Analytical study of systematic effects in data analysis for the muon (g–2) experiment at BNL

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Abstract

For the muon g–2 experiment at BNL a simple procedure was developed for analytical estimation of systematic shift of fit parameters of time distribution of decay electrons $G(t) = N_\circ \, e^{-t/\tau} \, [1 + A \, \cos(\omega \, t + \phi)]$. Some application of derived equations are discussed, including estimation of systematic shift due to finite histogram bin width.

Key words: Systematic errors, Fitting, Bin width, Constant background

1 Introduction

In our previous paper [1] we have derived equations for statistical errors and correlations of parameters of 5-parameter fit function used for the muon (g–2) experiment at BNL :

$$G(t) = N_\circ \, e^{-t/\tau} \; [1 + A \, \cos(\omega t + \phi)] \tag{1}$$

The five parameters in eq.1 are: normalization factor $N_\circ$, life time τ and amplitude A, frequency ω and phase ϕ of g–2 oscillations. In this article we discuss systematic shift of fit parameters due to unidentified small background.

MINUIT, etc. can not help us since we are interested in functional form...

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2 Systematic shift of fit parameters due to small unidentified background

A procedure, similar to that discussed in [1] for the statistical errors and correlations, can be developed for estimation of the systematic shift of fit parameters in a case a small background is present in the data. Suppose we have some low level background $h(t)$ admixed to the data, otherwise perfectly described by some multi-parameter function $f(x_1, x_2, ..., x_L; t)$, where $(x_1, x_2, ..., x_L) \equiv \boldsymbol{x}$ is $L-$dimensional vector (set) of parameters. The background might be small enough to not be observed "by eyes" or even to spoil the chi-squared considerably. Nevertheless fitting the histogram with a function $f(\boldsymbol{x}; t)$ alone will give us parameters x_i, shifted with respect to the "true" parameters $x_{i\circ}$ by some δx_i: $x_i = x_{i\circ} + \delta x_i$. These systematic shifts δx_i can be found from the chi-squared minimization requirement $\partial W/\partial x_i = 0$:

$$0 = \frac{\partial W}{\partial x_i} = \frac{\partial}{\partial x_i} \sum_n \frac{(f - \mathcal{N}_n)^2}{\sigma_n^2} = 2 \sum_n \frac{f - \mathcal{N}_n}{f} \; f_i' \approx 2 \sum_n \frac{f - f_\circ - h}{f} \; f_i' \approx$$
$$\approx 2 \sum_n \frac{\sum_{j=1}^{L} \left(f_j' \, \delta x_j \right) - h}{f} \; f_i' = 2 \sum_{j=1}^{L} \delta x_j \sum_n \frac{f_j' \, f_i'}{f} - 2 \sum_n \frac{h}{f} f_i' \tag{2}$$

where $f = f(\boldsymbol{x}; t_n)$, $f_\circ = f(\boldsymbol{x}_\circ; t_n)$, $f_i' = (\partial f/\partial x_i)$, t_n and $\mathcal{N}_n$ are center of n^{th} channel of the histogram and its content, respectively, and $\sigma_n = \sqrt{f}$ is statistical error associated with n^{th} channel. For sufficiently small histogram bin width b, summation in eq.2 can be replaced by integration: $\sum_n \rightarrow \int dt/b$ and we finally get a system of L linear equations:

$$\sum_{j=1}^{L} \delta x_j \int \frac{f_i' \, f_j'}{f} \, dt = \sum_{j=1}^{L} s_{ij} \; \delta x_j = \int h \, \frac{f_i'}{f} \, dt \; , \qquad i = 1, 2, ..., L \tag{3}$$

Coefficients $s_{ij} = \int \frac{f_i' \, f_j'}{f} \, dt$ in eq.3 are equal (up to normalization constant $1/b$) to elements of error matrix $\mathcal{A}$ discussed in [1]. We shall call matrix of coefficients s_{ij} as a shift matrix $\mathcal{S}$.

