How to include the information on Δm_s in CKM fits

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Abstract

In this note we discuss how to include the Δm_s information in the CKM fits starting from the amplitude spectrum given by the LEP Oscillation WG. The Likilihood method ratio as proposed in [1] will be carefully checked.

1 A brief introduction on the amplitude method

The study of $B_s^0 - \bar{B}_s^0$ oscillations is done by introducing the oscillation amplitude $\mathcal{A}$ as explained in [2]. Events have been distributed in two classes corresponding to oscillating and non-oscillating candidates, according to the information obtained by tagging the presence of a B or a $\bar{B}$ meson at the beam interaction time and at the b-hadron decay time. The expected decay-time distribution of events is obtained by using a function which contains time distributions for all components, taking into account the non-perfect event classification, the different behaviour of b-hadron decays, which depend on their type ($\bar{B}_d^0$, B^-, $\bar{B}_s^0$ or b-baryon), and background components originating from charm and light flavours. The time distribution for the oscillating (non-oscillating) signal is given by

$$\mathcal{P}(B_s^0 \to \bar{B}_s^0(B_s^0)) = \frac{1}{2\tau_{B_s^0}}(1 - (+)\mathcal{A}\cos\Delta m_s t)e^{-t/\tau_{B_s^0}} \; . \tag{1}$$

Such theoretical time distributions have been convoluted with the expected time resolution of the measurements. For each value of Δm_s and each analysis, the oscillation amplitude $\mathcal{A}$ is fitted. Fitted amplitude values have been combined [3] using a common set of determinations for external parameters and taking into account possible correlations between the different analyses. In this respect the use of the oscillation amplitude provides a simple framework to account for all these effects. The 95% C.L. limit on Δm_s corresponds to the value Δm_s^{lim} for which $\mathcal{A} + 1.645\ \sigma_{\mathcal{A}} = 1$, whereas the sensitivity Δm_s^{sens} corresponds to $1.645\ \sigma_{\mathcal{A}} = 1$. It is the value of Δm_s at which it is expected that the 95% limit will be set, if $\mathcal{A} = 0$.
In this paper we discuss how to use the Δm_s information as a constraint in CKM Fits.

2 Δm_s Information for CKM fits

In this section the two different methods to include Δm_s information in CKM Fits

2.1 Likelihood ratio $\mathcal{R}$ Method

Recently it has been proposed to use the log-likelihood function $\Delta \log \mathcal{L}^{\infty}(\Delta m_s)$ referenced to its value obtained for $\Delta m_s = \infty$ [4]. The log-likelihood values can be easily deduced from $\mathcal{A}$ and $\sigma_{\mathcal{A}}$ using the expressions given in [2]

$$\Delta \log \mathcal{L}^{\infty}(\Delta m_s) = \frac{1}{2}\left[\left(\frac{\mathcal{A}-1}{\sigma_{\mathcal{A}}}\right)^2 - \left(\frac{\mathcal{A}}{\sigma_{\mathcal{A}}}\right)^2\right] = \left(\frac{1}{2} - \mathcal{A}\right)\frac{1}{\sigma_{\mathcal{A}}^2} \; , \tag{2}$$

$$\Delta \log \mathcal{L}^{\infty}(\Delta m_s)_{mix} = -\frac{1}{2}\frac{1}{\sigma_{\mathcal{A}}^2} \; , \tag{3}$$

$$\Delta \log \mathcal{L}^{\infty}(\Delta m_s)_{nomix} = \frac{1}{2}\frac{1}{\sigma_{\mathcal{A}}^2} \; . \tag{4}$$

The last two equations give the average log-likelihood value when Δm_s corresponds to the true oscillation frequency (*mixing* case) and when Δm_s is far from the oscillation frequency ($|\Delta m_s - \Delta m_s^{true}| \gg \Gamma/2$, *no-mixing* case). Γ is the full width at half maximum

of the amplitude distribution in case of a signal; typically $\Gamma \simeq 1/\tau_{B_s^0}$.
Equation (2) allows to define the likelihood ratio R

$$\mathrm{R}(\Delta m_s) = e^{-\Delta \log \mathcal{L}^{\infty}(\Delta m_s)} = \frac{\mathcal{L}(\Delta m_s)}{\mathcal{L}(\Delta m_s = \infty)} \ . \tag{5}$$

