Introduction to Quantum Field Theory 10.2.03 - 20.2.03.

The Problems! by R.Zwicky

Convention: We will mainly work on the manifold $\mathbf{R}^4$ or let's better say $\mathbf{R}^{(1,3)}$ with metric $g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$. In the beginning of the exercises we will put ˆ on symbols representing operators.

1. **1-particle wavefunction or field ?**
   In the course we have seen two 'Scheinproblems' for the wavefunction interpretation of the Klein-Gordon equation:

   I. $j_0 = \rho \ngeq 0$
   Problems with probability interpretation?

   II. $E_{1-\text{particle}} = \pm E_p \quad , E_p \equiv \omega(\vec{p}) \equiv \sqrt{\vec{p}^2 + m^2}$
   Problems with stability?

   Let's see whether we can improve!

   (a) Show that: (by using a completeness relation)

   $$\langle x|\hat{p}^2 - m^2|\phi\rangle = K_m(x)\phi(x) \quad , K_m(x) \equiv -(\partial^2 + m^2)$$

   (b) Here we want to think of $\phi(x)$ as a wavefunction.

   $$K_m(x)\phi(x) = 0 \tag{1}$$
   $$\phi(x) = \int \frac{d^4p}{(2\pi)^4} e^{ip\cdot x}\tilde{\phi}(p) \tag{2}$$

   Work out the solution using this ansatz. The solution will allow you to perform one of the four integrations.
   Give all the energy eigenstates and their eigenvalues. ($\hat{p_0} = i\partial_0$)
   N.B If we demand covariance of the equation (1) then the field or 1-particle wavefunction $\phi(x)$ is a scalar!

   (c) If this would be the end of the story (no interaction) what restriction on the initial conditions would give us a physically meaningfull (stable) theory? 'Scheinproblem' II.

   (d) So we have to do something! Let us begin to think of $\phi(x)$ as a field. Convince yourself that the equation (1) is the result of of the following Lagrangian.

   $$S = \int_M \mathcal{L} = \frac{1}{2}\int_M [(\partial\phi)^2 - m^2\phi^2] \tag{3}$$

   Use or derive the Euler-Lagrange equation from the variation $\phi \to \phi + \delta\phi$. To understand the notation, this explicitely means:

   $$0 = \delta S = \frac{d}{d\epsilon}_{\epsilon=0}\int_M \mathcal{L}[\phi + \epsilon\delta\phi] \quad \delta\phi_{\partial M} = 0 \tag{4}$$

(e) Let us check what the energy of this field is all about. We will indicate two ways of getting an energy-momentum tensor $T^{\mu\nu}$:
Choose the second method if time is short.

i. Variation w.r.t. $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$ (known from General Relativity) gives the symmetric energy-momentum tensor.

$$\delta S_{matter} = \frac{1}{2} \int_M T^s_{\mu\nu} \delta g^{\mu\nu} \sqrt{-g} \tag{5}$$

You will need or want to derive the identities: $\delta g = g g^{\mu\nu} \delta g_{\mu\nu}$ and $\delta g_{\alpha\beta} = -g_{\alpha\gamma} g_{\beta\delta} \delta g^{\gamma\delta}$

ii. Variation w.r.t. $x \to x + \delta x$ to get the canonical energy momentum tensor.

$$T^c_{\mu\nu} = \frac{\delta \mathcal{L}}{\delta(\partial^\mu \phi)} \partial_\nu \phi - g_{\mu\nu} \mathcal{L} \tag{6}$$

The above result is an immediate consequence of Noether's theorem for a Lagrangian invariant under a symmetry up to a total derivative. Let's first derive Noether's theorem and then its modified version:
Noether's theorem Given a Lagrangian density $\mathcal{L}[\phi, \partial\phi]$ invariant under a symmetry δ (without use of the equation of motion) then the conserved current j_μ, $\partial \cdot j = 0$ is given by:

$$j_\mu = \frac{\delta \mathcal{L}}{\delta(\partial^\mu \phi)} \delta\phi = \pi_\mu \delta\phi \tag{7}$$

How to derive this ?

$$\begin{aligned} 0 \quad &= \quad \delta \mathcal{L} = \frac{\delta \mathcal{L}}{\delta \phi} \delta\phi + \frac{\delta \mathcal{L}}{\delta(\partial^\mu \phi)} \delta(\partial^\mu \phi) \\ &\stackrel{E-L}{=} \quad \partial_\alpha \left(\frac{\delta \mathcal{L}}{\delta(\partial_\alpha \phi}\right) \delta\phi + \frac{\delta \mathcal{L}}{\delta(\partial^\mu \phi)} \delta(\partial^\mu \phi) = \partial_\mu j^\mu \end{aligned}$$

Modified version: If $\delta\mathcal{L} = \partial_\mu \Omega^\mu$ then Noether current is modified $j_\mu \to j_\mu - \Omega_\mu$ and now there's almost nothing left to work yourself to (6)!
N.B. Looking at the T_{00} component we recognize it to be the Legendre transformation of the Lagrangian density in Minkowski space, which defines the Hamilton (energy) density.

The total energy at a given instant of time is given as:

$$H = \int_{M_n} T(n, n) \to \int d^3x T_{00} \geq 0 \tag{8}$$

where M_n is a spacelike hypersurface and n is the normal vector to it. In the second equality we went back to flat Minkowski space! The inequality which we should check gives us more comfort. Since $T_{\mu\nu}$ is not a measurable quantity but only its integral, we can always redefine it by $T_{\mu\nu} \to T_{\mu\nu} + \partial^\alpha E_{\alpha\mu\nu}$ with $E_{\alpha\mu\nu}$ an antisymmetric (3,0)-tensor [1]! Actually the results of two methods given above often differ in such a manner (e.g. QCD).
N.B. The energy momentum tensor obtained by metric variation is guaranteed to be symmetric, whereas for the canonical method this is not the case.

[1] In coordinate free language the T tensor is equivalent to $T + d\omega$ where ω is a 1-form!

(f) Let's calculate the Greensfunction of the Klein-Gordon operator.

$$K_m(x)G(x-y) = \delta^{(4)}(x-y) \tag{9}$$

This can be solved by making the ansatz

$$G(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{ip\cdot(x-y)} G(p) \tag{10}$$

The naive solution for $G(p) = \frac{1}{p^2-m^2}$ is immediate. But performing the integral over p demands us to make choices of how to go around the poles in th p_0-plane! Essentially you will find 3 inequivalent ways of doing this! $m^2 \to m^2 + (-i0, -ip_0, ip_0)$ [2], this corresponds to the Greensfunction (G_F, G_A, G_R), where F stands for Feynman (causal), A for advanced and R for retarded.
For simplicity let's go to two space-time dimesions. Perform the p_0-integration with the residue technique. For the remaining p_1-integration use a integral representation of the Bessel function:

$$K_0(z) = \int_0^\infty dt e^{-z\cosh(t)} \quad , Re[z] = 0 \tag{11}$$

$$K_0(\pm iz) = -\frac{\pi}{2}[N_0(z) \pm iJ_0(z)] \tag{12}$$

For calculational convenience it might be useful to remember Lorentz covariance of $G(x-y)$. You should end up with the following results:

$$G_F(x) = \frac{-i}{2\pi} K_0(m\sqrt{-x^2+i0}) \tag{13}$$

$$G_R(x) = \Theta(+x_0)G(x) \tag{14}$$

$$G_A(x) = \Theta(-x_0)G(x) \tag{15}$$

$$G(x) = 2\Re[G_F(x)] = \frac{1}{2}\Theta(x^2)\epsilon(x_0)J_0(m\sqrt{x^2}) \tag{16}$$

A more precise suggestion: First do the G_F function by assuming $x^2 > 0$ and then either guessing its covariant form or also doing the much easier case $x^2 < 0$. The other functions can then be found by analysing the integration paths.