3 Systematic shift of (g–2) frequency due to small background

The error matrix for g–2 distribution (eq.1) was calculated in [1] in approximation of neglecting terms of relative order of $(\omega\tau)^{-1}$, which is about 0.01 for the current muon g–2 experiment. It was shown that parameters $N_\circ$ and τ correlate and so do ϕ and ω. It was also shown that these correlations as well as statistical errors $\sigma_{N_\circ}$ and σ_ϕ depend on the choice of time origin. In the special time frame (which we call center of gravity frame) where t=0 time is one life time τ later then the histogram start time, correlations vanish, all parameters become statistically independent and $\sigma_{N_\circ}$, σ_ϕ reach their mininal values. In this case error matrix $\mathcal{A}$ becomes diagonal, and so the shift matrix $\mathcal{S}(s_{ij})$. That makes calculations easier without affecting result. In matrix form eq.3 reads :

$$
\begin{pmatrix}
\frac{e\,\tau}{N_\circ} & 0 & 0 & 0 & 0 \\
0 & \frac{N_\circ\,e}{\tau} & 0 & 0 & 0 \\
0 & 0 & \frac{N_\circ\,e\,\tau}{2} & 0 & 0 \\
0 & 0 & 0 & \frac{N_\circ\,A^2 e\,\tau^3}{2} & 0 \\
0 & 0 & 0 & 0 & \frac{N_\circ\,A^2 e\,\tau}{2}
\end{pmatrix}
\times
\begin{pmatrix}
\delta N_\circ \\ \delta\tau \\ \delta A \\ \delta\omega \\ \delta\phi
\end{pmatrix}
=
\begin{pmatrix}
\frac{1}{N_\circ}\int_{-\tau}^{\infty} h(t)\,dt \\
\frac{1}{\tau^2}\int_{-\tau}^{\infty} t\,h(t)\,dt \\
\int_{-\tau}^{\infty} \frac{h(t)\,\cos(\omega t+\phi)}{1+A\,\cos(\omega t+\phi)}\,dt \\
-A\int_{-\tau}^{\infty} \frac{t\,h(t)\,\sin(\omega t+\phi)}{1+A\,\cos(\omega t+\phi)}\,dt \\
-A\int_{-\tau}^{\infty} \frac{h(t)\,\sin(\omega t+\phi)}{1+A\,\cos(\omega t+\phi)}\,dt
\end{pmatrix}
\tag{4}
$$

and hence

$$\delta N_\circ = \frac{1}{e\,\tau}\int_{-\tau}^{\infty} h(t)\,dt \tag{5}$$

$$\delta\tau = \frac{1}{e\,N_\circ\,\tau}\int_{-\tau}^{\infty} t\,h(t)\,dt \tag{6}$$

$$\delta A = \frac{2}{e\,N_\circ\,\tau}\int_{-\tau}^{\infty} \frac{h(t)\,\cos(\omega t+\phi)}{1+A\,\cos(\omega t+\phi)}\,dt \tag{7}$$

$$\delta\omega = -\,\frac{2}{e\,N_\circ\,A\,\tau^3}\int_{-\tau}^{\infty} \frac{t\,h(t)\,\sin(\omega t+\phi)}{1+A\,\cos(\omega t+\phi)}\,dt \tag{8}$$

$$\delta\phi = -\,\frac{2}{e\,N_\circ\,A\,\tau}\int_{-\tau}^{\infty} \frac{h(t)\,\sin(\omega t+\phi)}{1+A\,\cos(\omega t+\phi)}\,dt \tag{9}$$

Calculation of relevant integrals for elements of error matrix $\mathcal{A}$ can be found in Appendix II of [1].

We note that eqs.7, 8 and 9 are not valid for smooth, compare to (g–2) oscillations, function $h(t)$ since integrals like $\int h(t)\sin(\omega t+\phi)dt$ in this case might be of order $\epsilon \equiv (\omega\tau)^{-1} = 0.01$ and must be neglected in adopted approximation. In Appendix I we derive equation for $\delta\omega$ for the case of arbitrary $h(t)$:

$$\delta\omega = -\,\frac{2}{e\,N_\circ\,A\,\tau^3}\int_{-\tau}^{\infty} \frac{t\,h(t)\,\sin(\omega t+\phi)}{1+A\,\cos(\omega t+\phi)}\,dt\,+$$

$$+\frac{2}{e^2 N_\circ A\,\tau^4}\int_{-\tau}^{\infty} h(t)\,dt \int_{-\tau}^{\infty} t\,e^{-t/\tau}\sin(\omega t+\phi)\,dt + \tag{10}$$
$$+\frac{2}{e^2 N_\circ A\,\tau^6}\int_{-\tau}^{\infty} t\,h(t)\,dt \int_{-\tau}^{\infty} t^2\,e^{-t/\tau}\sin(\omega t+\phi)\,dt$$

For smooth $h(t)$, the last two terms in eq.10 become comparable to the otherwise leading term in eq.8.