The function R corresponds to the ratio of probability densities for different Δm_s values. The absolute Δm_s probability density, instead, remains undefined because $\mathcal{L}(\Delta m_s)$ stays constant for Δm_s values much larger than the sensitivity; in this region R is equal to unity.
This method has been adopted in [1]

2.2 Modified χ^2 Method

In the first CKM fits using the complete amplitude spectrum the χ^2 w.r.t 1 was used:

$$\chi^2 = (1 - \mathcal{A}/\sigma_{\mathcal{A}})^2 \tag{6}$$

This method has two main drawbacks:

- the sign of the deviation of the amplitude with respect to the value $\mathcal{A} = 1$ was not used, whereas it is expected that an evidence for a signal would manifest itself by giving an amplitude value which is simultaneously compatible with $\mathcal{A} = 1$ and incompatible with $\mathcal{A} = 0$;
- values of $\mathcal{A} > 1$ are disfavoured w.r.t. $\mathcal{A} = 0$, while it is expected that, because of statistical fluctations, the amplitude value corresponding to the "true" Δm_s value could be higher than 1. This problem was solved in the early days of the use of Δm_s in CKM fits putting $\mathcal{A} = 1$ whenever higher.

A modified χ^2 has been introduced ad hoc to solve the second problem:

$$\chi^2 = 2 \cdot \left[Erfc^{-1}\left(\frac{1}{2} Erfc\left(\frac{1-A}{\sqrt{2}\sigma_A}\right)\right)\right]^2 \tag{7}$$

This method was adopted in [6] for including Δm_s information in the CKM fit. This paper contains also the claim on the danger of using an unlimited Likelihood from the Likelihood Ratio Method described in the previous section.

3 Comparison of the two methods using the World Average Amplitude Spectrum

The amplitude as a function of Δm_s and the corresponding $\Delta \log \mathcal{L}^{\infty}(\Delta m_s)$ are shown in Figure 1-a),b). The corresponding Likelihoods obtained using the Likelihood ratio and the Modified χ^2 methods are shown in Figure 1-c). It is clear that the two methods give very different Likelihood functions. In particular the Modified χ^2 method give large values for the Likelihood function in a quite broad Δm_s region, which implies a less tight constraint from Δm_s information in the CKM fit (and in particular for the determination of $\bar{\rho}$ and γ parameters).
Thus it is important to establish which is the correct method.

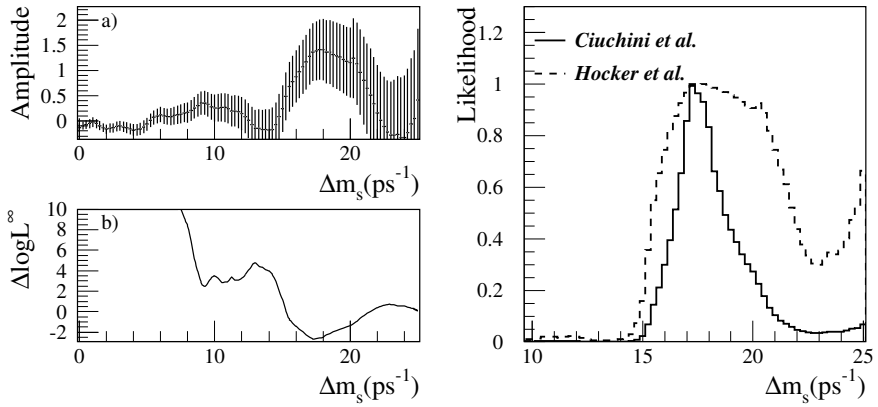

Figure 1: World average amplitude analysis: a) amplitude spectrum, b) $\Delta \log \mathcal{L}^{\infty}(\Delta m_s)$, c) comparison between R $\equiv$ Likelihood ratio method and the Modified χ^2 method.

4 the Toy Montecarlo

The behaviour of $\sigma(\mathcal{A})$ as a function of Δm_s can be reproduced by tuning the parameters of a fast simulation (toy-MC). The method used here is similar to the one presented in [5]. The errors on the amplitude can be written as:

$$\sigma^{-1}(\mathcal{A}) = \sqrt{N}\,\eta_{B_s}\,(2\epsilon - 1)\,W(\sigma_L, \sigma_P, \Delta m_s)$$

where N is the total number of events, η_{B_s} the purity of the sample in B_s decays, ϵ the purity of the tagging at the decay time, σ_L is the B_s flight length uncertainty and σ_P the relative uncertainty on its momentum. The parameters σ_L, σ_P and the global factor that multiply W have been obtained by adjusting the simulated error distribution on the one measured with real events.

