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N=4 amplitudes, collinear limits, and special kinematics

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with

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arXiv:1205.3448 and 1109.0058 , 1007.1805

N=4 superamplitude and R

Colour-ordered n-point amplitudes in planar $\mathcal{N}=4$ SYM:

superamplitude $\longrightarrow$ $$A_n = \sum_{k=0}^{n-4} A_{n,k}$$ $\longleftarrow$ N^{k}MHV amplitude of degree $(\eta)^{8+4k}$

Factor out tree-level superamplitude and IR divergences from loops:

$$\boxed{A_n = A_n^{\text{tree}}\, M_n^{\text{BDS}}\, R_n}$$

R_n is the *reduced* superamplitude (remainder f-n) $R_n = \sum_{k=0}^{n-4} R_{n,k}$

$R_{n,k}$ are finite and dual-conformally invariant

MHV case: $R_{n,0}$ are functions of conformal cross-ratios u_{ij}.

Multi-collinear limits: (m+1) coll. momenta

Full superamplitude factorises:

$$\boxed{A_n \to A_{n-m} \times \mathrm{Split}_m}$$

Taylor-expanding in Grassmann η's get for each N^{k}MHV:

$$\begin{aligned} A_{n,k} \quad &\to \quad A_{n-m,k} \times \mathrm{Split}_{m,0} + A_{n-m,k-1} \times \mathrm{Split}_{m,1} + \ldots \\ &= \quad \sum_{p=0}^{k} A_{n-m,k-p} \times \mathrm{Split}_{m,p} \end{aligned}$$

What about R_n?

Multi-collinear limits: (m+1) coll. momenta

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What about R_n? $\qquad R_n \to R_{n-1}$ $\qquad$ m=1

$$\lim_{k\,\text{fixed}} R_{n,k} \;\to\; R_{n-2,k} \times \mathsf{split}_{2,0} = R_{n-2,k} \times R_{6,0}$$ m=2

$$\lim_{k\,\text{fixed}} R_{n,k} \;\to\; R_{n-m,k} \times R_{m+4,0}$$ m general

Bern-Dixon-Kosower-Roiban-Spradlin-Vergu-Volovich
Anastasiou-Brandhuber-Heslop-VVK-Spence-Travaglini; $\qquad$ Heslop-VVK

Multi-collinear limits: (m+1) coll. momenta

General multi-collinear limit for super-R_n (with no restrictions on preserving helicity degree of the ampltitude:

Goddard-Heslop-VVK'12

$$\boxed{R_n \;\to\; R_{n-m} \times R_{m+4}}$$

based on d. conf. inv. of R
and m.-coll. factorisation of A

Proof:

1. A_n has universal collinear factorisation; so does M_{BDS}, hence

$$R_n \;\to\; R_{n-m} \times \mathsf{split}_m$$

2. Take maximal multi-coll. limit $m = n - 4$:

$$R_{m+4} \;\to\; R_4 \times \mathsf{split}_m = \mathsf{split}_m$$

3. But our $(m+1)$-coll. limit can also be achieved by a *superconformal transformation.*

Alday-Gaiotto-Maldacena-Sever-Vieira'10
Goddard-Heslop-VVK'12 (super-conf.)

Therefore $R_{m+4} \to R_{m+4}$.

4. Hence $\mathsf{split}_m = R_{m+4}$

Multi-collinear limits: (m+1) coll. momenta

$$R_n \;\to\; R_{n-m} \times R_{m+4}$$

The super-R_n can be expanded in η's :

$$\begin{aligned} R_{n,k} \quad & \to R_{n-m,k} \times R_{m,0} + R_{n-m,k-1} \times R_{m,1} + \dots \\ & = \sum_{p=0}^{k} R_{n-m,k-p} \times R_{m,p} \end{aligned}$$

From now on use the linear realisation of multi-collinear limits by taking the logarithm of the super-remainder f-n