In the rest of this article we discuss several applications of eqs.8 and 10 for muon (g–2) experiment.

4 Frequency shift due to laser hole

For detector's gain and timing calibration in our experiment we use laser pulses, which admix to the data stream for a fraction of data taking runs. The laser pulses have same amplitude and shape as decay electron pulses. The only feature that allow us to separate them is a fixed time where the laser pulses are sent. In an experimental histogram the laser pulses appear as narrow and strong peaks atop of relatively smooth (g–2) distribution eq.1. For calibration purpose these peaks are cut out and analyzed. A question arises as can we use the rest of the histogram for determination of ω and other parameters of (g–2) distribution ?

Eq.8 allows to estimate systematic shift of ω due to presence of narrow gap in the histogram after cutting off the laser peak. An absence of events in a narrow time interval Δ_t, much less than period of (g–2) oscillations, is equivalent to "negative" background $h(t) = -N_\circ e^{-t/\tau}[1 + A\cos(\omega t+\phi)]$ in this interval. Then $\delta\omega(gap)$ can be found as

$$\delta\omega(gap) = -\frac{2}{eN_\circ A\tau^3}\left[\frac{t\,\left(-N_\circ e^{-t/\tau}[1+A\cos(\omega t+\phi)]\right)\,\sin(\omega t+\phi)}{1+A\cos(\omega t+\phi)}\Delta_t\right] =$$
$$= \frac{2\Delta_t}{eA\tau^3}\,t\,e^{-t/\tau}\sin(\omega t+\phi) \tag{11}$$

According to eq.11, $\delta\omega(gap)$ oscillates with (g–2) frequency ω and shifted in phase with respect to $G(t)$ in eq.1 by quarter of period. Fig.1 shows amplitude of oscillations of $\delta\omega/\omega$ as a function of time for $\Delta_t = 1$ ns.

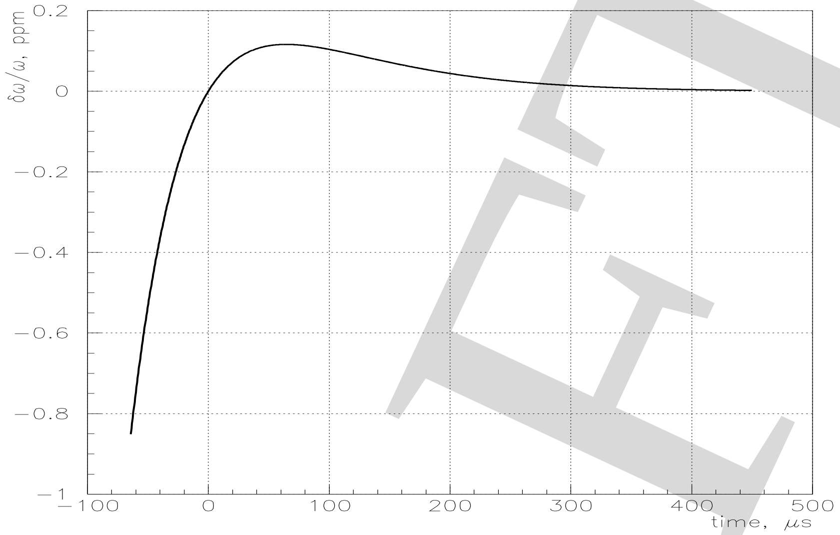

Fig. 1. $\delta\omega/\omega$ as a function of time for $\Delta_t = 1$ ns : amplitude of $\sin(\omega t+\phi)$ oscillations.

