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# Quantum Field Theory WS 05/06

## Exercise 10

1. **Gauge fixing**

   The classical Lagrange density $\mathcal{L}$ of the free electromagnetic field can be modified according to

   $$\mathcal{L}' = \mathcal{L} - \frac{\alpha}{2}(\partial_\kappa A^\kappa)^2 \,. \tag{1}$$

   Here the arbitrary parameter $\alpha$ serves as a Lagrange multiplier ensuring the Lorentz gauge condition $\partial_\kappa A^\kappa = 0\,$.

   (a) From Eq. (1) derive the canonical momentum $\pi^\mu$ and the equation of motion for $A^\mu$.

   (b) For $\alpha = 1$ convince yourself that the Lagrange density can be written as

   $$\mathcal{L}'' = -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu \,, \tag{2}$$

   where we safely neglected a divergence term. From Eq. (2) determine the Hamilton density of the electromagnetic field.

2. **Quantization in the Lorentz gauge**

   Following the lecture notes canonically quantize the electromagnetic field in Lorentz gauge. Start from the gauge fixed Lagrangian of Eq. (1) for the choice $\alpha = 1$.

   (a) Given the equal-time commutators for the field operator and its conjugate momentum

   $$\begin{aligned}\left[\hat{A}^\mu(t,\vec{x}), \hat{\pi}^\nu(t,\vec{y})\right] &= ig^{\mu\nu}\delta^3(\vec{x}-\vec{y})\,,\\ \left[\hat{A}^\mu(t,\vec{x}), \hat{A}^\nu(t,\vec{y})\right] &= [\hat{\pi}^\mu(t,\vec{x}), \hat{\pi}^\nu(t,\vec{y})] = 0\,,\end{aligned}$$

   convince yourself, that the Lorentz gauge condition, being imposed as an operator identity, is in contradiction with the field commutators.

   (b) Use the Fourier expansion of the field operator to deduce the commutation relations of the creation and annihilation operators and express the Hamilton operator in terms of $\hat{a}(\vec{k},\lambda)$ and $\hat{a}^\dagger(\vec{k},\lambda)$.

   (c) The spectrum of the Hamiltonian is not strictly positive due to the presence of scalar photons ($\lambda = 0$). Resolve this issue following the arguments of Gupta and Bleuler. Therefore impose the constraint $\partial^\mu \hat{A}^{(+)}_\mu|\psi\rangle \stackrel{!}{=} 0$ for the positive energy part of $\hat{A}^\mu$ and all physical states $|\psi\rangle$.

   (d) Derive the photon propagator $iD_F^{\mu\nu}(x-y) = \langle 0 \left| T\left[\hat{A}^\mu(x)\hat{A}^\nu(y)\right]\right| 0\rangle$.

(turn over)

3. **Feynman propagator: The simple way**

The Feynman propagator in momentum-space can be obtained from a given Lagrange density by inverting the Fourier transformed of the differential operator. According to this rule determine the Feynman propagator for:

(a) A neutral massive vector field whose Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu \,.$$

Is this ansatz gauge invariant? What is the associated equation of motion? What can be concluded for $\partial_\mu A^\mu$ when studying the divergence of the EoM?

(b) The free photon field, with the Lagrangian given by Eq. (1), for $\alpha \neq 0$.

**Hint:** The recipe is, bring the Lagrange density to the form $\mathcal{L} = \frac{1}{2} A^\mu D_{\mu\nu}(x) A^\nu$, with $D_{\mu\nu}(x)$ the differential operator in position-space. The propagator $D^{-1\nu\sigma}$ is then defined according to

$$D_{\mu\nu}(k) D^{-1\nu\sigma}(k) = g_\mu{}^\sigma \,,$$

and can be obtained by comparing coefficients for the general tensor structure

$$D^{-1\nu\sigma} = a(k^2) g^{\nu\sigma} + b(k^2) k^\nu k^\sigma \,.$$