N.B. This last exercise can be done in a more rigorous fashion with the theory of distributions. (c.f. Generalised Functions, Chilov & Gel'fand)

Summarising on our progress on the 'Scheinproblems'. We have seen that when we change from the wave-function to the field interpretation the field energy $H = \int d^3x T_{00} \geq 0$, even $T_{00} \geq 0$. So 'Scheinproblem' II seems to disappear in classical field theory.
'Scheinproblem' I persists. [3] To see this notice that people worked with a complex scalar field in analogy to the Schrödinger wavefunction. The action would look like:

$$S = \frac{1}{2}\int [\partial\phi^* \cdot \partial\phi - m^2\phi^*\phi] \tag{17}$$

This action is invariant under: $\phi \to e^{i\alpha}\phi$. Deriving the corresponding Noether current with formula (7).

$$j_\mu = \phi^* \overleftrightarrow{\partial}_\mu \phi \tag{18}$$

from where $j_0 \ngeq 0$.
N.B. If $\phi^*{=}\phi$ then $j_\mu = 0$. This should give us a hint on how to solve 'Scheinproblem' I!

[2] Actually there's a fourth method $m^2 \to m^2 + i0$, but this one is just the complex conjugate of the first one!
[3] We saw in the lecture that 'Scheinproblem' I disappears for the (classical) Dirac equation!

2. **Canonical quantization of the scalar field.**
In this exercise we will quantize the scalar field ϕ. For historical reasons this is sometimes called second quantisation. An important lesson from the course and the exercises so far is that the solution to the Klein-Gordon equation can be written as:

$$\phi(x) = \int d\mu(\vec{k})[f_k(x)\phi_+(k) + f_k^*(x)\phi_-(k)] \tag{19}$$

Where $d\mu(\vec{k}) = \frac{d^3\vec{k}}{2E_k}$, $f_k(x) = \frac{e^{-i\bar{k}\cdot x}}{(2\pi)^{3/2}}$, $\bar{k} = (E_k, \vec{k})$ and ϕ_+ (ϕ_-) are to be thought of as the coefficient of the positive (negative) energy solutions. In order to harmonise with the literature we write $\phi_- = a^\dagger$ and $\phi_+ = a$ from now on.
It is useful to introduce to the following Lorentz invariant scalar product for functions satisfying the Klein-Gordon equation: [4]

$$\langle f|g\rangle := i\int d^3x f(x)^* \overleftrightarrow{\partial_0}\, g(x) = \int \frac{d\mu(k)}{(2\pi)^3}\tilde{f}^*(k)\tilde{g}(k) \tag{20}$$

N.B. The Lorentz invariance is more transparent from the second equality !

(a) The usefulness of this scalar product is:

$$\langle f_q|\phi\rangle = a(q) \tag{21}$$
$$\langle f_q^*|\phi\rangle = a^\dagger(q) \tag{22}$$

Derive one of these relations by using the representation of the scalar product in x-space!

(b) Show that the canonical commutation relations ..

$$[\hat{\phi}(x), \hat{\pi}(y)]_{x_0=y_0} = i\delta^{(3)}(\vec{x}-\vec{y}) \tag{23}$$
$$[\hat{\phi}(x), \hat{\phi}(y)]_{x_0=y_0} = 0 \tag{24}$$
$$[\hat{\pi}(x), \hat{\pi}(y)]_{x_0=y_0} = 0 \tag{25}$$

... imply (by using the (21) and $\pi = \frac{\delta\mathcal{L}}{\delta(\partial_0\phi)}$)

$$[\hat{a}(\vec{k}), \hat{a}^\dagger(\vec{l})] = 2E_k\delta^{(3)}(\vec{k}-\vec{l}) \tag{26}$$
$$[\hat{a}(\vec{k}), \hat{a}(\vec{l})] = 0 \tag{27}$$
$$[\hat{a}^\dagger(\vec{k}), \hat{a}^\dagger(\vec{l})] = 0 \tag{28}$$

Just show the first commutator relation !!
N.B. On a technical level they should be interpreted as the quantisation of a system with infinite degrees of freedom. [5]

(c) Now that we have operator relations, we are seeking for a representation space! [6]
Physically it seems plausible that the operator coefficient for the negative energy wave-function annihilates the vacuum $|0\rangle$!

$$\hat{a}(\vec{k})|0\rangle = 0|0\rangle = 0 \tag{29}$$

[4] 1. $f(x)^* \overleftrightarrow{\partial_0}\, g(x) := [f(x)^* \overleftarrow{\partial_0}\, g(x) - f(x)^* \overrightarrow{\partial_0}\, g(x)]$, 2. $f(x) =: \frac{d^4k}{(2\pi)^4}\tilde{f}(k)e^{-ik\cdot x}$

[5] But as mentioned above one has also got used to the term of second quantisation.

[6] Von Neumann's theorem (1931) tells us that all representations for the canonical commutation relation are unitary equivalent. The theorem breaks down for a system of infinite degrees of freedom.

This is the so called **Fock-representation**. You are familiar with it from the quantum mechanical harmonic oscillator. Now we have infinitely many of them. From quantum mechanics we have learned that a good basis is generated by the particle number operators $\hat{N}(\vec{k}) := \hat{a}^\dagger(\vec{k})\hat{a}(\vec{k})$ with $[\hat{N}(\vec{k}), \hat{N}(\vec{l})] = 0$.
Express the Hamiltonian in the following way.

$$\hat{H} = \int d^3x \hat{T}_{00} = \int d^3x \frac{1}{2}[(\partial_0\hat{\phi})^2 + (\vec{\partial}\hat{\phi})^2 + m^2\hat{\phi}^2] \quad (30)$$

$$= \int d^3p[c_1\hat{N}(\vec{p}) + c_2\delta^{(3)}(0)] \quad (31)$$

We have succeeded in diagonalising the Hamiltonian. We choose the Fock space as a representation space.

(d) What should we do with this infinite constant? Kick it away!
More seriously since the ordering procedure is ambiguous when we pass from a classical to a quantum system, we take the liberty of ordering things in a more physical ourselves a convenient way. The so-called normal ordering: 'Move all annihilation operators $\hat{a}$ to the right of the creation operators $\hat{a}^\dagger$.'

$$N[\hat{H}] \equiv :\hat{H}: = \int d^3p[c_1\hat{N}(\vec{p})] \quad (32)$$

Simply calculate:

$$\langle 0|N[\hat{H}]|0\rangle \quad \langle\vec{q}|N[\hat{H}]|\vec{q'}\rangle \quad , |\vec{q}\rangle = a^\dagger(\vec{q})|0\rangle$$

understanding this is already understanding halfway the important **Wick-theorem**. So it seems that the negative energy states can't do us any harm in the free quantum theory! [7]

(e) **Causality** (this exercise is optional and almost equal to the following!) Investigate the causality of the Pauli-Jordan function.

$$\Delta(x) = [\hat{\phi}(x), \hat{\phi}(0)] = \langle 0|[\hat{\phi}(x), \hat{\phi}(0)]|0\rangle \quad (33)$$

You do not need to work out things in great detail. It is sufficient to write $\Delta(x) = \int d^3p f(x,p)$ to establishing the following two points:

- $\Delta(0, \vec{x}) = 0$
- Δ is a Lorentz covariant function

These two facts allows us to reconcile with special relativity !

$$\Delta(x) = 0 \quad x^2 < 0 \quad \text{i.e.spacelike} \quad (34)$$

N.B. $\Delta(x) = iG(x)$ c.f. exercise 3. It might be interesting to look at the behaviour of $\Delta(x)$ inside the lightcone.