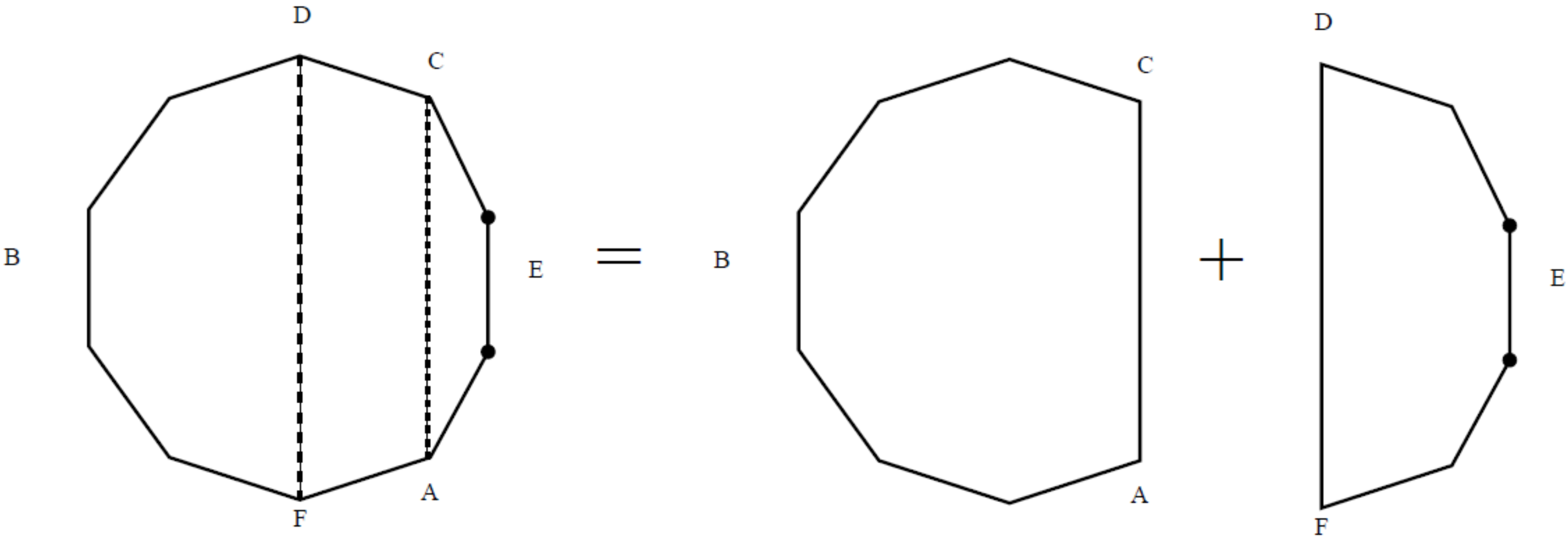
$\mathcal{R}_n := \log R_n$

$$\boxed{\mathcal{R}_n \;\to\; \mathcal{R}_{n-m} + \mathcal{R}_{m+4}}$$

Multi-collinear limits: (m+1) coll. momenta

$$\mathcal{R}_n \to \mathcal{R}_{n-m} + \mathcal{R}_{m+4}$$

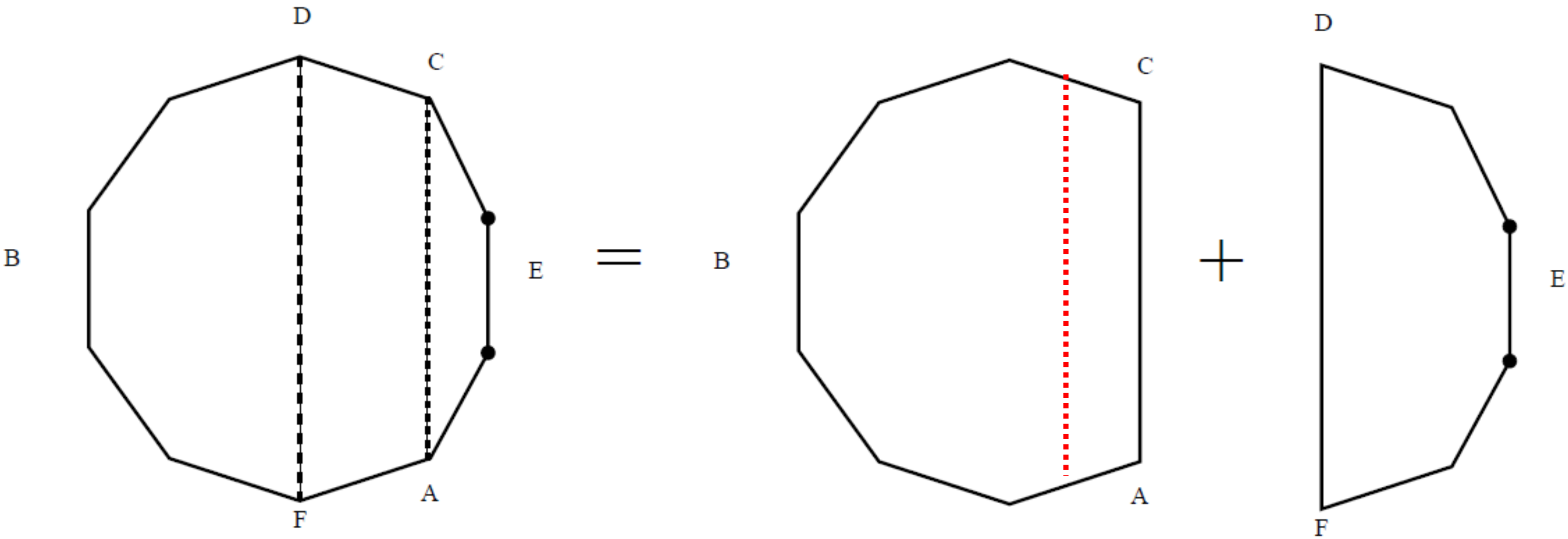
Conformal cross-ratios $u_{ij} = \frac{x_{i,j+1}^2 x_{i+1,j}^2}{x_{i,j}^2 x_{i+1,j+1}^2}$ 'connect' edge i with edge j.



Multi-collinear limits: (m+1) coll. momenta

$$\mathcal{R}_n \rightarrow \mathcal{R}_{n-m} + \mathcal{R}_{m+4}$$

Conformal cross-ratios $u_{ij} = \frac{x^2_{i,j+1} x^2_{i+1,j}}{x^2_{i,j} x^2_{i+1,j+1}}$ 'connect' edge i with edge j.

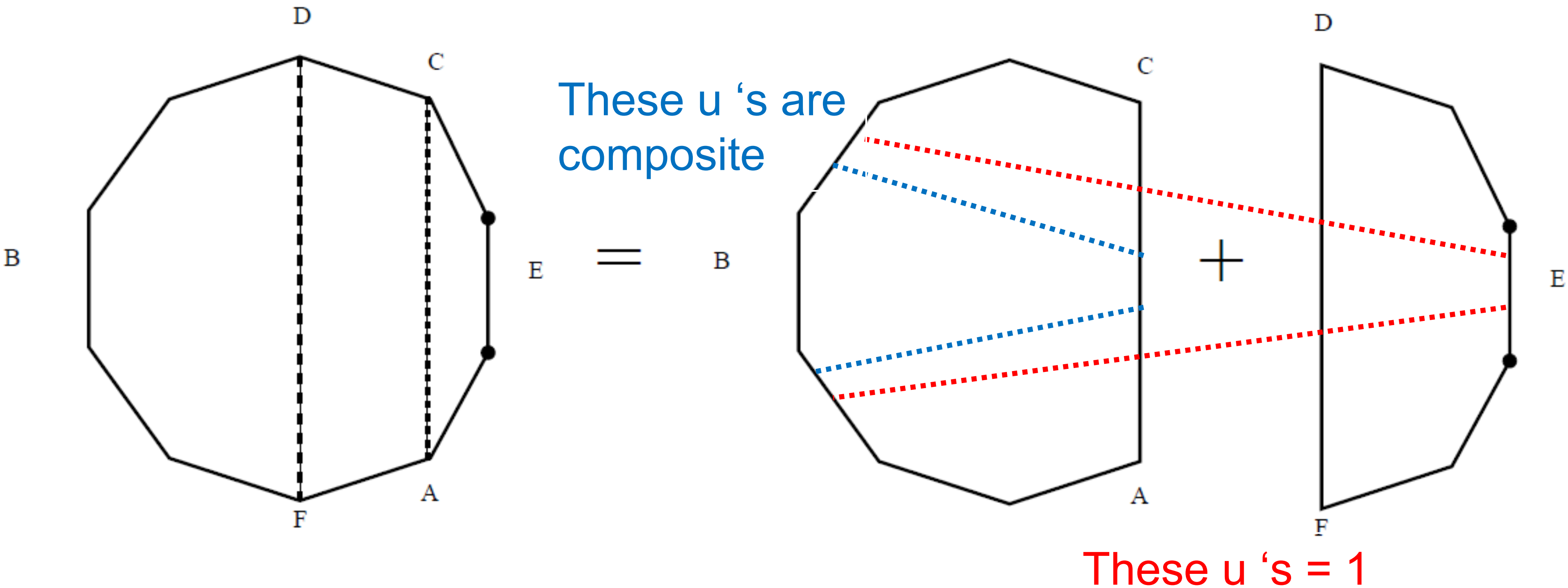


This u = 0

Multi-collinear limits: (m+1) coll. momenta

$$\mathcal{R}_n \to \mathcal{R}_{n-m} + \mathcal{R}_{m+4}$$

Conformal cross-ratios $u_{ij} = \frac{x^2_{i,j+1} x^2_{i+1,j}}{x^2_{i,j} x^2_{i+1,j+1}}$ 'connect' edge i with edge j.



u 's are useful at low n but limits become complicated at high n. X's are better!

