BooNE Neutrino Oscillations-preliminary

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Neutrino Oscillations

In the mini-BooNE neutrino oscillation experiment at Fermilab, an 8 GeV proton beam hits a target producing pions and kaons. Some decay in a 50 (25) m decay length producing neutrinos. The detector is 500 m downstream. Most neutrinos crossing the detector are ν_μ. We search for ν_μ oscillating into ν_e. The oscillation probability is:

$$p_{\phi,\Delta m^2}(E) = \sin^2 2\phi \sin^2\left(\frac{1.27\Delta m^2 L}{E}\right),$$

- ⋆ where neutrino energy E is in MeV
- ⋆ Δm^2 is in eV^2
- ⋆ L is in meters
- ⋆ ϕ and Δm^2 are parameters to be determined.

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There are 3 kinds of background:

- ⋆ Real ν_e from μ decay and K_{e3} decay.
- ⋆ $\nu_\mu N \to \mu^- N'$ events mistakenly identified as ν_e events.
- ⋆ $\nu_\mu N \to \nu_\mu \pi^0 N'$ events mistakenly identified as ν_e events.

  We will know (with some error) the number and spectrum of these background events.
  In two years of running at nominal intensity, we expect 1500 intrinsic ν_e background, 500 π^0 mis-id background, and 500 ν_μ mis-id background.

  There may or may not be a signal. If LSND is confirmed, we expect about 1000 oscillation signal events.

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Setting Limits

For setting limits, first consider the simplified case of a simple Poisson distribution. Several methods have been proposed over the last few years. We will consider a few of them.

- ⋆ Unified Method (Feldman and Cousins).
- ⋆ Conditional Confidence Intervals
- ⋆ Frequentist interpretation of Bayes intervals

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Unified Method

Let f_μ be the PDF of a Poisson distribution (μ).
If θ signal and b background events are expected,

$$L(\theta|n) = f_{b+\theta}(n) = \frac{1}{n!}(b+\theta)^n e^{-(b+\theta)}.$$

Let

$$\hat{\theta} = \max(n-b, 0),$$

$$R_\theta(n) = \frac{L(\theta|n)}{L(\hat{\theta}|n)} = \left(\frac{b+\theta}{n}\right)^n e^{n-(b+\theta)},$$

and

$$P_\theta\{n : R_\theta(n) \geq c_\theta\} \geq 1-\alpha.$$

Then the Unified intervals are

$$\{\theta : R_\theta(n) \geq c_\theta\}$$

Note: Extensive tables in Feldman and Cousins *Phys. Rev.* **D57**, (1998), 3773-3789.

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Advantages:

- ⋆ Always finds a non-null interval, even if experimental result is improbable.
- ⋆ Goes automatically from one sided limit to two-sided interval.
- ⋆ Intervals are invariant under a change of variable.
- ⋆ Works for normal distribution as well as Poisson. (Measurement of a parameter $\theta \geq 0$ with a normal measurement error).

Disadvantage:

- ⋆ Can give anomalously low limit. An early use of the method was for the Karmen experiment which expected $1.25 \pm .25$ neutrino oscillation signal events given results of a previous experiment (LSND) and expected 3 background events. Observing 0 events they quoted a 90% UCB as 1.08. The correct bound is 2.3 events.

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Conditional Confidence Intervals

If N =number observed = n, then B =number background in observation$\leq n$. If we were told that $B \leq m$ apriori, then the likelihood would be

$$L(\theta|n) = \frac{1}{F_b(m)} f_{b+\theta}(n)$$

If $n \leq m$ and

$$L(\theta|n) = \frac{1}{F_b(m)} \sum_{k=0}^{m} f_b(k) f_\theta(n-k)$$

for $n > m$, where f_μ and F_μ are the PDF and DF of Poisson(μ).

Try this likelihood with $m = n$ and apply Unified method to it.