[7] N.B. The constant $c_2\delta(0)$ is the well known zero point energy of the quantum harmonic oscillator. For a fermionic system $c_2 \to -c_2$. From this we learn that in supersymmetric theories unphysical divergences have tendencies to cancel among themselves!

3. **The Propagator is i× the Greensfunction !!** [8]

Now we want to study the Feynman propagator, which will be one of the main building blocks for constructing a perturbation theory. Evaluate the equation:

$$\Delta_F(x) \equiv \langle 0|T\phi(x)\phi(0)|0\rangle \tag{35}$$

where

$$T\phi(x)\phi(0) = \begin{cases} \phi(x)\phi(0) & x_0 > 0 \\ \phi(0)\phi(x) & x_0 < 0 \end{cases} \tag{36}$$

Plug (19) into (35) and then work out a comparison with:

$$G_F(x) \equiv \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip\cdot x}}{p^2 - m^2 + i0} \tag{37}$$

to obtain the final and important result:

$$\Delta_F(x) = iG_F(x) \tag{38}$$

N.B.

- In writing (35) we already assumed translational invariance
- we obtained: $K_m(x)\Delta_F(x) = i\delta^{(4)}(x)$. This is like a quantum equation of motion (Schwinger Dyson equation (from the path integral point of view) c.f. Peskin & Schröder chapter 9.6) From the operator point of view the δ-function arises because of the singular Θ-function in the definition of the propagator. Such a term is called a contact term.

[8] From now on we will suppress the $\hat{}$-symbol on operators for clearvoyance.

4. **Spinor Calculus - The simplest covariant equations**
A particle is characterised by the two invariants of the Poincare group, namely the mass and the spin. We want to work out spinor representations in this exercise. It is a well known fact (also shown in the course) that the orthochronos Lorentz group has a double covering. $L_+^\uparrow \sim SL(2,\mathbf{C})/\{\pm 1\}$. This mapping is to be understood as follows:

$$\overline{\Lambda(A)x} = A\overline{x}A^\dagger \quad ,\overline{a} \equiv a^\mu \sigma_\mu \quad ,\sigma_\mu = (\mathbf{1}_2, \vec{\sigma}) \tag{39}$$

N.B. In quantum mechanics you have seen the analogous relation for SU(2) double covering SO(3). For our purposes we take this as enough evidence that we are studying spin 1/2 representations.
$SL(2,\mathbf{C})$ is sometimes called the quantum mechanical Lorentz group!
It is our goal to study the fundamental representations of this double-covering space and find relativistic field equations for it! We will work in the fundamental representation of $SL(2,\mathbf{C})$. It is a fact that $A \in SL(2,\mathbf{C})$ is not unitary equivalent to A^*. Therefore we have two inequivalent representations transforming as:

$$u' = Au \quad u \in V \tag{40}$$
$$v' = A^* v \quad v \in \dot{V} \tag{41}$$

written in indices $u'_\alpha = A_\alpha^{\ \beta} u_\beta$, $v'_{\dot\alpha} = (A^*)_{\dot\alpha}^{\ \dot\beta} v_{\dot\beta}$ It's a *fact* that the epsilon tensor is the only invariant tensor in $SL(2,\mathbf{C})$, explicitely:

$$\epsilon_{\alpha\delta} = A_\alpha^{\ \beta} A_\delta^{\ \gamma} \epsilon_{\beta\gamma} \quad \epsilon_{01} = 1, \quad A \in SL(2,\mathbf{C}) \tag{42}$$

in matrix notation this means

$$A\epsilon A^T = \epsilon \tag{43}$$

it might be good to keep in mind that $(A^T)_\alpha^{\ \beta} = A_\beta^{\ \alpha}$ to understand the passage to matrix notation!

(a) Convince (show) yourself of this *fact* up to the statement of uniqueness.
Hence we know how to lower and raise indices for building invariant objects. ($-\epsilon_{\alpha\beta} = \epsilon^{\alpha\beta} = \epsilon_{\dot\alpha\dot\beta}$, $u^\alpha = \epsilon^{\alpha\beta} u_\beta$)

(b) Show that a 4 vector is a (1,1) tensor in $SL(2,\mathbf{C})$ by using (39).

$$\overline{x} = x_{\alpha\dot\beta} \quad ,\overline{x} \equiv \sigma_\mu x^\mu \tag{44}$$

Now we have the ingredients to write down a covariant dynamical equation (**Weyl equations**): [9]

$$\partial^{\alpha\dot\beta} u_\alpha = 0 \tag{45}$$
$$\partial_{\alpha\dot\beta} v^{\dot\beta} = 0 \tag{46}$$

(c) Now we want to go Dirac's way backwards! By adding a mass term we immediately see on grounds of indices that both equation need an additional fermion transforming under the conjugate representation.

$$i\partial^{\alpha\dot\beta} u_\alpha = m s^{\dot\beta} \tag{47}$$
$$i\partial_{\alpha\dot\beta} v^{\dot\beta} = m t_\alpha \tag{48}$$

[9] Those fermions are Weyl fermions. The two types correspond to left and right handed fermions.

From the transformation properties you might suspect that $u_\alpha(v^{\dot\beta})$ and $t_\alpha(s^{\dot\beta})$ are related. Act as if they were equal and plug one equation into the other! For this purpose you need to work out

$$a_{\alpha\dot\beta}a^{\alpha\dot\gamma} = \delta^{\dot\gamma}_{\dot\beta}a^2 \tag{49}$$

which is in matrix notation:

$$\bar{\bar{a}}\bar{a} = a^2\mathbf{1}_2 \tag{50}$$

where $\bar{\bar{a}} = a_\mu\bar{\sigma}^\mu$ c.f. (53) and we will accept $a^{\alpha\dot\beta} = \bar{\bar{a}}^T$ Work out the matrix equation (50) by further using $\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$. Now you can combine (47) and understand why 'm' is really the mass term.

You should get:

$$K_m(x)v^{\dot\beta} \;=\; 0 \tag{51}$$

$$K_m(x)u_\alpha \;=\; 0 \tag{52}$$

which is just the relativistic dispersion relation.

N.B.

- More compactly we can write (47) with the Dirac 4-spinor the **Dirac equation**. $\Psi = (u_\alpha, v^{\dot\beta})^T$ and $\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \bar\sigma_\mu & 0 \end{pmatrix}$ with $\bar\sigma_\mu = (1, -\vec\sigma)$ as

$$(i\partial\!\!\!/ - m)\Psi = 0 \tag{53}$$

  The γ_μ indicated above is a representation (Weyl or chiral) of the Clifford Algebra for the four dimensional Minkowski metric $g_{\mu\nu}$ i.e. $\{\gamma_\mu, \gamma_\nu\} = 2\mathbf{1}_4 g_{\mu\nu}$.
- We notice that if we want a mass term for the fermions we need both representations. However for the Majorana fermion there is a way out. A Majorana fermion is a real fermion in the following sense: For every fermion u_α there is a charge conjugate (anti-particle) $(u_\alpha)_c = u^{\dot\alpha}$. The Majorana condition is $(u)_c = u$. Looking at equation (47) we can write the **Majorana equation**

$$i\partial^{\alpha\dot\beta}u_\alpha = mu^{\dot\beta} \stackrel{\text{maj}}{=} mu_\beta$$

  the equation for the other chiral component is obvious from (45).
- In non-relativistic quantum mechanics we didn't bother with two different representations of the spin group. This is because $SU(2)$ has only one representation in two dimensional vector space. Particles and anti-particles are not so naturally related in quantum mechanics. Therefore we sometimes say that anti-particles are relativistic quantum effects.
- The spinorial calculus will proof itself to be useful when we study supersymmetry.
- An excellent reference on spinorial calculus is Straumann's lecture notes on QM II or chapter 13 in Wald's book General Relativity.