Multi-collinear limits: (m+1) coll. momenta

$$\mathcal{R}_n \to \mathcal{R}_{n-m} + \mathcal{R}_{m+4}$$

In the case where the edges $2k+1$ edges $n-2k,\dots,n$ become collinear we have

$$u_{1,n-2k-1} \to 0, \quad u_{i,n-2k+1} \to 1, \quad \dots \quad u_{i,n-1} \to 1 \qquad (2 \le i \le n-2k-2)$$

and the remainder function should reduce as

$$\mathcal{R}_n(u_{ij}) \to \mathcal{R}_{n-2k}(\hat{u}_{ij}) + \mathcal{R}_{2k+4}(u'_{ij})$$

where the cross-ratios of the reduced remainders are related to the n-point cross-ratios as

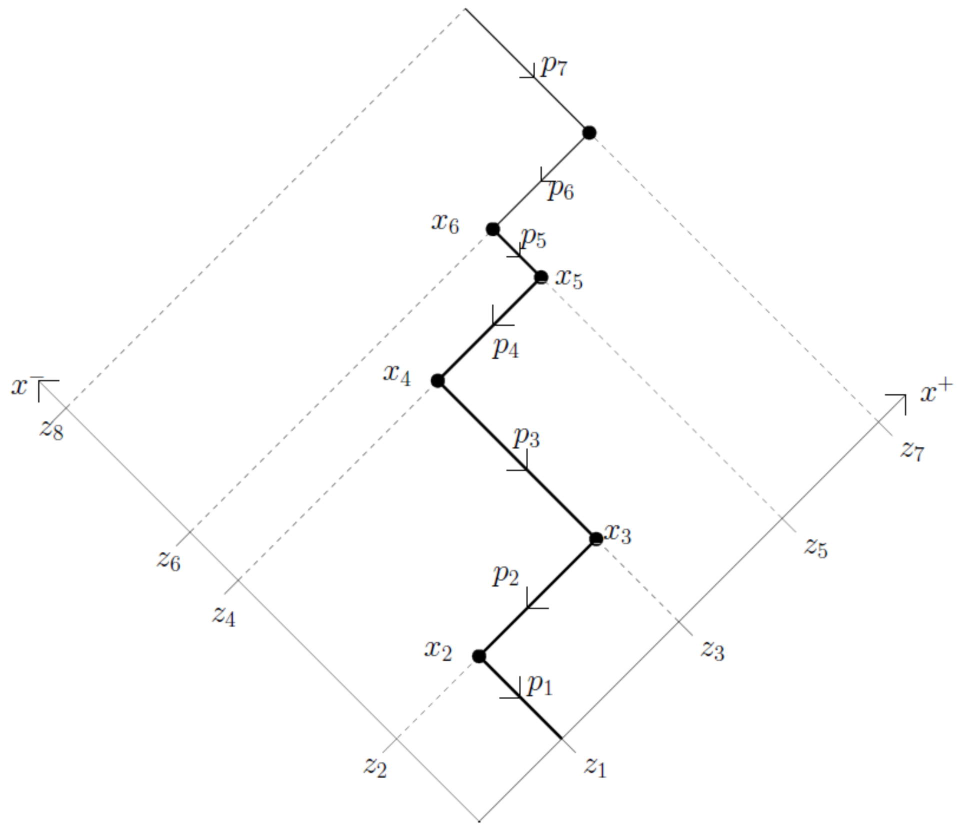
$$\begin{aligned} \hat{u}_{i,n-2k} &= u_{i,n-2k}\dots u_{i,n}, \qquad \hat{u}_{ij} = u_{ij} \qquad 1 \le i,j < n-2k \\ u'_{i,2} &= u_{i,2}\dots u_{i,n-2k-2}, \quad u'_{ij} = u_{ij} \qquad 0 \ge i,j \ge -2k \end{aligned}$$

u ‘s are useful at low n but limits become complicated at high n. X’s are better!

(1+1)-dimensional external kinematics

Wilson loop has a zig-zag shape.

Region momenta x_i (vertices of the contour) have the following form in light-cone coordinates (x_+, x_-)



$$x_i = \begin{cases} (z_{i-1}, z_i)\ , & i \text{ even} \\ (z_i, z_{i-1})\ , & i \text{ odd} \end{cases}$$

$$\begin{array}{ll} x_2 = (z_1, z_2)\ , & x_1 = (z_1, z_n) \\ x_4 = (z_3, z_4)\ , & x_3 = (z_3, z_4) \\ x_4 = (z_3, z_4)\ , & x_5 = (z_5, z_4) \\ \ldots & \ldots \end{array}$$

Collinear limits in (1+1)-kinematics

In 2d there are no non-trivial cross-ratios at 6-points, $\mathcal{R}_6$ is a (coupling dependent) constant.

Lowest non-trivial case is $\mathcal{R}_{8,0}$).

Define

const

$$\tilde{\mathcal{R}}_n := \mathcal{R}_n - \frac{n-4}{2}\mathcal{R}_6$$

so that

$$\tilde{\mathcal{R}}_n \to \tilde{\mathcal{R}}_{n-2}$$

and

$$\tilde{\mathcal{R}}_n^{(\ell)} \to \tilde{\mathcal{R}}_{n-m}^{(\ell)} + \tilde{\mathcal{R}}_{m+4}^{(\ell)}, \quad \text{for } m \geq 4$$

$\tilde{\mathcal{R}}_n$ is the natural object to use for collinear uplifts of amplitudes to higher number of points.

