals into current system
- ▷ Continue until all IBP (and LI) equations are exhausted

⋆ All remaining integrals cannot be eliminated $\Rightarrow$ Master Integrals

Master Integrals

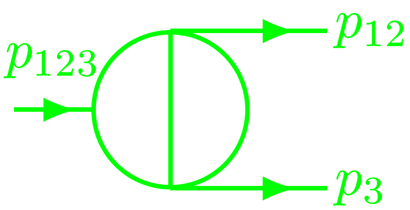
- Different techniques for calculation of MI
  - Mellin-Barnes transformations
  - Differential equations
- Multi-scale MI satisfy inhomogeneous differential equations in their external scales, e.g., E. Remiddi

$$s_{123}\frac{\partial}{\partial s_{123}}$$

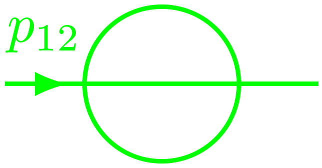


$$=$$

$$\frac{D-4}{2}\,\frac{2s_{123}-s_{12}}{s_{123}-s_{12}}$$



$$-\frac{3D-8}{2}\,\frac{1}{s_{123}-s_{12}}$$

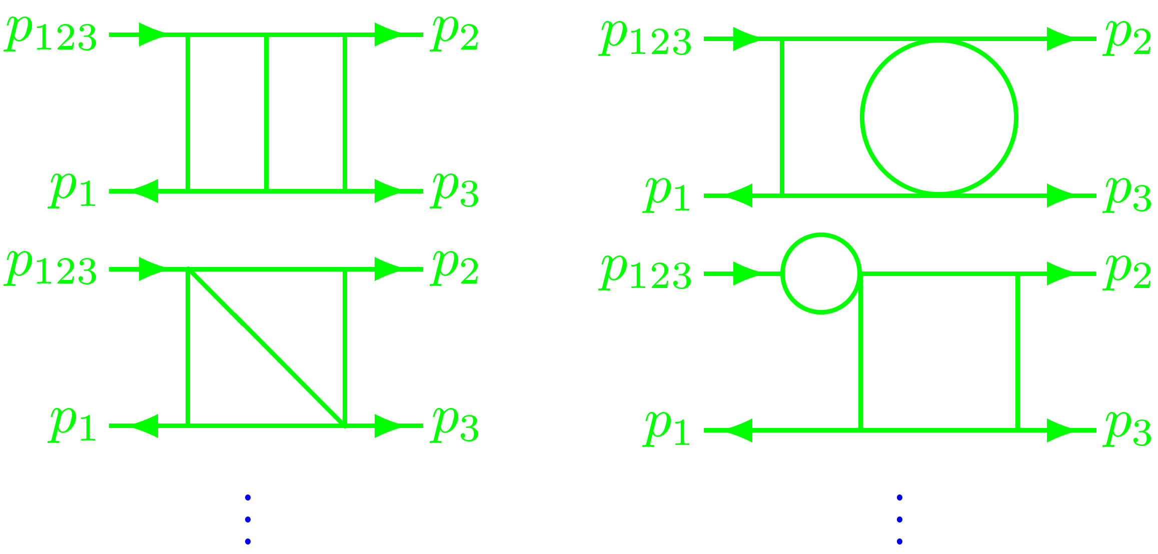


- MI found by integration of differential equations and application of boundary conditions
- Three-scale integrals (four-point functions one off-shell leg) require introduction of new MI