5. **Quantization of the Photon field *à* la Gupta Bleuler**
Quantization of gauge theories in not an easy job, because of the classical gauge freedom!

(a) Let's first tackle the canonical version. Get the canonical momenta

$$\pi_\mu = \frac{\delta \mathcal{L}}{\delta(\partial_0 A^\mu)} \tag{54}$$

from the Lagrangian:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad , F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

We would like to quantize our system with covariant canonical commutation relation: [10]

$$[A_\mu(x), \pi_\nu(y)]_{x0=y_0} = i g_{\mu\nu} \delta^{(3)}(\vec{x} - \vec{y}) \tag{55}$$

most obviously there's a problem since we obtain in particular $\pi_0 = 0$. Essentially two methods were developed for dealing with this problem:

- Exploit the gauge invariance and choose a gauge from the beginning. We impose the gauge condition in such a way that we can go through our program of canonical quantization. This method was pioneered by Fermi (1929), it's systematic approach, quantization under constraints was launched by Dirac (1950). Maybe in your course in quantum mechanics II you saw the example of quantising the photon field in the Coulomb gauge $(\vec{\partial} \cdot \vec{A}) = 0$
- Lorentz covariant method of BRST-quantization (1969,1975) developed for quantising non-abelian gauge theories. The abelian version which doesn't need a lot of the apparatus was found earlier by Gupta & Bleuler (1950)

(b) **Gupta-Bleuler** We begin by adding an explicit covariant gauge fixing term to the Lagrangian.

$$\mathcal{L}_{\mathcal{GF}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\alpha} (\partial_\mu A^\mu)^2 \tag{56}$$

One should make sure that the α parameter does not affect physical observables in the end! Let's bring this Lagrangian in a more convenient form by partial integration. [11]

$$\int \mathcal{L}_{\mathcal{GF}} = \frac{1}{2} \int A^\mu (g_{\mu\nu} \partial^2 - (1 - \alpha^{-1}) \partial_\mu \partial_\nu) A^\nu = \frac{1}{2} \int A^\mu \partial^2 P_{\mu\nu}(\alpha) A^\nu \tag{57}$$

(c) Derive the equation of motion either from (56) or (57) and the canonical momenta (54)!

(d) In the sequel we shall use the Feynman gauge $\alpha = 1$. As usual we use a Fourier ansatz for the vector potential.

$$A_\mu(x) = \sum_{\lambda=0}^{3} \int d\mu(\vec{k}) [a(\vec{k}, \lambda) \epsilon_\mu(\vec{k}, \lambda) f_k(x) + a^\dagger(\vec{k}, \lambda) \epsilon^*_\mu(\vec{k}, \lambda) f^*_k(x)] \tag{58}$$

Notice that in this ansatz the λ runs over four polarisations! We have to do something in the end to get rid of two of these four degrees of freedom. Since from Maxwell's theory or

[10] The covariance refers to the vector potential index A_μ, in the configuration space sense the canonical formalism is not covariant since it singles out x_0 w.r.t $\vec{x}$.

[11] N.B. For $\alpha \to \infty$ (removing the gauge fixing term) $P_{\mu\nu}$ is a projector! In particular projectors can't be inverted and that's one way to see why we have to fix the gauge ! For obvious reasons the $\alpha = 1$ Feynman gauge is a very popular gauge.

Lorentz group analysis we know that the photon has only two helicity states. Show that the relations below imply the canonical commutation relations (55):

$$[a(\vec{k},\lambda), a^\dagger(\vec{k'},\lambda')] = -2E_k g^{\lambda\lambda'}\delta^{(3)}(\vec{k}-\vec{k}') \tag{59}$$

and,

$$\sum_{\lambda=0}^{3} \epsilon_\mu(\vec{k},\lambda)\otimes\epsilon_\nu(\vec{k},\lambda')^* g^{\lambda\lambda'} = -g_{\mu\nu} \tag{60}$$

(e) This exercise is not necessary for the rest or simply supplementary. In case time is short jump to the next one! Show that the vector potential ansatz (58) satisfies:

$$[A_\mu(x), A_\nu(y)] = -ig_{\mu\nu}\Delta_0(x-y) \tag{61}$$

The covariance jumps at your eyes! Where $\Delta_0(x) = iG(x)_{m=0}$ is an old friend, the Pauli Jordan function:

$$\Delta_0(x) = i\int \frac{d^4k}{(2\pi)^3}\delta(k^2)e^{-ik\cdot x}\epsilon(k_0) \tag{62}$$

(f) Now we have to do something about our too many of degrees of freedom. The Hilbert space is too big, actually it is not even a Hilbert space since there are zero-norm states which might be expected looking at (59) or (61). We follow a rather pedestrian approach! Now you might think that it would be a good idea to define the physical Hilbert space as $(\partial\cdot A)|\text{phys}\rangle = 0$, but this is too strong. Derive an immediate contradiction from (61)!

(g) The correct way to do it is to impose the gauge constraint only weakly $\langle\text{phys'}|(\partial\cdot A)|\text{phys}\rangle$ or equivalently $(\partial\cdot A)^-|\text{phys}\rangle = 0$. [12] This is called the Gupta-Bleuler condition and was the difficult thing to discover.
Probably you are puzzled by something or let's say you got confused. Looking at (58) we seemed to have four d.o.f. Then we applied the gauge constraint so we expect to be left with three! Invent some argument why we are only left with two d.o.f.
Hint:You might want to look at the gauge condition in momentum space.
(A proper analysis can for instance be found in Bogoliubov & Shirkov chapter 8.5)

[12] From a modern point of view this is just the BRST condition. Looking at the equations of motion we see that Maxwell's equation are satisfied for physical matrix elements

6. **Free Dirac theory - getting familiar with 4-Spinors !**
So far we have seen that a 4-spinor is built out of two 2-spinors, one of them transforms in the fundamental and the other one in the complex conjugate representation of $SL(2,\mathbf{C})$.

$$\psi_{\text{Dirac}} = (u_\alpha, v^{\dot\beta})^T_{\text{exo}} = (\phi_a, \chi^{\dot a})^T_{\text{course}} \tag{63}$$

This representation corresponds to the Weyl representation of the γ-matrices. For this exercise it is sufficient to think of $\Psi \in \mathbf{C}^4$ and obeying the Dirac equation.

$$(i\partial\!\!\!/ - m)\Psi(x) = 0 \tag{64}$$

which derives from the Lagrangian $\mathcal{L}_D = \bar\Psi(i\partial\!\!\!/ - m)\Psi$, the γ-matrices satisfy the Clifford Algebra $\{\gamma_\mu, \gamma_\nu\} = 2\mathbf{1}_4 g_{\mu\nu}$ and $\bar\Psi \equiv \Psi^\dagger\gamma_0$. The field Ψ solving the Dirac equation is usually expanded as:

$$\Psi(x) = \sum_{r=1}^{2}\int d\mu(k)[a(k,r)u(k,r)f_k(x) + b^\dagger(k,r)v(k,r)f_k(x)^*] \tag{65}$$

In order that the above ansatz is a solution the u and v vector have to obey: [13]

$$(p\!\!\!/ - m)u(p) = 0 \quad (p\!\!\!/ + m)v(p) = 0 \tag{66}$$

This equation is best analysed in the rest frame:

$$(\gamma_0 m - m)u(m,\vec 0) = 0 \quad (\gamma_0 m + m)v(m,\vec 0) = 0 \tag{67}$$