5 Frequency shift due to fluctuation in one channel

Find frequency shift due to one sigma fluctuation in a single channel of histogram:

$$
\begin{aligned}
\delta\omega &= -\frac{2}{eN_\circ A\tau^3}\left[\frac{t\ \sqrt{N_\circ e^{-t/\tau}[1+A\cos(\omega t+\phi)]}\ \ \sin(\omega t+\phi)}{1+A\cos(\omega t+\phi)}\ b\right] = \\
&= -\frac{2\,b\,t}{eA\tau^3}\sqrt{\frac{e^{-t/\tau}}{N_\circ[1+A\cos(\omega t+\phi)]}}\ \sin(\omega t+\phi) = \\
&= -\frac{2\,t\,e^{-t/2\tau}\sin(\omega t+\phi)}{A\tau^2}\sqrt{\frac{b}{N\,e\,\tau[1+A\cos(\omega t+\phi)]}} = \\
&= -\sigma_\omega\left(\frac{t}{\tau}\,e^{-t/2\tau}\sin(\omega t+\phi)\sqrt{\frac{2\,b}{e\tau[1+A\cos(\omega t+\phi)]}}\right)
\end{aligned}
\tag{12}
$$

6 Effect of finite bin width

In a standard minimization program such as MINUIT, value of a fit function $f(t)$ is being taken as a value of $f(t)$ at center of histogram bins, while the content of the histogram is in fact integral over the whole bin. That cause

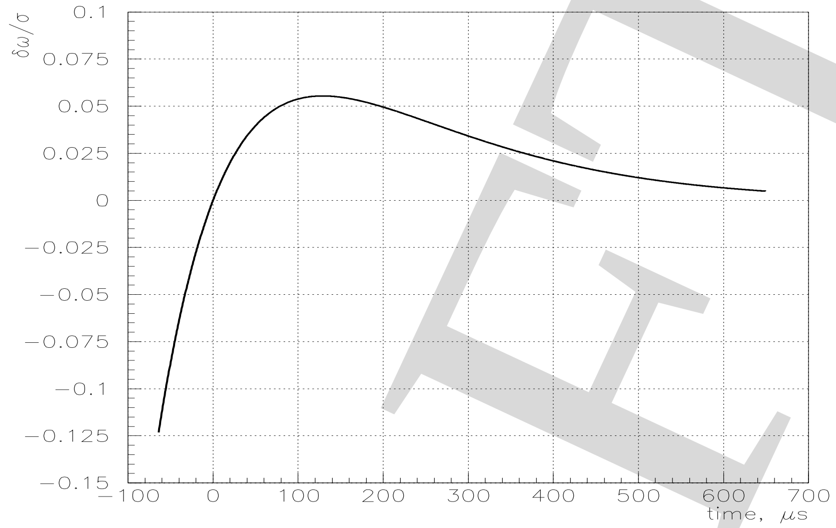

Fig. 2. $\delta\omega/\sigma_\omega$ as a function of time for $b = 0.5$ μs : amplitude of $\frac{-\sin(\omega t+\phi)}{\sqrt{1+A\cos(\omega t+\phi)}}$ oscillations

some systematic shift of parameters of optimization, which can be estimated analytically.

Let $f(t_\circ)$ be a value of a fit function $f(t)$ at the center of a bin. Average of $f(t)$ over bin size b will be:

$$\overline{f(t)} = \frac{1}{b}\int\limits_{-b/2}^{b/2} f(t_\circ + \xi)\, d\xi \approx \frac{1}{b}\int\limits_{-b/2}^{b/2} \left[f(t_\circ) + \xi\, \dot{f}(t_\circ) + \frac{1}{2}\xi^2\, \ddot{f}(t_\circ)\right] d\xi =$$
$$= f(t_\circ) + \frac{b^2}{24}\ddot{f}(t_\circ) \tag{13}$$

Thus the effect of finite bin size is equivalent to small additive background $h(t) = \frac{b^2}{24}\ddot{f}(t)$. Here we denote $\dot{f}(t) = \frac{df}{dt}$ and $\ddot{f}(t) = \frac{d^2 f}{dt^2}$.