Figure 2 shows the agreement between the toy-MC calculation and real data up to Δm_s=25 ps^{-1} (the upper bound on Δm_s of the amplitude plot for which there are measurements).

An additional problem when using the world average amplitude spectrum to construct the likelihood function for Δm_s is that, in priciple, one would like to define the likelihood in $[0, \infty]$ while the amplitude spectrum is defined only up to a certain value (for the present world average the value is 25 ps^{-1}).
A procedure has to be introduce to continue $\sigma(\mathcal{A})$ and $\mathcal{A}$.
The continuation for $\sigma(\mathcal{A})$ is shown in Figure **??**. It has been verified (with an independent toy-MC) that the extrapolated errors approximate well the simulated error distribution.
The continuation of $\mathcal{A}$ is more delicate.
In particular it is more sensitive to the real amplitude spectrum. Nevertheless if $\Delta m_s^{sens} << \Delta m_s^{last}$, the significance S ($S = \mathcal{A}/\sigma_{\mathcal{A}}$) is approximately constant. Is then a good approximation to take $\mathcal{A}(\Delta m_s) = \mathcal{A}(\Delta m_s^{last})/\sigma_{\mathcal{A}}(\Delta m_s^{last})\sigma_{\mathcal{A}}(\Delta m_s)$. Although this procedure is reasonable, it should be stressed that it is very desirable to have all the amplitudes (with errors) up to the Δm_s value where the significance is reasonably stable.

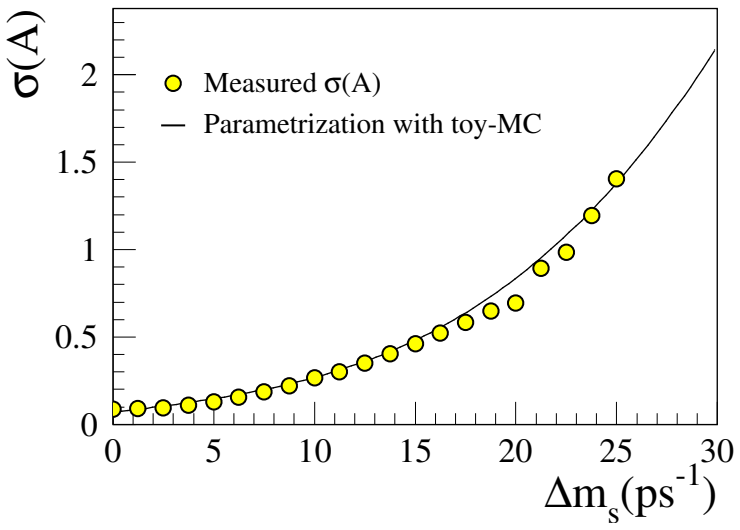

Figure 2: Comparison between the error distribution computed with a toy-MC (solid line) and the measured amplitude errors (circles).

5 Comparison of the methods in case of an oscillation signal

In this section we test the behaviour of the two methods in presence of a clear Δm_s oscillation signal.
For this reason we perform several Δm_s Toy-MC analyses with the same $\sigma(\mathcal{A})$ versus Δm_s behaviour as the world average analysis. For this case we have generated a Δm_s signal at 18.2 ps^{-1} (corresponding to value where there is the bump in the world average amplitude spectrum).
The results are shown in Figure 3. It is clear that the Likelihood ratio method is able to see the signal at the correct Δm_s value, whereas the χ^2 modifed method failed. The same exercise was repeated at different generated values of Δm_s giving always the same result.