The $\mathcal{H}_D = \gamma_0\vec\gamma\cdot\vec p + \gamma_0 m$ is a 4 dimensional hermitean operator and should therefore have 4 orthogonal eigenvectors with eigenvalues! The above equations tell us that we have found two, actually four solutions. Two are degenerate under $\mathcal{H}_{\mathcal{D}}$ and we label them by a further letter r! [14]

The general solution then can be obtained because of covariance by a Lorentz transformation on the spinors. $u(p) = S(\Lambda)u(m,\vec 0)$

(a) We choose the following orthnormalisations consistent with (65) and canonical quantization.

$$\begin{array}{ll} u(p,r)^\dagger u(p,s) = 2E_p\delta_{rs} & v(p,r)^\dagger u(p,s) = 0 \\ v(p,r)^\dagger v(p,s) = 2E_p\delta_{rs} & u(p,r)^\dagger v(p,s) = 0 \end{array}$$

By arguments of Lorentz covariance it is easy to show (do it): [15]

$$\bar u(p,r)u(p,s) = 2m\delta_{rs} \quad \bar v(p,r)v(p,s) = -2m\delta_{rs} \tag{68}$$

it might be useful to remember $\gamma_0^\dagger = \gamma_0$ and $S(\Lambda)^\dagger\gamma_0 = \gamma_0 S(\Lambda)^{-1}$

(b) A clever alternative to the Lorentz transformation trick is to make the obvious ansatz, as in the course!

$$\begin{array}{rcl} u(p) &=& N_u(p\!\!\!/ + m)u(m,\vec 0) \\ v(p) &=& N_v(p\!\!\!/ - m)v(m,\vec 0) \end{array}$$

where we used $(p\!\!\!/ - m)(p\!\!\!/ + m) = p^2 - m^2$ Now we can use (68) to determine (N_u, N_v).

[13] As long as we don't quantize a and b can be regarded as free coefficients to be determined by initial conditions.

[14] For further refinement one usually takes the hermitian Pauli-Lubanski tensor in the rest frame which is just the spin operator. $W_{PL}(p = (m,\vec 0)) = \vec S = \frac{1}{2}\begin{pmatrix} \vec\sigma & 0 \\ 0 & \vec\sigma \end{pmatrix}$ and $[\mathcal{H}_D, \vec S] = 0$

[15] In case you want to ask yourself a good question: Why does one(two) of the states have negative norm ?

(c) The projectors on positive and negative states can be written as:

$$\Lambda_+(p) = \frac{u(p)\otimes\bar{u}(p)}{\bar{u}(p)u(p)} = \sum_{r=1}^{2}\frac{u(p,r)\otimes\bar{u}(p,r)}{\bar{u}(p,r)u(p,r)} = \frac{\not{k}+m}{2m}$$
$$\Lambda_-(p) = \frac{v(p)\otimes\bar{v}(p)}{\bar{v}(p)v(p)} = \sum_{r=1}^{2}\frac{v(p,r)\otimes\bar{v}(p,r)}{\bar{v}(p,r)v(p,r)} = \frac{-\not{k}+m}{2m}$$

By construction these projectors have the obvious properties $\Lambda_+(p)u(p) = u(p)$,$\Lambda_+(p)v(p) = 0$, $\Lambda_-(p)v(p) = v(p)$ and $\Lambda_-(p)u(p) = 0$. Compute $\Lambda_\pm$ explicitely and verify that they obey the projector relations:

$$\Lambda_\pm^2 = \Lambda_\pm \quad \Lambda_+ + \Lambda_- = \mathbf{1}_4 \quad \mathrm{tr}[\Lambda_\pm] = 2$$

(d) (Optional) In the course it was beautifully explained by group theoretical arguments why we have to use anticommutators

$$\{a(p,r), a^\dagger(p',s)\} = 2E_p\delta_{rs}\delta^{(3)}(\vec{p}-\vec{p'})$$
$$\{b(p,r), b^\dagger(p',s)\} = 2E_p\delta_{rs}\delta^{(3)}(\vec{p}-\vec{p'})$$

to quantize the spinor fields. [16] Verify one or both anticommutation relations(canonical,covariant):

$$\{\Psi_\alpha(x), \pi_\beta(0)\}_{x_0=y_0} = i\delta_{\alpha\beta}\delta^{(3)}(\vec{x}) \tag{69}$$
$$\{\Psi_\alpha(x), \bar{\Psi}_\beta(0)\} = (i\not{\partial}+m)_{\alpha\beta}i\Delta_0(x) \tag{70}$$

(e) Work out the Hamiltonian from the Legendre transformation:

$$H = \int d^3xN[\pi\Psi - \mathcal{L}] = \int d^3xN[\bar{\Psi}i\gamma^0\partial_0\Psi] = \dots \tag{71}$$

Be aware of the following points!

- What would have happened if we would have used commutators?
- Check that the zero point energy has the opposite sign as for the scalar theory in exercise 2.
- Did we use the same normal ordering prescription as for bosons? What is the proper prescription ?

to get

$$H = \int d\mu(k)E_k[N_a(k) + N_b(k)]$$

with $N_a(k) = \sum_{r=1}^{2} a^\dagger(k,r)a(k,r)$ the number particle operator.
N.B. The Hamiltonian $H \geq 0$ and 'Scheinproblem' II seems also to be under control in the quantum theory!

(f) Analogous to the complex Klein Gordon field the Dirac Lagrangian $\mathcal{L}_D$ is invariant under the global transformations: $\Psi \to e^{i\alpha}\Psi$ and $\bar{\Psi} \to e^{-i\alpha}\bar{\Psi}$. The corresponding Noether current (7) is:

$$j^\mu = \bar{\Psi}\gamma^\mu\Psi \tag{72}$$

Now verify that the charge or probability density [17] is:

$$Q := \int d^3xN[j^0] = \int d\mu(k)[N_a(k) - N_b(k)] \tag{73}$$

[16] This is a consequence of the more general Spin Statitics connection that emerges in relativistic quantum field theory.
[17] No it is the charge and this solves the 'Scheinproblem' I!

7. **The Dirac Lagrangian in 2-spinor components**
Use (63) (Weyl representation) to establish:

$$\begin{aligned}\mathcal{L}_D &= \bar{\Psi}(i\gamma_\mu\partial^\mu - m)\Psi \\ &= \chi^{a*}i(\bar{\partial})_{a\dot{a}}\chi^{\dot{a}} + \phi^*_{\dot{a}}i(\bar{\bar{\partial}})^{\dot{a}a}\phi_a + m[\chi^{a*}\phi_a + \phi^*_{\dot{a}}\chi^{\dot{a}}]\end{aligned}$$

where we have seen that a 4 vector x_μ.

$$\bar{x} = (\sigma_\mu x^\mu)_{a\dot{a}}$$
$$\bar{\bar{x}} = (\bar{\sigma}_\mu x^\mu)^{\dot{a}a}$$

$\sigma_\mu = (1, \vec{\sigma})$,$\bar{\sigma}_\mu = (1, -\vec{\sigma})$. And the γ-matrix in the Weyl representation is:

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \bar{\sigma}_\mu & 0 \end{pmatrix}$$

also derive the equations of motion from the Lagrangian $\mathcal{L}_D$ in order to get the Dirac equations (47).