For the muon (g–2) 5-parameter function find the second derivative:

$$\dot{G}(t) = -N_\circ\, e^{-t/\tau} \left[\frac{1}{\tau} + \frac{A}{\tau}\cos(\omega t + \phi) + A\,\omega\, \sin(\omega t + \phi)\right] \tag{14}$$

$$\ddot{G}(t) = N_\circ\, e^{-t/\tau} \left[\frac{1}{\tau^2} + \frac{A}{\tau^2}\cos(\omega t + \phi) + \frac{A\,\omega}{\tau}\sin(\omega t + \phi) + \right.$$
$$\left. + \frac{A\,\omega}{\tau}\sin(\omega t + \phi) - A\,\omega^2\cos(\omega t + \phi)\right] =$$
$$= \frac{N_\circ}{\tau^2}\, e^{-t/\tau} \left[1 + A\left(1 - (\omega\,\tau)^2\right)\cos(\omega t + \phi) + 2\,A\,\omega\,\tau\,\sin(\omega t + \phi)\right] =$$

$$= \omega^2 N_\circ e^{-t/\tau} \left[\epsilon^2 - A(1-\epsilon^2)\cos(\omega t+\phi) + 2A\epsilon\sin(\omega t+\phi)\right] \quad (15)$$

where $\epsilon = (\omega\tau)^{-1} = 0.011$. In the leading approximation on ϵ, from eq.8 we have

$$\frac{\delta\omega}{\omega} \approx \frac{1}{\omega}\frac{2}{e N_\circ A\tau^3}\frac{b^2}{24}\omega^2 N_\circ A \int_{-\tau}^{\infty} t\, e^{-t/\tau} \frac{\cos(\omega t+\phi)\sin(\omega t+\phi)}{1+A\cos(\omega t+\phi)}\, dt \quad (16)$$

As shown in Appendix II, the fraction in the integral in eq.16 can be approximated as

$$\frac{\cos(\omega t+\phi)\sin(\omega t+\phi)}{1+A\cos(\omega t+\phi)} \approx \frac{a}{A}\left[\sin 2(\omega t+\phi) - a\sin(\omega t+\phi) - a\sin 3(\omega t+\phi)\right] \quad (17)$$

where $a = \frac{1-\sqrt{1-A^2}}{A} \approx 0.2$ for $A \approx 0.4$. Then integration in eq.16 can be done analytically. Explicitly, in the leading approximation on ϵ :

$$\int_{-\tau}^{\infty} t\, e^{-t/\tau} \sin n(\omega t+\phi)\, dt \approx -\frac{\tau\, e}{n\,\omega}\cos n(-\omega\tau+\phi) = -\frac{\tau\, e}{n\,\omega}\cos n(\omega\tau-\phi) \quad (18)$$

Then

$$\frac{\delta\omega}{\omega} \approx \frac{b^2}{24\,\tau^2}\left[-\frac{1}{2}\cos 2(\omega\tau-\phi) + 0.2\cos(\omega\tau-\phi) + \frac{0.2}{3}\cos 3(\omega\tau-\phi)\right] \quad (19)$$

Absolute value of sum of cosines in eq.19 has maximum at $(\omega\tau - \phi) = \pi$, therefore

$$\left|\frac{\delta\omega}{\omega}\right| \leq \frac{b^2}{24\,\tau^2}(0.5+0.2+0.07) = 3.2\cdot 10^{-2}\left(\frac{b}{\tau}\right)^2 \quad (20)$$

For $b = 0.15\,\mu$s and $\tau = 64.4\,\mu$s, $|\delta\omega/\omega| = 0.17$ ppm which is in good agreement with numerical simulations.

7 Frequency shift due to constant background

Suppose we have constant background $h(t) \equiv h = const$. From eq.10 for frequency shift due to smooth background we find:

$$
\begin{aligned}
\delta\omega = & \frac{2}{e^2 N_\circ A\tau^4} \int\limits_{-\tau}^{-\tau+T} h\,dt \int\limits_{-\tau}^{\infty} t\, e^{-t/\tau} \sin(\omega t+\phi)\,dt \\
& + \frac{2}{e^2 N_\circ A\tau^6} \int\limits_{-\tau}^{-\tau+T} t\,h\,dt \int\limits_{-\tau}^{\infty} t^2 e^{-t/\tau} \sin(\omega t+\phi)\,dt \\
& - \frac{2}{e N_\circ A\tau^3} \int\limits_{-\tau}^{-\tau+T} \frac{t\,h\,\sin(\omega t+\phi)}{1+A\cos(\omega t+\phi)}\,dt \approx \\
& \approx \frac{2}{e^2 N_\circ A\tau^4} \times h\,T \times \left(-\frac{\tau\, e}{\omega}\, \cos(\omega\tau-\phi)\right) \\
& + \frac{2}{e^2 N_\circ A\tau^6} \times \frac{h\,T\,(T-2\tau)}{2} \times \left(-\frac{\tau^2\, e}{\omega}\, \cos(\omega\tau-\phi)\right) \\
& - \frac{2}{e N_\circ A\tau^3} \int\limits_{-\tau}^{-\tau+T} \frac{h\,t\,\sin(\omega t+\phi)}{1+A\cos(\omega t+\phi)}\,dt = \\
& = -\epsilon \frac{h\,T^2}{e\,N_\circ A\tau^3}\, \cos(\omega\tau-\phi) \\
& - \frac{2\,h}{e N_\circ A\tau^3} \int\limits_{-\tau}^{-\tau+T} t\,\frac{2\,a}{A} \left[\sin(\omega t+\phi) - a\sin 2(\omega t+\phi) + a^2 \sin 3(\omega t+\phi)\ ...\right] dt =
\end{aligned}
\tag{21}
$$