Recall: Momentum supertwistors

It is useful to package the external data $\{p_i^\mu, \eta_i^A\}$ in terms of the region momenta $x_i^{\alpha\dot{\alpha}}$ and their fermionic components $\theta_i^{\alpha A}$:

$$\begin{aligned} p_i^{\alpha\dot{\alpha}} \equiv \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}} &= x_i^{\alpha\dot{\alpha}} - x_{i+1}^{\alpha\dot{\alpha}}\,, \qquad && \alpha, \dot{\alpha} = 1, 2 \\ \lambda_i^\alpha \eta_i^A &= \theta_i^{\alpha A} - \theta_{i+1}^{\alpha A}\,, && A = 1, \ldots, 4 \end{aligned}$$

where λ_i^α and $\tilde{\lambda}_i^{\dot{\alpha}}$ are the 2-component helicity spinors.

The chiral superspace coordinates $X_i = (x_i, \theta_i^A)$ define the vertices of the n-sided null polygon contour for the dual Wilson loop.

Momentum supertwistors transform linearly under $SU(2,2|4)$ dual superconformal transformations. They are defined via

$$\mathcal{Z}_i = (Z_i^a\,;\, \chi_i^A) = (\lambda_i^\alpha, x_{\dot{\alpha}\alpha\, i}\lambda_i^\alpha\,;\, \theta_{\alpha\, i}^A \lambda_i^\alpha)$$

where Z^a denote 4 bosonic, and χ^A are 4 fermionic components.

Drummond-Henn-Korchemsky-Sokatchev;
Hodges; Mason-Skinner

(1+1)-dimensional external kinematics

Momentum twistors in 2d have a checkered pattern:

$$Z_i = \begin{cases} (Z_i^1, 0, Z_i^3, 0) = (1, 0, z_i, 0) & i \text{ odd} \\ (0, Z_i^2, 0, Z_i^4) = (0, 1, 0, z_i) & i \text{ even} \end{cases} ,$$

which is a manifestation of $SU(2,2) \to SL(2)_+ \times SL(2)_-$ in 2d.

Lorentz-invariant intervals z_{ij} coincide with $SL(2)_\pm$-invariant two-bracket of twistors

$$z_{ij} = \langle\, ij \,\rangle := \begin{cases} Z_i^3 Z_j^1 - Z_i^1 Z_j^3 & i \text{ and } j \text{ odd} \\ Z_i^4 Z_j^2 - Z_i^4 Z_j^2 & i \text{ and } j \text{ even} \\ 0 & \text{otherwise} . \end{cases}$$

Also $\langle\, ijkl \,\rangle := \epsilon_{abcd} Z_i^a Z_j^b Z_k^c Z_l^d$ reduces in 2d $\langle\, 1234 \,\rangle = \langle\, 13 \,\rangle \langle\, 24 \,\rangle$.

Lightcone coordinates are interchangeable with twistors in 2d and only two-brackets of bosonic twistors (of the same parity) can appear.

(1+1)-dimensional external kinematics

For superamplitudes in 2d, it is natural to consider a supersymmetric reduction, $SU(2,2|4) \to SL(2|2)_+ \times SL(2|2)_-$, under which momentum *super*twistors

$$\mathcal{Z}_i = (Z_i^a\,;\,\chi_i^A) = \begin{cases} (Z_i^1, 0, Z_i^3, 0; \chi_i^1, 0, \chi_i^3, 0) & i \text{ odd} \\ (0, Z_i^2, 0, Z_i^4; 0, \chi_i^2, 0, \chi_i^4) & i \text{ even} \end{cases},$$

See also Caron-Huot & He

Note that the MHV-prefactor $\delta^{(8)}\left(\sum_{i=1}^{n} \lambda_i \eta_i\right)$ under this SU(4) splitting necessarily goes to zero.

But after dividing by this prefactor we can still compute meaningful quantities.

Alternatively, we can avoid using supersymmetric reduction.

Most general cross-ratios in special 2d kinematics are

$$u_{ij;kl} = \frac{\langle\, il\,\rangle\langle\, jk\,\rangle}{\langle\, ik\,\rangle\langle\, jl\,\rangle}\,, \qquad u_{ij;kl} = 1 - u_{il;kj}$$

Can be reduced to *fundamental* cross-ratios $u_{ij;kl} = \prod_{I=i+1}^{j-1} \prod_{K=k+1}^{l-1} u_{IK}$

$$u_{ij} = \frac{x_{i,j+1}^2 x_{i+1,j}^2}{x_{i,j}^2 x_{i+1,j+1}^2} = \frac{\langle\, i-1, j+1\,\rangle\langle\, i+1, j-1\,\rangle}{\langle\, i-1, j-1\,\rangle\langle\, i+1, j+1\,\rangle} = u_{i-1,i+1;j-1,j+1}$$

For $n = 8$ and $n = 10$, all non-trivial u_{ij} are of the form $u_i := u_{i,i+4}$, with $i = 1, \ldots, 4$ for the octagon and $i = 1, \ldots, 10$ for the decagon with the additional constraint:

$$\begin{array}{rclr} n = 8 & : & 1 - u_i = u_{i+2}\,, & i = 1, 2 \\ n = 10 & : & 1 - u_i = u_{i+2}\, u_{i-2}\,, & i = 1, \ldots, 10 \end{array}$$

At $n = 8$ points there are just four fundamental cross-ratios

$$u_1 = u_{1,5} \quad , \quad u_2 = u_{2,6} \quad , \quad u_3 = 1 - u_1 := v_1 \quad , \quad u_4 = 1 - u_2 := v_2$$

MHV amplitudes in special kinematics

The conjecture at the centre of the method (Heslop - VVK)

(the logarithms of) cross-ratios form the basis of the vector space on which the symbol of the amplitude is defined

The symbol

Goncharov, Goncharov-Spradlin-Vergu-Volovich, …

The *symbol* associates to any generalised polylogarithm, a tensor whose entries are rational functions of the arguments. The rank of the tensor is equal to the weight of the polylogarithm.

$$\text{Symb}\Big(\log x\Big) = x \ , \qquad \text{Symb}\Big(\text{Li}_n(x)\Big) = -(1-x) \otimes \overbrace{x \otimes \ldots \otimes x}^{n-1}$$

The symbol has the properties inherited from the logarithm

$$\begin{aligned} \ldots \otimes x\, y \otimes \ldots &= \ldots \otimes x \otimes \ldots + \ldots \otimes y \otimes \ldots \\ \ldots \otimes 1/x \otimes \ldots &= - \ldots \otimes x \otimes \ldots \end{aligned}$$

For the product of functions the symbol is given by taking the shuffle product of the symbol of each function

$$\text{Symb}(fg) = \text{Symb}(f) \sqcup\!\sqcup\, \text{Symb}(g) \ .$$

2-loop MHV amplitudes @ 8 points

- The corresponding weak coupling result was obtained by
  Del Duca - Duhr - Smirnov
- It has the amazingly simple form (simplified from longer expression than 6-points):

$$R_8^{\mathrm{DDS}} = -\frac{1}{2}\log(u_{15})\log(u_{26})\log(u_{37})\log(u_{48}) - \frac{\pi^4}{18} .$$

Heslop – VVK:

Assume the entries of the symbol are all u's

Consistent with all known results so far. Complicated momentum twistor expressions all reduce to u's in 2d

With this single simple assumption we can:

- Derive the 8-point 2-loop result $a(\log u_1 \log u_2 \log u_3 \log u_4) + b$

8-point 2-loops derivation

Collinear limits $u_1 \to 1$, $u_3 \to 0$.