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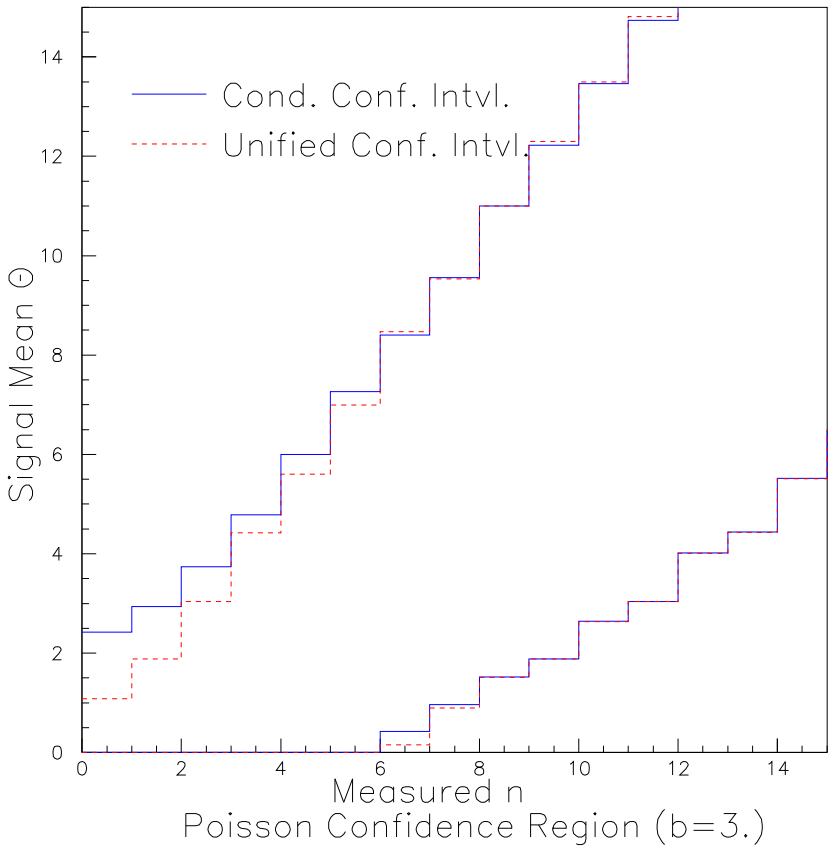
Unified and Conditional Intervals



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Advantages:

- ⋆ Always finds a non-null interval, even if experimental result is improbable.
- ⋆ Goes automatically from one sided limit to two-sided interval.
- ⋆ Does not give anomalously low limit if fewer events than expected from background are seen.
- ⋆ Intervals are invariant under a change of variable.

Disadvantage:

- ⋆ Does not work for normal distribution or for Poisson distribution with large background.

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Flat Prior Bayes Credible Interval

If Θ is given a uniform prior, let g and G denote the posterior density and DF of Θ. Thus

$$g(\theta|n) \propto f_{b+\theta}(n).$$

The Bayesian credible intervals are

$$[a_n, b_n] = \{\theta : g(\theta|n) \geq c_n\},$$

where

$$G(b_n|n) - G(a_n|n) = 1 - \alpha.$$

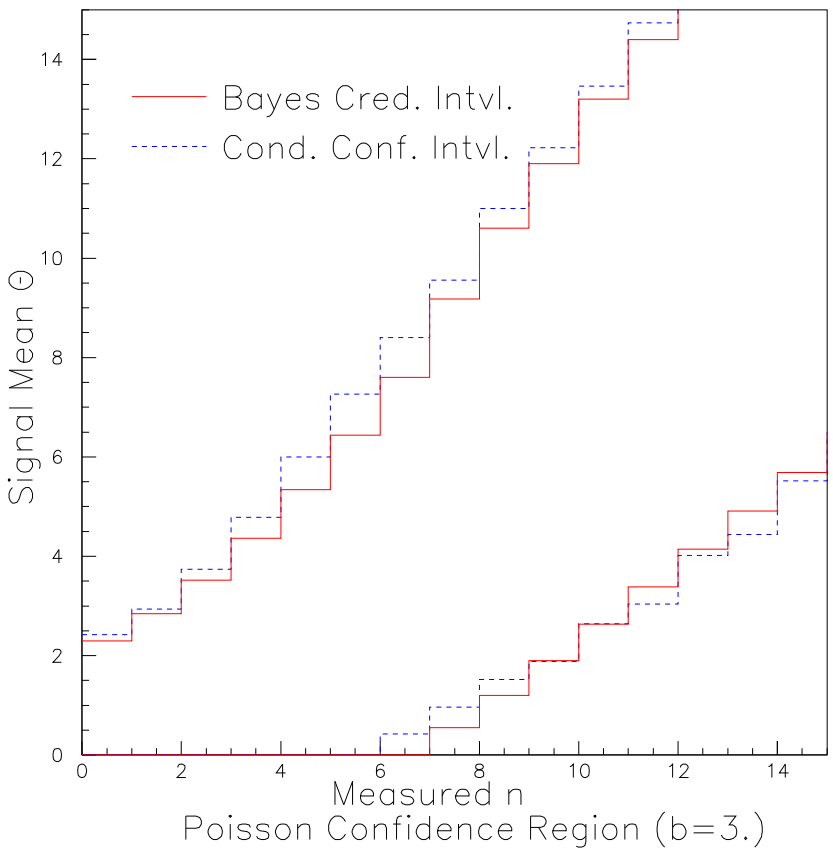
A useful identity can be shown to be:

$$P[\Theta > \theta|N = n] = P_\theta[N \leq n|B \leq n],$$

$$1 - G(\theta|n) = F_{b+\theta}(n)/F_b(n).$$

Bayesian Intervals

Flat Prior



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Frequentist interpretation: What is the frequentist confidence level of this Bayesian determined interval? For the normal case it can be shown that the confidence level is always

$$FCP \geq \frac{1-\alpha}{1+\alpha}.$$

This limit is very conservative. For $\alpha = 0.1$, this limit gives 0.8182, but numerical evaluations show ≥ 0.86.

In addition, small ad hoc modification can make the difference in values of the FCP and $1-\alpha$ very small indeed. Similar results hold for the Poisson case. For details see Roe and Woodroofe, *Phys. Rev.* **D63**, Jan. 2001,013009.

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Suppose nuisance variable σ for normal problem or b for Poisson problem treated in Bayesian manner: priors $1/\sigma^2$ for the normal and $e^{-\xi}$ with B Poisson ξ. For the normal case two independent random variables $X \sim$ Normal$[\theta, \sigma^2]$ and $W \sim \sigma^2 \chi^2_r$ are observed. 2nd variable gives measure of σ. Credible limits widened as expected. However, if examined from frequentist view, deviation of frequentist confidence from Bayesian credible value is lessened. (Zhang, Tonglin and Michael Woodroofe (2001), accepted by J. Statdist. Plan. Inf.)

This alleviates another issue. Mandelkern criticized the unified and the Bayesian methods since, for normal case, as physical results get substantially negative (which throws into question the form of the distribution), limits get sharper. With nuisance variables smeared, limits get wider for far negative results, as σ becomes less certain.

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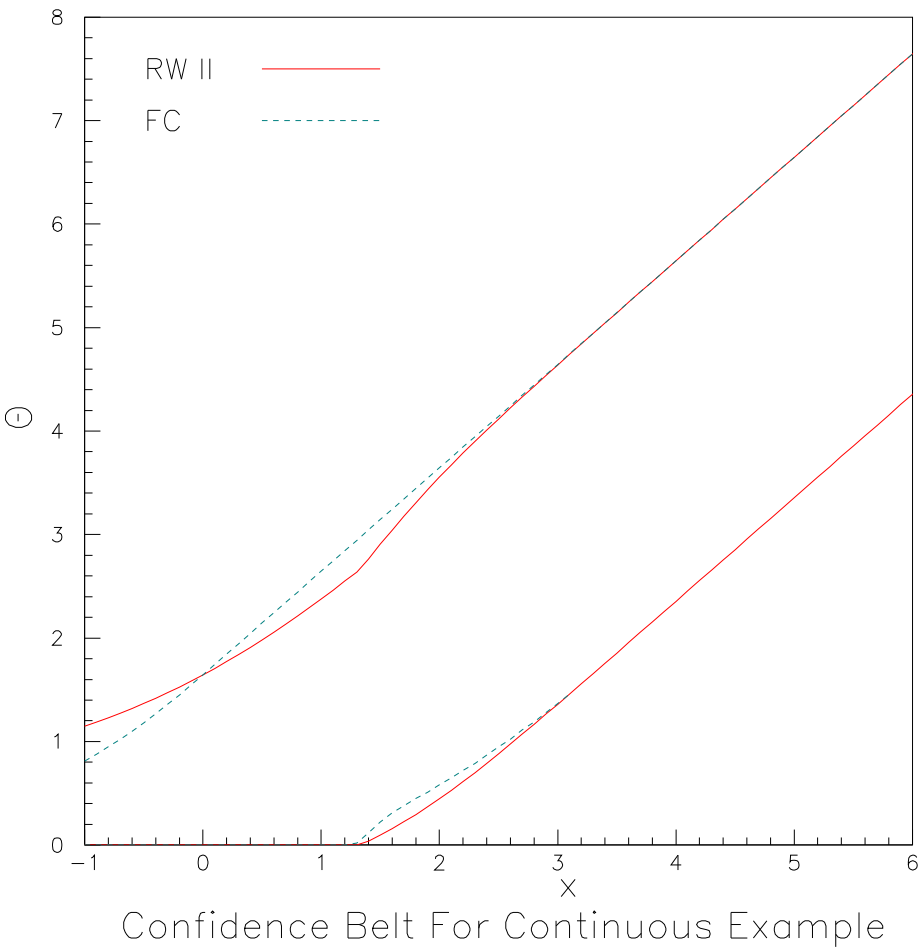