Calculate integral of the first term in series $(-a)^{n-1} \sin n(\omega t+\phi)$ in eq.21 :

$$
\begin{aligned}
& \int\limits_{-\tau}^{-\tau+T} t\, \sin(\omega t+\phi)\,dt \approx -\frac{t}{\omega} \cos(\omega t+\phi)\ \Big|_{-\tau}^{-\tau+T} = \\
= & -\frac{T-\tau}{\omega} \cos\left[\omega(T-\tau)+\phi\right] + \frac{-\tau}{\omega} \cos\left[\omega(-\tau)+\phi\right] = \\
= & -\frac{T}{\omega} \cos\left[\omega(T-\tau)+\phi\right] - 2\,\frac{\tau}{\omega}\, \sin\left(\frac{\omega\,T}{2}\right)\, \sin\left[\omega\left(\frac{T}{2}-\tau\right)+\phi\right]
\end{aligned}
\tag{22}
$$

The second term in the series after integration has order $a/2 \approx 0.1$ with restect to the first term which, in turn, is of order $\tau/T \sim 0.1$ with respect to the first term in eq.21. Therefore we neglect all terms but the fierst in series $(-a)^{n-1} \sin n(\omega t+\phi)$. Final equation for the frequency shift due to constant bacground is :

$$
\left|\frac{\delta\omega}{\omega}\right| < \frac{\epsilon^2}{e\,A}\ \frac{h}{N_\circ}\ \frac{T^2}{\tau^2} \left(1+\frac{\tau}{T}\right) \tag{23}
$$

8 Appendix I

In general case, systematic shifts of parameters of (g–2) distribution (eq.1) are been found from system of 5 linear equations $\partial W/\partial x_i = 0$, which in matrix form reads :

$$\mathcal{S} \times \delta X = \mathcal{B}\,, \tag{24}$$

$$\text{where} \qquad \delta X = \begin{pmatrix} \delta N \\ \frac{N_\circ}{\tau}\,\delta\tau \\ N_\circ\;\delta A \\ N_\circ\,A\,\delta\omega \\ N_\circ\,A\,\delta\phi \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} \int_{-\tau}^{\infty} h(t)\,dt \\ \int_{-\tau}^{\infty} t\,h(t)\,dt \\ \int_{-\tau}^{\infty} \frac{h(t)\cos\,\alpha}{1+A\cos\,\alpha}\,dt \\ -\int_{-\tau}^{\infty} \frac{t\,h(t)\sin\,\alpha}{1+A\cos\,\alpha}\,dt \\ -\int_{-\tau}^{\infty} \frac{h(t)\sin\,\alpha}{1+A\cos\,\alpha}\,dt \end{pmatrix} \quad \text{and} \tag{25}$$