Cyclicity and parity

$$u_1 \to u_2 \to u_3 \to u_4 \to u_1$$
$$u_1 \leftrightarrow u_4, \;\; u_2 \leftrightarrow u_3$$

Three independent symbols (ℓ loops $\Rightarrow$ weight 2ℓ polylogs)

$$\mathcal{R}_8^{(2)} = a\mathcal{R}_{8;a}^{(2)} + b\mathcal{R}_{8;b}^{(2)} + c\mathcal{R}_{8;c}^{(2)} + 2\mathcal{R}_6^{(2)}$$

$$\mathcal{S}\left(\mathcal{R}_{8;a}^{(2)}\right) = u_1 \otimes u_2 \otimes u_3 \otimes u_4 \; + \; \text{7 terms related by cyclic symmetry}$$
$$\mathcal{S}\left(\mathcal{R}_{8;b}^{(2)}\right) = u_1 \otimes u_2 \otimes u_4 \otimes u_3 \; + \; \text{7 terms related by cyclic symmetry}$$
$$\mathcal{S}\left(\mathcal{R}_{8;c}^{(2)}\right) = u_1 \otimes u_3 \otimes u_2 \otimes u_4 \; + \; \text{7 terms related by cyclic symmetry}\,.$$

(Collinear vanishing function needs all four u's.)

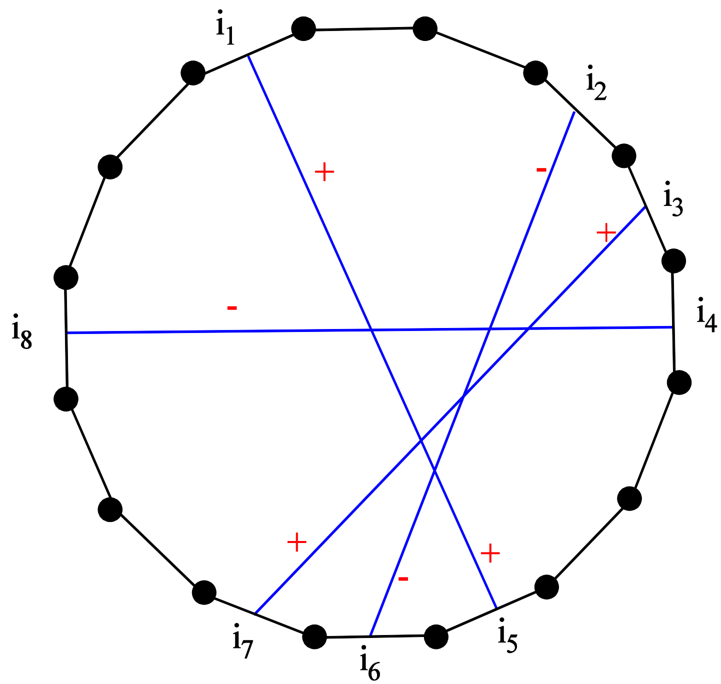
- Requiring this to be the symbol of a function (integrability constraint) puts $a = b = c$ giving the two loop result.

2-loop MHV amplitudes @ n points

$$R_n = -\frac{1}{2}\Big(\sum_{\mathcal{S}} \log(u_{i_1 i_5}) \log(u_{i_2 i_6}) \log(u_{i_3 i_7}) \log(u_{i_4 i_8})\Big) - \frac{\pi^4}{72}(n-4) \,,$$

$$\mathcal{S} = \Big\{ i_1, \ldots i_8 : 1 \leq i_1 < i_2 < \cdots < i_8 \leq n, \qquad i_k - i_{k-1} = \mathrm{odd} \Big\}$$

- This solution is unique (with our symbol assumption)
- This solution was checked numerically using our ABHKTS numerics



3-loop MHV amplitudes @ 8 points

Ansatz for symbol at 3-loops

$$\sum_{i_1 \dots i_6} \mathrm{const}_{i_1 \dots i_6} \cdot u_{i_1} \otimes u_{i_2} \otimes u_{i_3} \otimes u_{i_4} \otimes u_{i_5} \otimes u_{i_6} \,.$$

- Symbol must vanish in the collinear limit
- Symbol should respect cyclic and parity symmetry
- Leads to 195 free constants!
- Next impose that the symbol must be the symbol of a function (integrability constraint)
- remarkably this fixes 182 of these, leaving only 13
- Fix 6 more using OPE [Alday Gaiotto Maldacena Sever Vieira]

3-loop MHV amplitudes @ 8 points

Cross-ratios at 8 points are

$$u_1 := u_{15},\ u_2 := u_{26},\ u_3 := 1 - u_1,\ u_4 := 1 - u_2$$

Require that the 8-point amplitude is cyclically (and parity) symmetric, and that it vanishes in the collinear limit $z_8 \to z_6$, i.e. $u_1 \to 0\,, u_3 \to 1$ with u_2, u_4 unconstrained.