8. **Wick-Theorem (1950), for Bosons**
The Wick theorem is the most important structure in building a perturbation theory! Mastering the Wick theorem allows you to extract the Feynman rules for a given Lagrangian (theory). Before stating the theorem and demanding a proof we want to give a pedagogical example. Actually the induction base of the inductive proof! We can write the scalar field (19) as a sum of creator and annihilator parts:

$$\phi(x) = \phi^+(x) + \phi^-(x)$$

$\phi^{+(-)}$ are the creation (annihilator) parts! [18] and the normal ordering we saw in exercise 2 is simply

$$N[\phi(x)\phi(y)] = \phi^+(x)\phi^+(y) + \phi^+(x)\phi^-(y) + \phi^+(y)\phi^-(x) + \phi^-(x)\phi^-(y)$$

From the canonical commutation relations (26) it is clear that:

$$T\phi(x)\phi(0) = N[\phi(x)\phi(0)] + \mathbf{C} - \text{number}$$

using $\langle 0|\phi^+ = 0,\ \phi^-|0\rangle = 0$ the **C**-number is demystified to be $\Delta_F(x)$.

Wick Theorem (weak version): Notation: $\phi_a = \phi(x_a)$

$$[\phi_1 \dots \phi_n] = N[\phi_1 \dots \phi_n] \quad + \sum_{\text{pairs}\setminus 1} \Delta^+(p[1]-p[2]) \dots \Delta^+(p[2k-1]-p[2k]) \times N[\phi_1 \dots \check{\phi}_{p[1]} \dots \check{\phi}_{p[2k]} \dots \phi_n]$$

where $\check{\phi}$ means that the corresponding field is omitted in the string. Now you have all the equipment to proof the theorem by induction.
Almost ... you need to convince yourself that:

$$[\phi^-(x), \phi^+(0)] = \langle 0|\phi(x)\phi(0)|0\rangle =: \Delta^+(x) \tag{74}$$

having proofed Wick's weak version sufficient for causal perturbation approach (Epstein-Glaser) we proceed to argue for a stronger version necessary for a perturbation in the Dyson series.
Wick Theorem (strong version):

$$T[\phi_1 \dots \phi_n] = N[\phi_1 \dots \phi_n] \quad + \sum_{\text{pairs}\setminus 1} \Delta_F(p[1]-p[2]) \dots \Delta_F(p[2k-1]-p[2k]) \times N[\phi_1 \dots \check{\phi}_{p[1]} \dots \check{\phi}_{p[2k]} \dots \phi_n]$$

Note that $T[\dots]$ is only well defined for $x_i^0 \neq x_j^0$. Since it is completely symmetric in in the time arguments we may assume $x_1^0 > \dots > x_n^0$. But then we can use the weak version. Completing the arguments we have to convince ourselves of two points:

- For coincident points the RHS serves up to a certain point (c.f. below) as a definition of the LHS.
- $\Delta^+ \to \Delta_F$ when we open for all x_i^0 (no a priori ordering)

Some more remarks:

[18] N.B. **Warning!** The notation of $\pm$ is reversed as opposed to (19) since the positive energy parts 'carry' the annihilation operators and the negative energy parts the creation operators.

- Problem of coincident points is severe. $\Delta_F(x)$ is a well defined distribution, whereas $\Delta_F(x)^2$ is not and results in UV-divergences c.f. Causal Approach!!
- An analogous theorem can be derived for fermions where the pairing picks up the permutation sign c.f. any standard book on QFT.
- What we really need at the end of the day in standard perturbation theory is $\langle 0|T[\ldots]|0\rangle = \sum_{\text{completepairing}}(\Delta_F \ldots \Delta_F)$ and slight modification introduced by normal ordered interaction polynomials.

9. **From cross section to S-matrix elements, an example**
We want to derive the cross section for a $2 \to n$ particle process, two particles colliding and thereby producing n particles. The two colliding particles are described by wavepackets centred around $\bar{p}_1$ & $\bar{p}_2$. We go into a frame where the two momenta are collinear i.e. $\vec{\bar{p_1}}$ parallel $\vec{\bar{p_2}}$ excluding the pathetic case of equality! Such frames are for example the centre of mass or laboratory frames. Cross sections are not Lorentz invariant! [19] The incoming state is described by:

$$|I_1, I_2\rangle := \int d\mu(\vec{p_1}) d\mu(\vec{p_2}) I_1(\vec{p_1}) I_2(\vec{p_2}) |\vec{p}_1, \vec{p}_2\rangle \tag{75}$$

where $d\mu(\vec{p}) = \frac{d^3p}{(2\pi)^3 E_{\vec{p}}}$ is the invariant measure. The n outgoing particles shall go into regions F_i in phase space. From our basic understanding of quantum mechanics we can write down the transition probability:

$$P(I_1, I_2 \to F_1..F_n) = \int_F \prod_{i=1}^{n} d\mu(\vec{q_i}) |\langle \vec{q_1}..\vec{q_n}|S - 1|I_1, I_2\rangle|^2 \tag{76}$$

where we subtracted the trivial one from the S-matrix! The S-matrix is defined and parametrised as:

$$\begin{aligned}
{}_{in}\langle\{p_i\}|\{p_f\}\rangle_{out} &= {}_{in}\langle|\{p_i\}|S|\{p_f\}\rangle_{in} \\
\langle\{p\}|S - 1|\{q\}\rangle &=: i(2\pi)^4 \delta^{(4)}(\sum p - \sum q) M(p \to q)
\end{aligned}$$

But this is not the end of the story! There's an additional subtlety. In a scattering experiment it is certainly not possible to know the impact parameter b! The way out is that instead of the state $|I_1, I_2\rangle$ we really take an ensemble of states $|I_1^b, I_2\rangle$ where $I_1^b(\vec{p}) = e^{-i\vec{p}\cdot\vec{b}} I_1(\vec{p})$ from basic principles of quantum mechanics, all right ? Let the impact parameter lie in a domain B in a homogeneous manner [20], expressed in formulas we define our cross section:

$$\sigma(I_1, I_2 \to F_1..F_n) = \int_B d^2b P(I_1^b, I_2 \to F_1..F_n) \tag{77}$$

(a) Now we finally come to the point where we can do some work! Let's put all our stuff into (77) to obtain: (In a compact or sloppy notation?)

$$\begin{aligned}
\sigma \;=\; & (2\pi)^8 \int_B d^2b \int_F \prod_i d\mu(q_i) \int d\mu(p_1) d\mu(p_2) d\mu(p_1') d\mu(p_2') \times \\
& I_1(p_1)^* I_2(p_2)^* I_1(p_1') I_2(p_2') \delta(\sum p_i - \sum q_i) \delta(\sum p_i' - \sum q_i) \\
& e^{i\vec{b}\cdot(\vec{p_1}' - \vec{p_1})} M(p_1 + p_2 \to \sum q_i)^* M(p_1' + p_2' \to \sum q_i)
\end{aligned}$$

Where we have introduced (p_1', p_2') as dummy integration variables.

(b) It is possible to integrate over p' and b. Integrate first b yielding a two dimensional delta function. Use this delta function plus one of the other two to perform all six integrations over p_i' by assuming that $I_i(\vec{p_i})$ is sufficiently centered around $\vec{\bar{p_i}}$! You should obtain:

$$d\sigma(\bar{p}_1, \bar{p}_2 \to q_1..q_2) = \prod_i d\mu(q_i) \frac{(2\pi)^{10}}{4A} |M(\bar{p}_1, \bar{p}_2 \to q_1..q_2)|^2 \delta^{(4)}(\bar{p}_1 + \bar{p}_2 - \sum q_i)$$

[19] Cross sections of the type we calculate are only invariant under a subgroup of Lorentz transformations. Namely all rotation and boosts in the collinear direction.