$$\mathcal{S} = \int_{-\tau}^{\infty} e^{-t/\tau} \begin{pmatrix} (1+A\cos\,\alpha) & t\,(1+A\cos\,\alpha) & \cos\,\alpha & -t\,\sin\,\alpha & -\sin\,\alpha \\ t\,(1+A\cos\,\alpha) & t^2\,(1+A\cos\,\alpha) & t\,\cos\,\alpha & -t^2\,\sin\,\alpha & -t\,\sin\,\alpha \\ \cos\,\alpha & t\,\cos\,\alpha & \frac{\cos^2\alpha}{1+A\cos\,\alpha} & -\frac{t\,\sin\,\alpha\,\cos\,\alpha}{1+A\cos\,\alpha} & -\frac{\sin\,\alpha\,\cos\,\alpha}{1+A\cos\,\alpha} \\ -t\,\sin\,\alpha & -t^2\,\sin\,\alpha & -\frac{t\,\sin\,\alpha\,\cos\,\alpha}{1+A\cos\,\alpha} & \frac{t^2\,\sin^2\alpha}{1+A\cos\,\alpha} & \frac{t\,\sin^2\alpha}{1+A\cos\,\alpha} \\ -\sin\,\alpha & -t\,\sin\,\alpha & -\frac{\sin\,\alpha\,\cos\,\alpha}{1+A\cos\,\alpha} & \frac{t\,\sin^2\alpha}{1+A\cos\,\alpha} & \frac{\sin^2\alpha}{1+A\cos\,\alpha} \end{pmatrix} dt$$

Here we redefine vector δX and denote $(\omega t + \phi) \equiv \alpha$ for simplicity. Matrix $\mathcal{S}$ is symmetric and almost diagonal:

$$\mathcal{S} \approx \begin{pmatrix} e\,\tau & 0 & 0 & 0 & 0 \\ 0 & e\,\tau^3 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}e\,\tau & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}e\,\tau^3 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}e\,\tau \end{pmatrix} \tag{26}$$

According to the Cramer's rule, solution for eq.24 is

$$\delta x_j = \frac{S_{1j}b_1 + S_{2j}b_2 + S_{3j}b_3 + S_{4j}b_4 + S_{5j}b_5}{det(\mathcal{S})}, \qquad S_{ij} = (-1)^{i+j} M_{i,j} \tag{27}$$

where S_{ij} are cofactors of elements s_{ij} of matrix $\mathcal{S}$ and M_{ij} are its minors. We shall find solution for $\delta\omega$ to the leading order of $\epsilon = \int \mathcal{F}(t) \times \cos(\omega t + \phi)\,dt$, where $\mathcal{F}(t)$ is a smooth function of t. Note that all diagonal elements of matrix

$\mathcal{S}$ are of order ϵ° and off-diagonal elements are of order ϵ^1. In calculation of minors M_{ij} we shall only keep terms with less number of off-diagonal elements, explicitely for $y_4 \equiv N_\circ \, A \, \delta\omega$:

$$\begin{aligned} y_4 = (& - s_{22}\, s_{33}\, s_{55}\, s_{41}\, b_1 - s_{11}\, s_{33}\, s_{55}\, s_{42}\, b_2 - s_{11}\, s_{22}\, s_{55}\, s_{43}\, b_3 + \\ & + s_{11}\, s_{22}\, s_{33}\, s_{55}\, b_4 - s_{11}\, s_{22}\, s_{33}\, s_{45}\, b_1 \;) \,/\, det(\mathcal{S}) \end{aligned} \tag{28}$$

Determinant of matrix $(\mathcal{S})$ in leading order of ϵ is equial to the product of diagonal elements :

$$det(\mathcal{S}) \approx (e\,\tau) \times (e\,\tau^3) \times \left(\frac{1}{2} e\,\tau\right) \times \left(\frac{1}{2} e\,\tau^3\right) \times \left(\frac{1}{2} e\,\tau\right) = \frac{1}{8} e^5\,\tau^9 \tag{29}$$

From eqs.28,29 find $\delta\omega$:

$$\begin{aligned} N_\circ A\,\delta\omega = & \frac{2}{e^2\tau^4} \int_{-\tau}^{\infty} h(t)\,dt \int_{-\tau}^{\infty} t\,e^{-t/\tau} \sin(\omega t + \phi)\,dt \\ & + \frac{2}{e^2\tau^6} \int_{-\tau}^{\infty} t\,h(t)\,dt \int_{-\tau}^{\infty} t^2 e^{-t/\tau} \sin(\omega t + \phi)\,dt \\ & + \frac{4}{e^2\tau^4} \int_{-\tau}^{\infty} \frac{h(t)\cos(\omega t + \phi)}{1 + A\cos(\omega t + \phi)}\,dt \int_{-\tau}^{\infty} \frac{t\,e^{-t/\tau}\sin(\omega t + \phi)\cos(\omega t + \phi)}{1 + A\cos(\omega t + \phi)}\,dt \\ & - \frac{2}{e\tau^3} \int_{-\tau}^{\infty} \frac{t\,h(t)\sin(\omega t + \phi)}{1 + A\cos(\omega t + \phi)}\,dt \\ & + \frac{4}{e^2\tau^4} \int_{-\tau}^{\infty} \frac{h(t)\sin(\omega t + \phi)}{1 + A\cos(\omega t + \phi)}\,dt \int_{-\tau}^{\infty} \frac{t\,e^{-t/\tau}\sin^2(\omega t + \phi)}{1 + A\cos(\omega t + \phi)}\,dt \end{aligned} \tag{30}$$

