Exercise 4

1. Consider the quantized Klein-Gordon field $\hat{\phi}(x)$.

   (a) Show that the Pauli-Jordan function, defined via $i\Delta(x) = \left[\hat{\phi}(x), \hat{\phi}(0)\right]$, satisfies

   $$\frac{\partial}{\partial t}\Delta(x)|_{x_0=0} = -\delta^3(\vec{x})\,.$$

   (b) Define positive and negative frequent Δ-functions according to

   $$i\Delta^{\pm}(x-y) = \pm\int\frac{\mathrm{d}^4k}{(2\pi)^3}\,\theta(\pm k_0)\delta(k^2-m^2)\,e^{-i\,k(x-y)}\,.$$

   How can the Pauli-Jordan function and the Feynman propagator Δ_F be constructed from Δ^+ and Δ^-?[1]

   (c) Show that Δ_F is a Greens function to the Klein-Gordon equation. Is this also true for Δ^+ and Δ^-? How about the Pauli-Jordan function?

2. (a) Classically, the scattering angle for the scattering of a test particle off a spherically symmetric potential is a function of the projectile energy and the impact parameter only:

   $$\vartheta(E_0, b) = \pi - 2b\int\limits_{r_{\min}}^{\infty}\frac{\mathrm{d}r}{r^2\left[1 - V(r)/E_0 - b^2/r^2\right]^{1/2}}\,. \qquad (1)$$

   Starting from Eq. (1) calculate $\vartheta(E_0, b)$ for

   - the scattering of two hard balls with radii R_1 and R_2,
   - the case of Coulomb scattering.

   Use the relation

   $$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{b\,\mathrm{d}b}{\mathrm{d}\cos\vartheta}$$

   to derive the classical cross section for both processes.

   (b) In terms of nonrelativistic quantum mechanics the differential cross section reads

   $$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = |f(\vartheta)|^2 \quad \text{where} \quad f(\vartheta) = \frac{1}{k}\sum_l\frac{2l+1}{2\mathrm{i}}\left(e^{2\mathrm{i}\delta_l} - 1\right)P_l(\cos\vartheta)\,. \qquad (2)$$

   Use Eq. (2) to prove the optical theorem [2]

   $$\sigma = \frac{4\pi}{k}\,\mathrm{Im}\{f(0)\}\,.$$

(turn over)

[1]Hint: *Use the Fourier representation of the Heaviside function*

$$\theta(x) = \lim_{\varepsilon\to 0}\int\limits_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi i}\,\frac{e^{i\,\omega x}}{\omega - i\varepsilon}$$

[2]Hint: $\int_{-1}^{+1}\mathrm{d}z\,P_l(z)P_{l'}(z) = \frac{2}{2l+1}\,\delta_{ll'} \qquad P_l(\pm 1) = (\pm 1)^l$

3. Let $\hat{\phi}$ be an arbitrary quantum field. Show explicitly that the time ordered product of three field operators is given by

$$\begin{aligned} T[\,\hat{\phi}(x_A)\,\hat{\phi}(x_B)\,\hat{\phi}(x_C)\,] = {} & :\hat{\phi}(x_A)\,\hat{\phi}(x_B)\,\hat{\phi}(x_C): \\ & + \langle 0|T[\,\hat{\phi}(x_A), \hat{\phi}(x_B)\,]|0\rangle\,\hat{\phi}(x_C) \\ & + \langle 0|T[\,\hat{\phi}(x_B), \hat{\phi}(x_C)\,]|0\rangle\,\hat{\phi}(x_A) \\ & + \epsilon_{AB}\,\langle 0|T[\,\hat{\phi}(x_A), \hat{\phi}(x_C)\,]|0\rangle\,\hat{\phi}(x_B)\,, \end{aligned}$$

where the sign of the permutation ϵ_{AB} is $+1$ for bosons and -1 for fermions.