Confidence Belt For Continuous Example

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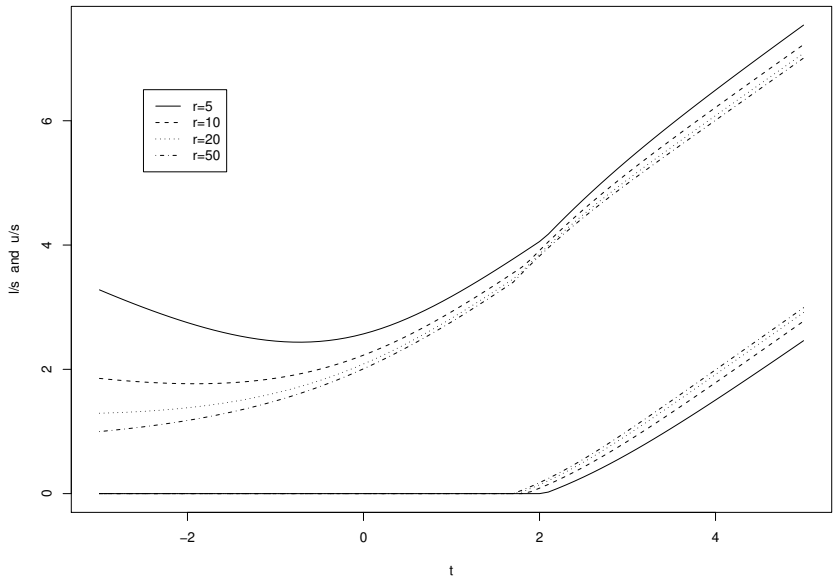
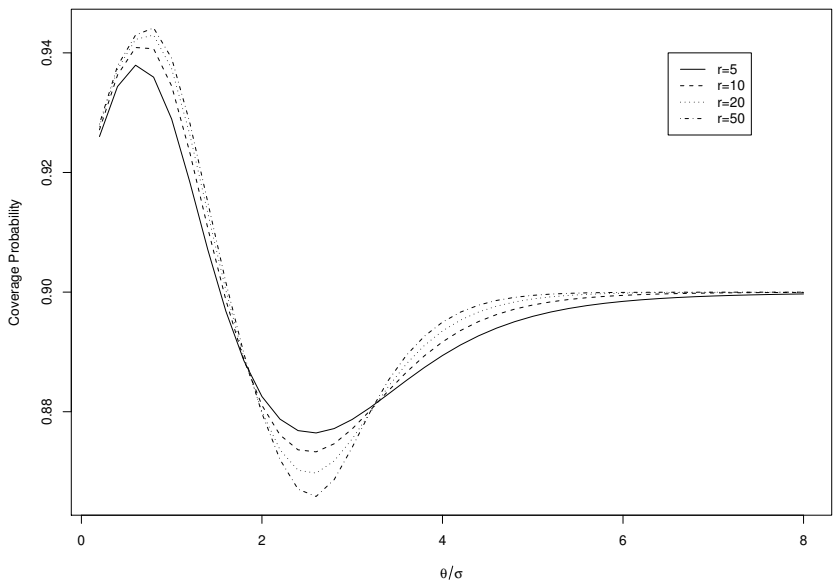
90% Credible Intervals for Various r



Frequentist Coverage Probability



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Advantages:

- ⋆ Always finds a non-null interval, even if experimental result is improbable.
- ⋆ Goes automatically from one sided limit to two-sided interval.
- ⋆ Does not give anomalously low limit if fewer events than expected from background are seen.
- ⋆ Does work for normal distribution.

Disadvantage:

- ⋆ Frequentist confidence limits not completely constant (0.86 to $> .9$).
- ⋆ Intervals are not invariant under a change of variable. (Is this critical? Errors from $\left(E\left\{\frac{-\partial^2 \ln L}{\partial\theta^2}\right\}\right)^{-1} = \sigma^2$ in Maximum Likelihood method are also not invariant.)

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Admissable P-Values

For testing $H_0 : \theta \geq \theta_0$ in the Poisson model $N = B + S$, the usual P-value is

$$p(n) = P_{\theta_0}[N \leq n] = F_{b+\theta_0}(n),$$

where F_μ is the Poisson DF with mean μ. Let

$$q(n) = P_{\theta_0}[N \leq n | B \leq n].$$

Example: if $b = 3,\ \theta_0 = 1$, and $n = 0$, then $p = e^{-4}$ and $q = e^{-1}$.