[20] Ideally the extension of this domain is much larger than the impact of the quantum forces under consideration. Problematic for QED, IR-divergences.

where,

$$\sigma(\bar{p}_1 + \bar{p}_2 \to F) = \int_F d\sigma(\bar{p}_1 + \bar{p}_2 \to q_1..q_2)$$

and A is the famous Møller factor:

$$A = E_{\bar{p}_1} E_{\bar{p}_2} |\bar{v}_1^{||} - \bar{v}_2^{||}| \quad , \bar{v}^{||} = \frac{\bar{p}^{||}}{E_{\bar{p}}}$$

N.B. The Møller factor is not Lorentz invariant. Although assuming a collinear frame it can be put into a Lorentz invariant form $A = \sqrt{(\bar{p}_1 \cdot \bar{p}_2)^2 - m_1^2 m_2^2}$.

The presentation strongly follows the one of Peskin & Schröder chapter 4.5.

10. **Derivation of Feynman rules for QED**

We would like to derive the Feynman rules for QED! That is we want to find easy rules to set up our integrals for S-martix elements at any given order of the coupling constant. The QED Lagrangian is given by a U(1) gauge field minimally coupled to a Dirac fermion.

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}(i\partial\!\!\!/ + eA\!\!\!/)\Psi = \mathcal{L}_0 + \mathcal{L}_I \tag{78}$$

where $c\!\!\!/ = \gamma_\mu c^\mu$ is the notation for the Feynman slash! The interaction Lagrangian is $\mathcal{L}_I = -\mathcal{H}_I = N[\bar{\Psi}eA\!\!\!/\Psi]$. **Normal ordering** in the interaction part is necessary (in order to avoid crazy self closing loops) and often suppressed like in (78), :). The Dyson Series (evolution operator in the interaction picture) obtained in the course is:

$$U_w(t,t') = Te^{-i\int_t^{t'}\int d^3\vec{x}\mathcal{H}_w^I} \tag{79}$$

The corresponding S-Matrix is defined as:

$$S := U_w(\infty, -\infty) \tag{80}$$

our goal is to work out S-matrix elements between physical states $|\phi_f\rangle$,$|\phi_i\rangle$. [21]

$$S_{fi} := \langle\phi_f|S|\phi_i\rangle \tag{81}$$

So what are the physical states? The Hilbert space a Fock space under consideration is given by $\mathcal{H} = \mathcal{H}_m \otimes \mathcal{H}_f$ where m stands for matter fermions and f for the photon field. This space is spanned by the generic states: [22]

$$\begin{aligned}&|k_1,\lambda_1;..;k_a,\lambda_a;p_1,r_1,..;p_b,r_b;\bar{p}_1,\bar{r}_1..;\bar{p}_c,\bar{r}_c\rangle = \\ &[a^\dagger(k_1,\lambda_1)..a^\dagger(k_a,\lambda_a)][u^\dagger(p_1,r_1)..u^\dagger(p_b,r_b)][v^\dagger(\bar{p}_1,\bar{r}_1)..v^\dagger(\bar{p}_c,\bar{r}_c)]|0\rangle\end{aligned}$$

I suggest to work out a specific example and from this experience we may abstract the general rules. Let's take the *vacuum polarisation* in the one-loop approximation. That is to say:

$$\langle k_1,\lambda_1;0;0|S|k_2,\lambda_2;0;0\rangle + c_0 + ec_1 + e^2c_2 + \dots \tag{82}$$

e^2c_2 is the one-loop vacuum polarisation! And even more concretely

$$\frac{1}{2}\int d^4xd^4y(+ie)^2\langle 0|a(k_1,\lambda_1)T[\bar{\Psi}A\!\!\!/\Psi](x)[\bar{\Psi}A\!\!\!/\Psi](y)|a^\dagger(k_2,\lambda_2)|0\rangle \tag{83}$$

The way to systematically tackle this is Wick's theorem. In our approach we have to combine the two versions of Wick's theorem. [23] We have to keep in mind:

- Since we take Fock vacuum expectation values we only need completely contracted terms of the Wick sum! In our language a contraction of the operators A and B means $\langle 0|TAB|0\rangle$ or $\langle 0|AB|0\rangle$ for the strong or weak version respectively.
- Due to their anticommuting nature our fermionic friends pick up the sign for the overall contraction!
- There are no contractions inside normal ordered products!

[21] Actually it is better to define $S := U_w(\infty,-\infty)/\langle 0|U_w(\infty,-\infty)|0\rangle$ in order to avoid vacuum graphs in higher orders. Such a construction is automatic in the LSZ approach!

[22] What assumption are made for the states below. N.B. we are working in the interaction picture!

[23] In the LSZ approach we only need the strong version

(a) We need the following relations (proof at least one of them)

$$
\begin{aligned}
(2\pi)^{3/2}\langle 0|a(k,\lambda)A^{\mu}(x)|0\rangle &= -\,g^{\lambda\lambda}e^{*}(k,\lambda)^{\mu}e^{ik\cdot x}\\
(2\pi)^{3/2}\langle 0|A^{\mu}(x)a^{\dagger}(k,\lambda)|0\rangle &= -\,g^{\lambda\lambda}e(k,\lambda)^{\mu}e^{-ik\cdot x}\\
(2\pi)^{3/2}\langle 0|a(p,r)\bar{\Psi}(x)|0\rangle &= \bar{u}(p,r)e^{ik\cdot x}\\
(2\pi)^{3/2}\langle 0|\Psi(x)a^{\dagger}(p,r)|0\rangle &= u(p,r)e^{-ik\cdot x}\\
(2\pi)^{3/2}\langle 0|b(p,r)\Psi(x)|0\rangle &= v(p,r)e^{ik\cdot x}\\
(2\pi)^{3/2}\langle 0|\bar{\Psi}(x)b^{\dagger}(p,r)|0\rangle &= \bar{v}(p,r)e^{-ik\cdot x}
\end{aligned}
$$

(b) We will go on and evaluate (83) in the Feynman gauge $\alpha = 1$. [24]

$$
e^2 c_2 = (2\pi)^4\delta^{(4)}(k_1-k_2)\frac{\epsilon^{*}_{\mu}(k_1,\lambda_1)}{(2\pi)^{3/2}}\frac{\epsilon_{\nu}(k_2,\lambda_2)}{(2\pi)^{3/2}}\underbrace{e^2\int\frac{d^4k}{(2\pi)^4}\mathrm{tr}[\gamma^{\mu}\tilde{S}_F(k+k_1)\gamma^{\nu}\tilde{S}_F(k)]}_{\equiv i\Pi_2^{\mu\nu}(q)} \quad (84)
$$

(c) Now you got it all in front of your eyes. Give the Feynman rules in momentum space.

You need the following definitions to build up your theory:

$$
\begin{aligned}
\Psi(x) &= \sum_{r=1}^{2}\int d\mu(\vec{k})[a(\vec{k},r)u(\vec{k},r)f_k(x) + b^{\dagger}(\vec{k},r)v(\vec{k},r)f_k(x)^{*}]\\
\bar{\Psi}(x) &= \sum_{r=1}^{2}\int d\mu(\vec{k})[a^{\dagger}(\vec{k},r)\bar{u}(\vec{k},r)f_k(x)^{*} + b(\vec{k},r)\bar{v}(\vec{k},r)f_k(x)]\\
A_{\mu}(x) &= \sum_{\lambda=0}^{3}\int d\mu(\vec{k})[a(\vec{k},\lambda)\epsilon(\vec{k},\lambda)f_k(x) + a^{\dagger}(\vec{k},\lambda)\epsilon(\vec{k},\lambda)^{*}f_k(x)^{*}]
\end{aligned}
$$

and ...

$$
\begin{aligned}
&\langle 0|T\Psi(x)\bar{\Psi}(0)|0\rangle = S_F(x) = i\int\frac{d^4p}{(2\pi)^4}\frac{(\not{p}-m)e^{-ip\cdot x}}{p^2-m^2+i0}\\
&\langle 0|TA_{\mu}(x)A_{\nu}(0)|0\rangle = -(g_{\mu\nu}-(1-\alpha^{-1})\frac{\partial_{\mu}\partial_{\nu}}{\partial^2})\Delta_F(x)\\
&\Delta_F(x) = i\int\frac{d^4p}{(2\pi)^4}\frac{e^{-ip\cdot x}}{p^2-m^2+i0}
\end{aligned}
$$

[24] In the result (84) we obtained momentum momentum conservation. Helicity/spin conservation is not yet obvious but will become obvious in the next exercise when we further evaluate the integral!