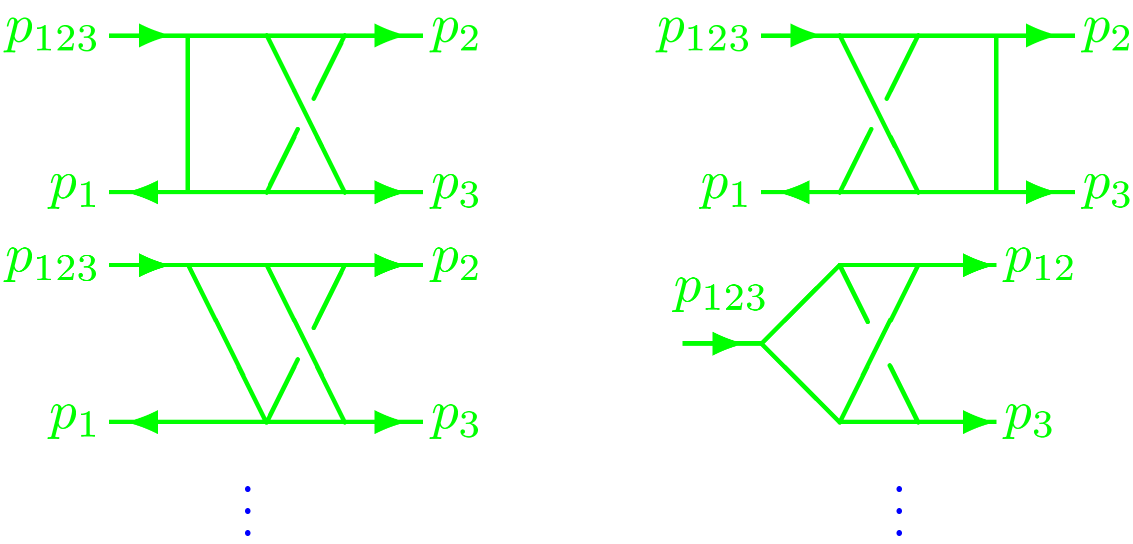
- More MI than four-point on-shell case, 24 in total
- More complicated ⇒ 2 dimensional Harmonic Polylogarithms

T. Gehrmann, E. Remiddi, hep-ph/0008287, hep-ph/0101124

- 16 Planar MI, e.g.,



- 8 Non-planar MI, e.g.,



Harmonic Polylogarithms

- Two-scale integrals contain only polylogarithms, $\mathrm{Li}_n(x)$, $\mathrm{S}_{n,p}(x)$
- Harmonic polylogarithms (HPL) extend the polylogarithms, $H(\vec{m};x)$
- Start by defining,

$$\begin{aligned} H(1;x) &\equiv -\ln(1-x) \\ H(0;x) &\equiv \ln(x) \\ H(-1;x) &\equiv \ln(1+x) \end{aligned}$$

and $f(1;x) \equiv \frac{1}{1-x}$, $f(0;x) \equiv \frac{1}{x}$, $f(-1;x) \equiv \frac{1}{1+x}$

- HPL are generated by

$$H(a,\vec{b};x) = \int_0^x dx\, f(a;x) H(\vec{b};x)$$

- HPL
  - are linearly independent
  - close under $\int_0^x dx \left(\frac{1}{x}, \frac{1}{1-x}, \frac{1}{1+x}\right) H(\vec{b};x)$
- 3 scale integrals $\Rightarrow$ 2 dimensional HPL, close under

$$\int_0^y dy \left(\frac{1}{y}, \frac{1}{1-y}, \frac{1}{1-y-z}, \frac{1}{y+z}\right) H(\vec{b}(z);y)$$

General Algorithm

QGRAF

QCD model

FORM

Feynman rules

MAPLE

LI & IBP identities $\rightarrow$ Laporta algorithm

All tensor integrals in terms of MI

$$\mathcal{I}(1,1,1,-2,1,-1,1,1,1,s,t,M) = c_1 \; + \cdots + c_n$$

Sum of all diagrams (tree $\times$ loop) in terms of tensor integrals

$$\langle \mathcal{M}^{(0)} | \mathcal{M}^{(2)} \rangle = i_1 \times \mathcal{I}(1,1,1,-2,1,-1,1,1,1,s,t,M) + \cdots$$

Sum of diagrams in terms of MI

$$\langle \mathcal{M}^{(0)} | \mathcal{M}^{(2)} \rangle = c_1 \; + \cdots + c_n$$

ϵ expansions of MI

Final result in terms of ϵ

$$\langle \mathcal{M}^{(0)} | \mathcal{M}^{(2)} \rangle = \sum_{i=1}^{4} \frac{\alpha_i}{\epsilon^i} + finite$$