What is a P-Value?

Think of p and q as estimators of the indicator function of H_0; that is,

Risk:

$$r(p;\theta) = E_\theta\{[p(N) - 1_{H_0}(\theta)]^2\}.$$

The following theorem can then be proven:

Theorem: If $b > 0$, then p is an inadmissable p-value w.r.t. squared error loss, and q is admissable.

Consider next the less trivial case where there is partial information for each event as to what kind of event it is. We can model this as a non-homogeneous Poisson process or as a "marked Poisson" process, with the energy of the events as "marks" with intensity

$$\lambda_{\phi,\Delta m^2}(e,x) = N[\theta(e)g(e,x) + bh(e,x)],$$

where g and h are normalized to integrate to 1. θ depends on ϕ and Δm^2; g depends on Δm^2. (ϕ is usually called θ, but I have changed the name here to avoid confusion.) N is proportional to the overall length of the run. The normalization of θ and b are then independent of run length. g and h are also dependent on x, an event shape parameter, such as the output of a neural net distinguishing ν_e from ν_μ or neutral current events with a π^0.

The "marked Poisson" process corresponds to binning with bin width $\to 0$.

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If $0 < e_1 < e_2 < e_3 \cdots < e_n < \infty$, probability of observing n events with $e_i \leq E \leq e_i + de_i$, and $x_i \leq X \leq x_i + dx_i, \quad i = 1, \cdots, n$ is

$$L = \left(\prod_{i=1}^{n} \lambda_{\phi,\Delta m^2}(e_i, x_i) \times de_i dx_i \right) \times e^{-\Lambda(\phi,\Delta m^2)},$$

where,

$$\Lambda(\phi, \Delta m^2) = \int_{E_{min}}^{E_{max}} \int \lambda_{\phi,\Delta m^2}(e, x) dx de.$$

If we use a Bayes method with uniform prior, the marginal PDF is

$$f(n, x) =$$

$$\prod_{k=1}^{n} h(x_k) \int_0^{\infty} e^{-(b+\theta)} \prod_{k=1}^{n} [b + \theta r(x_k)] e^{-(b+\theta)} d\theta,$$

where $r(x_k) = g(x_k)/h(x_k)$.

The posterior PDF is

$$q(\theta|n, x) = \frac{f_\theta(n, x)}{f(n, x)}.$$

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The appropriate Bayes credible interval/limit can then be found by iteration techniques. The present situation is a little more complicated than that described in Roe-Woodroofe, in that g is now a function of the parameters of interest, and θ is a function of e.

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Determining Values

If the LSND results are correct, we expect of the order of 1000 signal events and, overall, 2500 background events.

Since $g = g_{\Delta m^2}$, define different variables

$$\theta(\phi, \Delta m^2) g_{\Delta m^2}(e, x) =$$

$$a(e, x) \sin^2 2\phi \sin^2 \left(\frac{1.27 \Delta m^2 L}{e} \right),$$

where $a(e, x)$ is *not* a function of $\phi, \Delta m^2$. If LSND is correct we will not be near a boundary. The maximum likelihood method should work well here.

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Let $\omega = \ln L$,

$$L = \prod_{k=1}^{n} N[\theta(e_k)g(e_k, x_k) + bh(e_k, x_k)]e^{-\Lambda},$$

where

$$\Lambda = \int_{E_{min}}^{E_{max}} \int N dx de$$
$$\left[a(e,x)\sin^2 2\phi \sin^2\left(\frac{1.27\Delta m^2 L}{e}\right) + bh(e,x)\right].$$

$$\omega = n \ln N - \Lambda + \sum_{k=1}^{n}$$
$$\left[a(e_k,x_k)\sin^2 2\phi \sin^2\left(\frac{1.27\Delta m^2}{e_k}\right) + bh(e_k,x_k)\right].$$

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Find minimum of $-\omega$ (MINUIT). Then run a grid of nearby points and find the value of L_{test} such that 95% of the time the L found would be higher than L_{test}. Plot out region(s) of $\phi, \Delta m^2$ where L_{test} corresponds to the L obtained in the experiment.

Energy smearing can be included in the MC experiments, as can the effect of uncertainties in nuisance variables g, h, b. The latter would be included as systematic errors, somewhat alleviating the conceptual problem of mixing Bayes and frequentist concepts.

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