Note that if function $h(t)$ is not smooth, then 4^{th} term in eq.30 is of order ϵ^0 and other terms are of order ϵ^1 and hence can be neglected. In this case solution is same as listed in eq.8. If function $h(t)$ is smooth, then 1^{st}, 2^{nd} and 4^{th} terms are of order ϵ^1 and 3^{rd}, 5^{th} terms are of order ϵ^2 and hence can be neglected. Thus 3^{rd} and 5^{th} terms can be neglected in any case and final solution is :

$$\begin{aligned} \delta\omega = & \frac{2}{e^2 N_\circ A \tau^4} \int_{-\tau}^{\infty} h(t)\,dt \int_{-\tau}^{\infty} t\,e^{-t/\tau} \sin(\omega t + \phi)\,dt \\ & + \frac{2}{e^2 N_\circ A \tau^6} \int_{-\tau}^{\infty} t\,h(t)\,dt \int_{-\tau}^{\infty} t^2 e^{-t/\tau} \sin(\omega t + \phi)\,dt \end{aligned} \tag{31}$$

$$- \frac{2}{eN_\circ A\tau^3} \int\limits_{-\tau}^{\infty} \frac{t\, h(t) \sin(\omega t + \phi)}{1 + A\cos(\omega t + \phi)}\, dt$$

9 Appendix II

To calculate integral

$$\int t\, e^{-t/\tau} \; \frac{\sin(\omega t + \phi)\, \cos(\omega t + \phi)}{1 + A\, \cos(\omega t + \phi)} \; dt \tag{32}$$

we decompose the fraction in eq.32 into series $C_n \, \sin n(\omega t + \phi)$. For that we use slightly modified eq.417.4 from [2] :

$$\frac{\sin\alpha}{1 + 2a\cos\alpha + a^2} = \sin\alpha - a\sin 2\alpha + a^2 \sin 3\alpha - a^3 \sin 4\alpha + ... \quad [a^2 < 1] \tag{33}$$

We assume $0 < a < 1$. To match eq.32 we first modify denominator in eq.33 :

$$1 + 2a\cos\alpha + a^2 = (1 + a^2) \times (1 + \frac{2a}{1 + a^2} \cos\alpha) = (1 + a^2) \times (1 + A\cos\alpha) \tag{34}$$

We denote $\frac{2a}{1+a^2} = A,\ 0 < A < 1$. The inverse relation, satisfying our assumptions, is $a = \frac{1-\sqrt{1-A^2}}{A}$. For g–2 experiment $A \approx 0.4$ and hence $a \approx 0.2$. Then fraction in eq.32 can be written as

$$\begin{aligned}
\frac{\sin\alpha\, \cos\alpha}{1 + A\, \cos\alpha} &= (1 + a^2) \sum_{n=1} (-a)^{n-1} \cos\alpha\, \sin n\alpha = -\frac{1 + a^2}{a} \sum_{n=1} (-a)^n \cos\alpha\, \sin n\alpha = \\
&= -\frac{1 + a^2}{2a} \sum_{n=1} (-a)^n [\, \sin(n-1)\alpha + \sin(n+1)\alpha\,] = \\
&= \frac{a}{A} \sum_{n=0} (-a)^n [\, \sin n\alpha + \sin(n+2)\alpha\,] = \\
&= \frac{a}{A} \left(-a\sin\alpha + \sum_{n=2} \left[(-a)^n + (-a)^{n-2} \right] \sin n\alpha \right)
\end{aligned} \tag{35}$$

In leading approximation on a we have

$$\frac{\sin\alpha\, \cos\alpha}{1 + A\, \cos\alpha} = \frac{a}{A} (\sin 2\alpha - a\sin\alpha - a\sin 3\alpha) \tag{36}$$

References

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