3-loop remainder function:

$$\boxed{\tilde{\mathcal{R}}_8 = \sum_{\sigma,\tau} a_{\sigma\tau} f_\sigma^+(u_1) f_\tau^+(u_2)\ , \qquad a_{\sigma\tau} = a_{\tau\sigma}}$$

where $f_\sigma^+(u)$ is a generalised polylogarithm and

$$f_\sigma^+(u) = +f_\sigma^+(1-u)\ , \qquad f_\sigma^+(0) = 0$$

3-loop MHV amplitudes @ 8 points

$$\tilde{\mathcal{R}}_8^{(3)} = \sum_{\sigma,\tau} a_{\sigma\tau} f_\sigma(u_1) f_\tau(u_2) \ , \qquad a_{\sigma\tau} = a_{\tau\sigma}$$

weight 2 : $f^{+}_{\text{weight}\,2}(u) = \log(u)\log(v)$ $\quad f^{-}_{\text{weight}\,2}(u) = \text{Li}_2(u) - \text{Li}_2(v)$

weight 4 a :

$$\begin{aligned}
\text{Symb}[f^{\pm}_{a1}] &:= u \otimes u \otimes u \otimes v \pm v \otimes v \otimes v \otimes u \\
\text{Symb}[f^{\pm}_{a2}] &:= u \otimes u \otimes v \otimes u \pm v \otimes v \otimes u \otimes v \\
\text{Symb}[f^{\pm}_{a3}] &:= u \otimes v \otimes u \otimes u \pm v \otimes u \otimes v \otimes v \\
\text{Symb}[f^{\pm}_{a4}] &:= v \otimes u \otimes u \otimes u \pm u \otimes v \otimes v \otimes v
\end{aligned}$$

weight 4 b :

$$\begin{aligned}
\text{Symb}[f^{\pm}_{b1}] &:= u \otimes u \otimes v \otimes v \pm v \otimes v \otimes u \otimes u \\
\text{Symb}[f^{\pm}_{b2}] &:= u \otimes v \otimes u \otimes v \pm v \otimes u \otimes v \otimes u \\
\text{Symb}[f^{\pm}_{b3}] &:= u \otimes v \otimes v \otimes u \pm v \otimes u \otimes u \otimes v
\end{aligned}$$

weight 3 c :

$$\begin{aligned}
\text{Symb}[f^{\pm}_{c1}] &:= u \otimes u \otimes v \pm v \otimes v \otimes u \\
\text{Symb}[f^{\pm}_{c2}] &:= u \otimes v \otimes u \pm v \otimes u \otimes v \\
\text{Symb}[f^{\pm}_{c3}] &:= u \otimes v \otimes v \pm v \otimes u \otimes u
\end{aligned}$$

3-loop MHV amplitudes @ 8 points

Near-collinear OPE limit (Gaiotto-Maldacena-Sever-Vieira) gives

$$\lim_{u_1 \to 0} \mathcal{R}_8^{(3)}(u_1, u_2, u_3, u_4) = \log^2(u_1)\log(u_3) \cdot F_3(u_2, u_4) + O(\log(u_1))$$

we find an equivalent form for this so that its arguments are just the cross-ratios u_2 and u_4

$$\begin{aligned} F_3(u_2, u_4) = \quad & 2\text{Li}_3(u_2) + \left(\text{Li}_2(u_4) - \tfrac{\pi^2}{6}\right)\log(u_2) + \tfrac{3}{2}\log(u_4)\log^2(u_2) \\ + \quad & 2\text{Li}_3(u_4) + \left(\text{Li}_2(u_2) - \tfrac{\pi^2}{6}\right)\log(u_4) + \tfrac{3}{2}\log(u_2)\log^2(u_4) - 2\zeta_3 \end{aligned}$$

Now we notice that $F_3(u_2, u_4) = f_{c1}^{+}(u_2, u_4) + 2f_{c2}^{+}(u_2, u_4) + f_{c3}^{+}(u_2, u_4)$ which is consistent with our symbol construction!

3-loop MHV amplitudes @ 8 points

The result is:

$$\begin{aligned}\tilde{\mathcal{R}}_8^{(3)} = \quad & \log u_1 \log(1-u_1)\Big[\alpha_1\, f^+_{a3}(u_2) + \alpha_2\, f^+_{a4}(u_2) + \alpha_3\, f^+_{b2}(u_2) + \alpha_4\, f^+_{b3}(u_2)\Big] \\ & +\alpha_5 f^+_{c2}(u_1) f_{c2}(u_2) + \alpha_6 f^+_{c2}(u_1) f^+_{c3}(u_2) + \alpha_7 f^+_{c3}(u_1) f^+_{c3}(u_2) \\ & +f^+_{c1}(u_1)\Big[\tfrac{1}{2} f^+_{c1}(u_2) + 2 f^+_{c2}(u_2) + f^+_{c3}(u_2)\Big] \\ & +(u_1 \leftrightarrow u_2)\end{aligned}$$

3-loop MHV amplitudes @ 8 points

All the functions appearing at 3-loops at 8-points (eaily reconstructed from their symbols):

$$
\begin{aligned}
f^{+}_{a3}(u,v) &= 3\mathrm{Li}_4(u) - \mathrm{Li}_3(u)\log(u) + 3\mathrm{Li}_4(v) - \mathrm{Li}_3(v)\log(v) - \frac{\pi^4}{30}\,, \\
f^{+}_{a4}(u,v) &= -\mathrm{Li}_4(u) - \mathrm{Li}_4(v) + \frac{\pi^4}{90} \\
f^{+}_{b2}(u,v) &= (\mathrm{Li}_3(u) - \zeta_3)\log(v) - \mathrm{Li}_2(u)\mathrm{Li}_2(v) + \log^2(u)\log^2(v) + (\mathrm{Li}_3(v) - \zeta_3)\log(u) \\
f^{+}_{b3}(u,v) &= -(\mathrm{Li}_3(u) - \zeta_3)\log(v) + \mathrm{Li}_2(u)\mathrm{Li}_2(v) - \frac{1}{2}\log^2(u)\log^2(v) - (\mathrm{Li}_3(v) - \zeta_3)\log(u) \\
f^{+}_{c1}(u,v) &= -\mathrm{Li}_3(u) - \left(\mathrm{Li}_2(v) - \frac{\pi^2}{6}\right)\log(u) - \frac{1}{2}\log(v)\log^2(u) \\
&\quad -\mathrm{Li}_3(v) - \left(\mathrm{Li}_2(u) - \frac{\pi^2}{6}\right)\log(v) - \frac{1}{2}\log(u)\log^2(v) + \zeta_3 \\
f^{+}_{c2}(u,v) &= 2\mathrm{Li}_3(u) + \left(\mathrm{Li}_2(v) - \frac{\pi^2}{6}\right)\log(u) + \log(v)\log^2(u) \\
&\quad +2\mathrm{Li}_3(v) + \left(\mathrm{Li}_2(u) - \frac{\pi^2}{6}\right)\log(v) + \log(u)\log^2(v) - 2\zeta_3 \\
f^{+}_{c3}(u,v) &= -\mathrm{Li}_3(v) - \mathrm{Li}_3(u) + \zeta_3
\end{aligned}
$$