$e^+e^- \to 3$ Jets

▷ Matrix element takes the form,

$$|\mathcal{M}\rangle = \sqrt{4\pi\alpha_q}\sqrt{4\pi\alpha_s} \times \left[|\mathcal{M}_0\rangle + \left(\frac{\alpha_s}{2\pi}\right)|\mathcal{M}_1\rangle + \left(\frac{\alpha_s}{2\pi}\right)^2|\mathcal{M}_2\rangle + \mathcal{O}(\alpha_s^3)\right]$$

▷ Require the NNLO contribution,

$$\mathcal{A}^{NNLO} = \langle\mathcal{M}^{(1)}|\mathcal{M}^{(1)}\rangle + \langle\mathcal{M}^{(0)}|\mathcal{M}^{(2)}\rangle + \langle\mathcal{M}^{(2)}|\mathcal{M}^{(0)}\rangle$$

▷ In terms of Feynman diagrams

Tree level	2 diagrams
One-loop	13 diagrams
Two-loop	229 diagrams

▷ Need to separate into finite piece and poles

$$\mathcal{A}^{NNLO} = \mathcal{P}oles + \mathcal{F}inite$$

The $\boldsymbol{I}(\epsilon)$ Operator

S. Catani and M.H. Seymour

- Coefficients of poles have universal predicable structure;

$$|\mathcal{M}^1\rangle = \boldsymbol{I}|\mathcal{M}^0\rangle + |\mathcal{M}^{1,fin}\rangle$$

$$\boldsymbol{I}(p_i,\epsilon) \sim \frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)} \sum_i^n \nu_i^{sing}(\epsilon) \sum_{j\neq i}^n \boldsymbol{T}_i \cdot \boldsymbol{T}_j \left(-\frac{\mu^2}{2p_i \cdot p_j}\right)^\epsilon$$

- $\boldsymbol{T}_i$ Colour charge operator, if_{abc} for gluons and t^a_{bc} for (anti)quark
- $\boldsymbol{T}_i^2$ Casimir operator, C_A for gluons and C_F for (anti)quark
- Divergences are generated by the $\nu_i^{sing}(\epsilon)$ function;

$$\nu_i^{sing}(\epsilon) = \frac{1}{\epsilon^2} + \frac{\gamma_i}{\epsilon}$$

- with γ_i being $\frac{\beta_0}{C_A}$ for gluons and $\frac{3}{2}$ for (anti)quark

Two Loop Divergences

S. Catani

- ▷ Extend formalism to two loop calculations

$$|\mathcal{M}^2\rangle = \boldsymbol{I}^{(1)}|\mathcal{M}^1\rangle + \boldsymbol{I}^{(2)}|\mathcal{M}^0\rangle + |\mathcal{M}^{2,fin}\rangle$$

- ▷ $\boldsymbol{I}^{(1)}$ is the 1-loop insertion operator
- ▷ $\boldsymbol{I}^{(2)}$ is new 2-loop insertion operator;

$$\begin{aligned}\boldsymbol{I}^{(2)}(\epsilon) &= -\frac{1}{2}\boldsymbol{I}^{(1)}(\epsilon)\left(\boldsymbol{I}^{(1)}(\epsilon) + \frac{2\beta_0}{\epsilon}\right) \\ &+ e^{-\epsilon\gamma_E}\frac{\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)}\left(\frac{\beta_0}{\epsilon} + K\right)\boldsymbol{I}^{(1)}(2\epsilon) + \boldsymbol{H}^{(2)}\end{aligned}$$

with K and β_0 constants and $\boldsymbol{H}^{(2)}$ contains a single pole and is process and renormalisation shcheme dependent

- ▷ $\boldsymbol{I}^{(2)}(\epsilon)$ acting on the one-loop matrix element produces poles $1/\epsilon^n$ with $n = 1, \ldots, 4$