11. **Rendez-vous with the vacuum polarisation**
We are meeting an old friend from yesterday (84):

$$i\Pi_2^{\mu\nu}(q) = -e^2 \int \frac{d^4k}{(2\pi)^4} \mathrm{tr}[\frac{(\not k + m)\gamma^\mu(\not k + \not q + m)\gamma^\nu}{(k^2 - m^2 + i0)((k+q)^2 - m^2 + i0)}] \tag{85}$$

let's begin to work out this integral by first evaluating the trace. You will need:

$$\mathrm{tr}[\gamma_\alpha\gamma_\beta] = 4g_{\alpha\beta} \tag{86}$$

$$\mathrm{tr}[\gamma_\alpha\gamma_\beta\gamma_\gamma\gamma_\delta] = 4[g_{\alpha\beta}g_{\gamma\delta} - g_{\alpha\gamma}g_{\beta\delta} + g_{\alpha\delta}g_{\beta\gamma}] \tag{87}$$

Now the denominator is a really unhandy thing. Feynman found a way to deal with such problems. Use the famous Feynman parametrisation:

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{(xA + (1-x)B)^2} \tag{88}$$

Then it is useful to complete squares to $(k + xq)^2$ in the denominator ... and change variables to $p = k + xq$. Because of even-odd type symmetries you only need to retain the terms in even powers of p!

$$i\Pi_2^{\mu\nu}(q) = -4e^2 \int_0^1 dx \int \frac{d^4p}{(2\pi)^4} \frac{2p^\mu p^\nu - g^{\mu\nu}p^2 - 2x(1-x)q^\mu q^\nu + g^{\mu\nu}(m^2 + x(1-x)q^2)}{(p^2 - M^2)^2}$$

$M^2 \equiv m^2 - x(1-x)q^2$. This is the standard integral for the vacuum polarisation. It is not difficult to see that this is divergent! We proceed by introducing a regularisation. We choose the **dimensional regularisation**. [25] The basic idea is that if in the above integral we could change the space time dimension from $d = 3 + 1$ to $d < 1 + 1$ then the above integral would be convergent. Let's play this game and substitute $\frac{d^4p}{(2\pi)^4} \to \frac{d^dp}{(2\pi)^d}$ in the integral. Now that we have a formally convergent integral we can justify a further manipulation: the *Wick rotation.* We deform the contour in the p_0-plane to the imaginary axis (and I think by now you know in which sense you have to turn the contour !) and then substitute $p^0 \to ip_E^0$ and get:

$$i\Pi_2^{\mu\nu}(q) = -i4e^2 \int_0^1 dx \int \frac{d^dp_E}{(2\pi)^d} \frac{[-\frac{2}{d}g^{\mu\nu}p_E^2 + g^{\mu\nu}p_E^2 - 2x(1-x)q^{\mu\nu} + g^{\mu\nu}(m^2 + x(1-x)q^2)]}{(p_E^2 + M^2)^2}$$

where $p_E^2 = (p_E^0)^2 + \vec{p}^2$ and in the nominator we performed the effective substitution $p_E^\mu p_E^\nu \to \frac{1}{d}g^{\mu\nu}p_E^2$ (Think about the argument yourself). We have a type of integral doesn't depend on the angles in the integration so we can go polar:

$$\int d^dk f(k^2) = \int d\Omega_d \int_0^\infty dk k^{d-1} f(k^2) = \frac{2(\pi)^{d/2}}{\Gamma[d/2]} \int_0^\infty dk k^{d-1} f(k^2)$$

I hope you followed and checked every step until now. We will use the general integral:

$$\Gamma[d,a,b] \equiv \int \frac{d^dk}{(2\pi)^d} \frac{(k^2)^a}{(k^2 + A^2)^b}$$
$$= \frac{1}{(4\pi)^2}(\frac{A^2}{4\pi})^{d/2-2}(A^2)^{a-b+2}\frac{\Gamma[d/2+a]\Gamma[b-a-d/2]}{\Gamma[b]\Gamma[d/2]}$$

[25] This regularisation has the advantage that it is gauge invariant. Generally one has the tendency to choose a regularisation which doesn't brake symmetries or at least the most important ones. Usually the symmetries are restored after renormalisation, if this is not the case then we say that the theory has an anomaly!

If you like you can derive it by using the definition of the beta function:

$$B(a,b) = \frac{\Gamma[a]\Gamma[b]}{\Gamma[a+b]} = \int_0^1 dyy^{a-1}(1-y)^{b-1} \tag{89}$$

putting things together we get:

$$i\Pi_2^{\mu\nu}(q) = (q^2 g^{\mu\nu} - q^\mu q^\nu) i\Pi_2(q^2) \tag{90}$$

where we see that the projector appears nicely and guarantees us 'quantum' gauge invariance, the Ward identity $q_\mu \Pi_2^{\mu\nu}(q) = 0$. Furthermore we can read of spin/helicity conservation of the photon by looking at equation (84). $\Pi_2(q^2)$:

$$\Pi_2(q^2) = \frac{-8e^2}{(4\pi)^{d/2}} \int_0^1 dxx(1-x) \frac{\Gamma[2-d/2]}{M^{4-d}} \tag{91}$$

We pretended something like $d = 4 - \epsilon$ and that ϵ was sufficient positive. Now the crucial point is that the Γ function has a unique analytic continuation. So this will be our criterion. We will now perform a Laurent series in the Γ function and not surprisingly the divergences come in the disguise of poles. The Laurent series of the Γ function is given by: [26]

$$\Gamma[x] = \frac{1}{x} - \gamma + O(x) \tag{92}$$

There's an additional subtlety hidden in Pandora's box. Since we have changed the dimensionality of the integral we must compensate for it by multiplying the overall expression by a dimensionfull parameter μ^ϵ (μ has the dimension of mass). [27] Consequently expanding in ϵ: (recall $d = 4 - \epsilon$) [28]

$$\Pi_2(q^2) = -\frac{e^2}{2\pi} \int_0^1 dxx(1-x)\left(\frac{2}{\epsilon} - \log[\frac{M^2}{\mu^2}] - \gamma + \log[4\pi]\right) + O(\epsilon) \tag{93}$$

The minus sign is crucial! Since what we have calculated is responsible for the charge renormalisation. The negative sign induces screening. Non abelian gauge theories have the opposite sign and therefore contain antiscreening, also known as 'asymptotic freedom'!!

A physical interpretation of Π_2 can be found in Peskin & Schröder chapter 7.5.

By the way the Γ function is defined by:

$$\Gamma[z] := \int_0^\infty t^{z-1} e^{-t} \quad , \Re[z] > 0 \tag{94}$$

and the rest of the job is done by analytic continuation and once again one picks up simple poles on the negative real axis. This is best seen in the Weierstrass' representation:

$$\Gamma[z] = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+z)} + \int_1^\infty dtt^{z-1} e^{-t} \forall z \tag{95}$$

and Bye Bye !

[26] γ is the Euler's constant and given by 0.577..

[27] μ is the renormalisation scale for the d dimensional regularisation.

[28] often one uses the fine structure constant $\alpha = \frac{e^2}{4\pi}$