6 Test of the coverage of the two methods applied to a CKM Fits

The Δm_s likelihood from Likelihood Ration method is unlimited. A criticism was made in [6] claming on the danger of using such information in a CKM fits.
For this reason we did the following study. We have performed a simplified CKM fits where the only constraint used are: Δm_d and Δm_s. In this way we measure the quantity $R_t(=\sqrt{(1-\rho)^2+\eta^2})$ which is a side of the Unitarity Triangle. The system of constraints for the quantity R_t is:

$$\Delta m_d = a^2 R_t^2$$
$$\Delta m_d / \Delta m_s = b^2$$

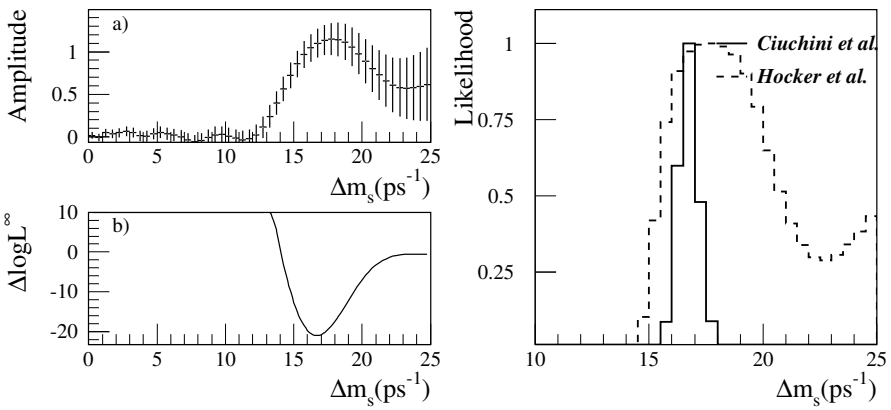

Figure 3: Toy-MC analyses with the same $\sigma(\mathcal{A})$ versus Δm_s behaviour as the world average analysis : a) amplitude spectrum, b) $\Delta \log \mathcal{L}^{\infty}(\Delta m_s)$, c) comparison between R $\equiv$ Likelihood ratio method and the Modified χ^2 method.

or

$$\Delta m_d = a^2 R_t^2$$
$$\Delta m_s = a^2/b^2$$

where a and b are gaussian distributed parameters (simulating the effect of the theoretical parameter $f_B\sqrt{B}$ and ξ).
Several experiments has been generated, each of them caracterized by the following set of parameters:

R_t
a_{theo} extracted from the a distribution
b_{theo} extracted from the b distribution
$\Delta m_d(theo)$ computed from R_t, a and b
$\Delta m_s(theo)$ computed from R_t, a and b
$\Delta m_d(exp)$ from $\Delta m_d(theo)$ smeared by the experimental resolution
a complete Amplitude spectrum from a toy-experiment generated with $\Delta m_s(theo)$

For each experiment a fit on R_t was performed and it was checked if the generated value of R_t was inside the 68%,95% and 99% C.L. fitted regions. This exercise was reapeted 1000 times, with similar and independent fits.
The results if the Likelihood ratio or the modified χ^2 are adopted for using the Δm_s informations in the fit are shown in Table 1,2.
The Likelihood ratio method give always the correct results. It is not the case of the Modifed χ^2 method.

7 Conclusions

In this paper we studied how to convert the amplitude spectrum as a function of Δm_s in a Likelihood function as a function of Δm_s, which can be used in CKM fits. For this

	68%	95%	99%
$\Delta m_s = 10$	67.5 ± 1.5	93.1 ± 0.8	98.1 ± 0.4
$\Delta m_s = 18.2$	71.4 ± 1.4	96.1 ± 0.6	99.6 ± 0.2
$\Delta m_s = 25$	69.5 ± 1.5	96.4 ± 0.6	99.3 ± 0.3
average			

Table 1: Results obtained with the Likelihood Ratio Method.

	68%	95%	99 %
$\Delta m_s = 10$	48.6 ± 1.6	83.8 ± 1.2	94.3 ± 0.7
$\Delta m_s = 18.2$	64.6 ± 1.5	93.0 ± 0.8	99.2 ± 0.3
$\Delta m_s = 25$	77.5 ± 1.5	98.2 ± 0.4	99.7 ± 0.2

Table 2: Results obtained with the modified χ^2.

purpose the Likelihood ratio method was proposed in [1]. In this paper we demonstrated that in case of an oscillation signal, this method give the correct measurement of Δm_s. Furthermore we have also demonstarted that the Likelihood as a function of Δm_s obtained using this method can be safely used as a Δm_s constraint in a CKM fit.

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