General formula for the n-point uplift

The n-point MHV amplitude for $\ell \geq 1$, at any loop order which is a general solution to all multi-collinear limits is given by

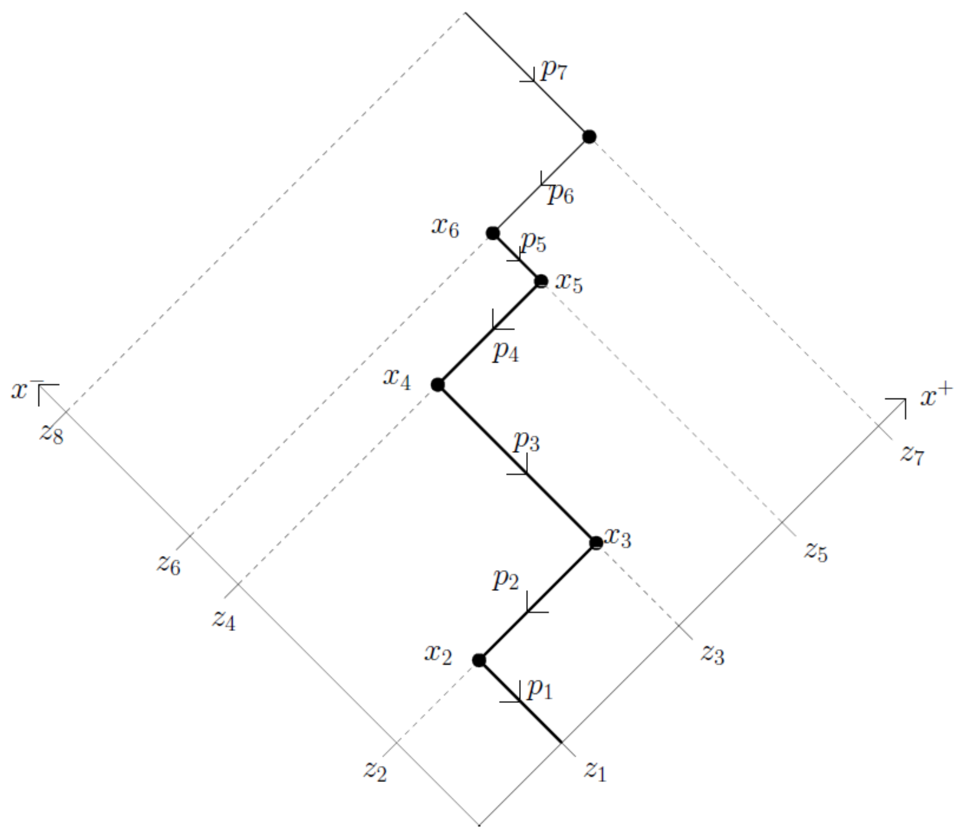
$$
\begin{aligned}
\tilde{\mathcal{R}}_n^{(\ell)}(z_1, z_2, \ldots, z_n) \quad &= \sum\nolimits_{1 \leq i_1 < i_2 < i_3 < i_4 \leq n} S_8^{(\ell)}(x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4})(-1)^{i_1 + \ldots i_4} \; + \\
&+ \sum\nolimits_{1 \leq i_1 < \ldots < i_5 \leq n} S_{10}^{(\ell)}(x_{i_1}, x_{i_2}, \ldots, x_{i_5})(-1)^{i_1 + \ldots i_5} \; + \\
&+ \sum\nolimits_{1 \leq i_1 < \ldots < i_6 \leq n} S_{12}^{(\ell)}(x_{i_1}, x_{i_2}, \ldots, x_{i_6})(-1)^{i_1 + \ldots i_6} \; + \\
&+ \ldots + \\
&+ \sum\nolimits_{1 \leq i_1 < \ldots < i_{2\ell} \leq n} S_{4\ell}^{(\ell)}(x_{i_1}, x_{i_2}, \ldots, x_{i_{2\ell}})(-1)^{i_1 + \ldots i_{2\ell}}
\end{aligned}
$$

more precisely, $i_1 + 1 < i_2\,,\; i_2 + 1 < i_3\,,\; i_3 + 1 < i_4\,, \ldots$

Recall: (1+1)-external kinematics

We can specify the zig-zag shape of the 2d contour by specifying every second vertex in x variables.

E.g. at 8-points $S_8(x_2, x_4, x_6, x_8)$ or $S_8(x_1, x_3, x_5, x_7)$



$$x_i = \begin{cases} (z_{i-1}, z_i)\ , & i \text{ even} \\ (z_i, z_{i-1})\ , & i \text{ odd} \end{cases}$$

$$\begin{array}{ll} x_2 = (z_1, z_2)\ , & x_1 = (z_1, z_n) \\ x_4 = (z_3, z_4)\ , & x_3 = (z_3, z_4) \\ x_4 = (z_3, z_4)\ , & x_5 = (z_5, z_4) \\ \dots & \dots \end{array}$$

General formula for the n-point uplift

Minimal case: 8-point amplitude

$$\boxed{\tilde{\mathcal{R}}_8(z_1, z_2, \ldots, z_8) = S_8(x_2, x_4, x_6, x_8) + S_8(x_1, x_3, x_5, x_7)}$$

8-pt amplitude additional contr.

Divide S_8 into two parts, so that,

$$\boxed{\begin{aligned} S_8(x_2, x_4, x_6, x_8) &= \frac{1}{2}\mathcal{R}_8(z_1, z_2, \ldots, z_8) + T_8(x_2, x_4, x_6, x_8) \\ S_8(x_1, x_3, x_5, x_7) &= \frac{1}{2}\mathcal{R}_8(z_1, z_2, \ldots, z_8) + T_8(x_1, x_3, x_5, x_7) \end{aligned}}$$

T_8 is not determined by the amplitude $\mathcal{R}_8$. It follows that

$$T_8(x_2, x_4, x_6, x_8) + T_8(x_1, x_3, x_5, x_7) = 0$$

This condition is guaranteed by the flip symmetry of T_8 together with the *anti*-symmetry under $z_i \to z_{i+1}$,

$$T_8(x_1, x_3, x_5, x_7) = T_8(x_1^f, x_3^f, x_5^f, x_7^f) = -T_8(x_2, x_4, x_6, x_8)\,.$$

where flipped variables are $x^f = (x_-, x_+)$ for every $x = (x_+, x_-)$

General formula for the n-point uplift

$$\boxed{\tilde{\mathcal{R}}_8(z_1, z_2, \ldots, z_8) = S_8(x_2, x_4, x_6, x_8) + S_8(x_1, x_3, x_5, x_7)}$$