▷ Can now extract poles and hence finite piece

$$
\begin{aligned}
\mathcal{P}oles = 2\Re\Big[\; & \frac{1}{2}\langle \mathcal{M}^{(0)}|\boldsymbol{I}^{(1)}(\epsilon)\boldsymbol{I}^{(1)}(\epsilon)|\mathcal{M}^{(0)}\rangle \\
& + \langle \mathcal{M}^{(0)}|\boldsymbol{I}^{(1)}(\epsilon)|\mathcal{M}^{(1)fin}\rangle \\
& - \frac{\beta_0}{\epsilon}\,\langle \mathcal{M}^{(0)}|\boldsymbol{I}^{(1)}(\epsilon)|\mathcal{M}^{(0)}\rangle \\
& + e^{-\epsilon\gamma}\frac{\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)}\left(\frac{\beta_0}{\epsilon}+K\right)\langle \mathcal{M}^{(0)}|\boldsymbol{I}^{(1)}(2\epsilon)|\mathcal{M}^{(0)}\rangle \\
& + \langle \mathcal{M}^{(0)}|\boldsymbol{H}^{(2)}(\epsilon)|\mathcal{M}^{(0)}\rangle \Big]
\end{aligned}
$$

▷ Decompose $\mathcal{F}inite$ into colour factors,

$$
\begin{aligned}
\mathcal{F}inite^{(2\times 0)}(x,y,z) \quad = \quad & V\Big[N^2 A_{20}(y,z) + B_{20}(y,z) \\
& + \frac{1}{N^2}C_{20}(y,z) + N N_F D_{20}(y,z) \\
& + \frac{N_F}{N}E_{20}(y,z) + N_F^2 F_{20}(y,z) \\
& + N_{F,\gamma}\left(\frac{4}{N}-N\right)G_{20}(y,z)\Big]
\end{aligned}
$$

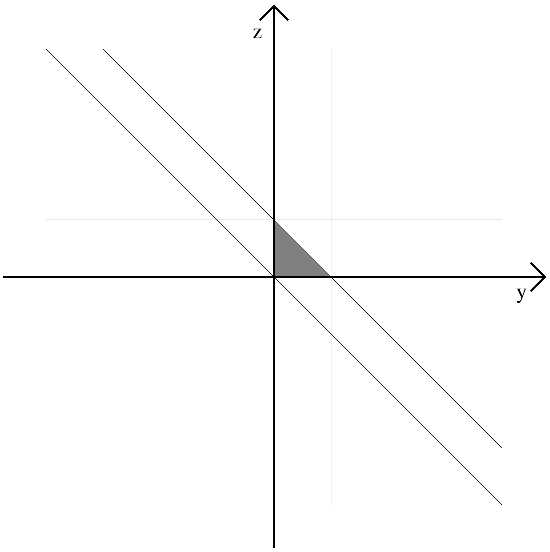
Future Work

▷ Matrix elements for $e^+e^- \to q\bar{q}g$ calculated

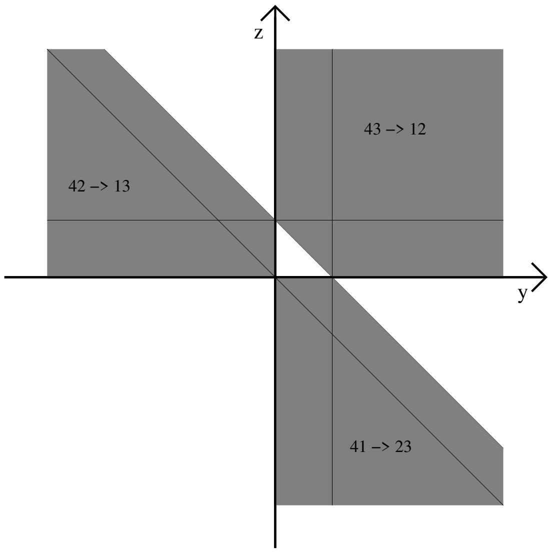
LG, T. Gehrmann, E. W. N. Glover, A. Koukoutsakis, E. Remiddi

▷ Need to extend results to other processes by crossing and analytic continuation

▷ Required for $pp \to V + 1$ Jet and $ep \to (2+1)$ Jets



3 Jet



DIS

T. Gehrmann, E. Remiddi - in progress