Divide S_8 into two parts, so that,

$$\boxed{\begin{aligned} S_8(x_2, x_4, x_6, x_8) &= \frac{1}{2}\mathcal{R}_8(z_1, z_2, \ldots, z_8) + T_8(x_2, x_4, x_6, x_8) \\ S_8(x_1, x_3, x_5, x_7) &= \frac{1}{2}\mathcal{R}_8(z_1, z_2, \ldots, z_8) + T_8(x_1, x_3, x_5, x_7) \end{aligned}}$$

these are obtained from f $^+$ functions
as explained before

these are obtained from f $^-$ functions
-- same symbolic construction

General formula for the n-point uplift

Next is the 10-point amplitude:

$$\begin{aligned}\tilde{\mathcal{R}}_{10}(z_1, z_2, \ldots, z_n) \quad &= \textstyle\sum_{1\le i_1<i_2<i_3<i_4\le 10} S_8(x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4})(-1)^{i_1+\ldots i_4} \\ &+S_{10}(x_2, x_4, x_6, x_8, x_{10}) \; - \; S_{10}(x_1, x_3, x_5, x_7, x_9)\end{aligned}$$

For $S_{10}(x_2, x_4, x_6, x_8, x_{10}) - S_{10}(x_1, x_3, x_5, x_7, x_9)$ to be cyclically symmetric in z-variables, S_{10} has to be *anti*-symmetric under the flip symmetry.

Together with T_8's these contributions from S_{10}'s give precisely the collinear-vanishing part of the 10-point amplitude

Properties of S_m

- S_m are conformally invariant functions of m z-variables or equivalently $m/2$ x-variables $S_m(z_1, \ldots z_m) = S_m(x_2, x_4, \ldots, x_m)$

- They are symmetric under cyclic symmetry and parity up to a minus sign in x-variables (but not necessarily in z),

$$S_m(x_2, x_4, \ldots x_m) = S_m(x_4, x_6, \ldots, x_2) = (-1)^{m/2} S_m(x_m, x_{m-2}, \ldots, x_2)$$

- S_m required to satisfy flip (anti)-symmetry

$$S_m(x_{i_1}, x_{i_2}, \ldots x_{i_{m/2}}) = (-1)^{m/2} S_m(x^f_{i_1}, x^f_{i_2}, \ldots x^f_{i_{m/2}}) \ .$$

- S_m must vanish in the collinear limit $z_m \to z_{m-2}$ ie $x_m \to x_{m-1}$

$$\lim_{x_m \to x_{m-1}} S_m(x_2, \ldots, x_{m-2}, x_m) = 0 \qquad \text{(collinear limits)}$$

More geometrical way of saying this

$$S_m(x_i, \ldots, x_j, x_k) = 0 \qquad \text{if} \qquad x_j, x_k \text{ become lightlike separated} \ .$$

Again: our general MHV formula

The n-point MHV amplitude for $\ell \geq 1$, at any loop order which is a general solution to all multi-collinear limits is given by

$$
\begin{aligned}
\tilde{\mathcal{R}}_n^{(\ell)}(z_1, z_2, \ldots, z_n) \quad &= \sum\nolimits_{1\leq i_1<i_2<i_3<i_4\leq n} S_8^{(\ell)}(x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4})(-1)^{i_1+\ldots i_4} \; + \\
&+ \sum\nolimits_{1\leq i_1<\ldots<i_5\leq n} S_{10}^{(\ell)}(x_{i_1}, x_{i_2}, \ldots, x_{i_5})(-1)^{i_1+\ldots i_5} \; + \\
&+ \sum\nolimits_{1\leq i_1<\ldots<i_6\leq n} S_{12}^{(\ell)}(x_{i_1}, x_{i_2}, \ldots, x_{i_6})(-1)^{i_1+\ldots i_6} \; + \\
&+ \ldots + \\
&+ \sum\nolimits_{1\leq i_1<\ldots<i_{2\ell}\leq n} S_{4\ell}^{(\ell)}(x_{i_1}, x_{i_2}, \ldots, x_{i_{2\ell}})(-1)^{i_1+\ldots i_{2\ell}}
\end{aligned}
$$

more precisely, $i_1 + 1 < i_2\,,\; i_2 + 1 < i_3\,,\; i_3 + 1 < i_4\,,\ldots$

Collinear limit

Consider the simplest collinear limit, $z_n \to z_{n-2}$ ie $x_n \to x_{n-1}$. Using

$$\lim_{x_n \to x_{n-1}} S_m(i, j \ldots k) = S_m(i, j, \ldots k) \quad \text{for} \quad i, j, \ldots k \neq n-1, n$$

$$\lim_{x_n \to x_{n-1}} [S_m(i, j \ldots k, n-1) - S_m(i, j, \ldots k, n)] = 0$$

one can see that

$$\lim_{x_n \to x_{n-1}} \tilde{\mathcal{R}}_n(z_1, \ldots z_n) = \tilde{\mathcal{R}}_{n-2}(z_1, \ldots, z_{n-2})$$

as required under collinear limits.

General formula for the n-point uplift

- Multi-collinear limits also work out correctly:

  use the structure of the sum and conformal properties.

- There is only a finite number of functions S_m at each fixed loop order , $m < 4$ (number of loops) +1

  <= S_m has to vanish in all multi-collinear limits, and combined with weight = 2 (number of loops) and our symbols-of-u's-only assumption, this is possible only for $m < 4$ (number of loops) +1

Summary + more

- Multi-collinear limits for super-amplitudes R

- (1+1)-dimensional external kinematics

- Central assumption for the symbol of the amplitude being made out of fundamental u's in this kinematics

- Reproduced 2-loop 8-point MHV
- General n-point uplift of 2-loop MHV – logs only – very compact result!

- Obtained 3-loop 8-point MHV in terms of 7 explicitly reconstructed functions

- General S-formula for MHV n-point amplitudes (any n, any loop-order) in terms of S_m functions which are constructible with the e.g. symbol-of-u's assumption.

+ more

- Same general S-formula holds for loop-level non-MHV amplitudes
  S(x) -> S(X=x,θ)

- Tree-level NMHV amplitudes derived in this kinematics separately
  (very compact expressions)

- NMHV amplitudes can be obtained from the (known) MHV expressions with a certain manipulation of the MHV symbol
  -- to appear soon

  Consistency and connections with the Q-bar formula of Caron-Huot & He.