Exercise 6

1. **One-loop diagrams in $\lambda_0\phi^4$ theory**
   Draw all distinct Feynman diagrams contributing to the four-point function $G^{(4)}(x_1, x_2, x_3, x_4)$ at second order in the coupling constant λ_0. Show that in momentum space all the connected diagrams are proportional to

$$J(q^2) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\varepsilon} \frac{i}{(k-q)^2 - m^2 + i\varepsilon}\,, \tag{1}$$

   with q^2 equal to either $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$ or $u = (p_1 - p_4)^2$.

2. **Feynman parameters**
   Convince yourself that

$$\frac{1}{AB} = \int_0^1 \mathrm{d}x\mathrm{d}y\, \delta(x+y-1) \frac{1}{[xA + yB]^2} = \int_0^1 \mathrm{d}x \frac{1}{[xA + (1-x)B]^2}\,. \tag{2}$$

   Use Eq. (2) to prove that

$$\frac{1}{AB^n} = \int_0^1 \mathrm{d}x\mathrm{d}y\, \delta(x+y-1) \frac{n y^{n-1}}{[xA + yB]^{n+1}} \tag{3}$$

   and

$$\frac{1}{A_1 A_2 \ldots A_n} = \int_0^1 \mathrm{d}x_1 \ldots \mathrm{d}x_n\, \delta\left(\sum_{i=1}^n x_i - 1\right) \frac{(n-1)!}{[x_1 A_1 + x_2 A_2 \ldots x_n A_n]^n}\,.$$

3. **Revealing the infinities**

   (a) Use Eq. (2) to combine the two denominators in Eq. (1) and complete the square of the resulting denominator by shifting to a new loop momentum. Transform the integrand from Minkowski to Euclidean space-time defined by

$$p_{E\mu} = p_E^\mu = (p_{E1}, p_{E2}, p_{E3}, p_{E4}) = (\vec{p}_E, p_{E4})\,,$$

   with $p_{E4} = -ip_0$ being real, $\vec{p}_E = \vec{p}$ and therefore $p_E \cdot p_E = -p_0^2 + \vec{p}^{\,2} \geq 0\ \forall\, p$. To accomplish this the p_0 integration has been analyticly continuated from the real to the complex axis, i.e. $p_0 \to ip_0$, beforehand.

(turn over)

Do the resulting momentum integral by using four-dimensional spherical coordinates [1]. For the $\bar{k}$ integral use a finite cut-off Λ as upper limit. The remaining Feynman parameter integral can be solved using

$$\int_0^1 \mathrm{d}x \ln\left[1 + \frac{4}{a}x(1-x)\right] = -2 + \sqrt{1+a}\ln\left(\frac{\sqrt{1+a}+1}{\sqrt{1+a}-1}\right), \quad \text{where} \quad a > 0\,.$$

(b*) Go back to part (a). After combining the two denominators and shifting to the new loop momentum, replace one of the resulting factors according to

$$\frac{1}{k^2 - \tilde{m}^2 + i\varepsilon} \to \frac{1}{k^2 - \tilde{m}^2 + i\varepsilon} - \frac{1}{k^2 - \Lambda^2 + i\varepsilon} = \frac{\tilde{m}^2 - \Lambda^2}{[k^2 - \tilde{m}^2 + i\varepsilon]\,[k^2 - \Lambda^2 + i\varepsilon]}\,.$$

This procedure is known as Pauli-Villars regularization, and the introduced parameter Λ^2 has to fulfill $\Lambda^2 \gg m^2$. Introduce a second Feynman parameter according to Eq. (3), transform to Euclidean space-time and evaluate the resulting integrals. This time the momentum integral can be performed from zero up to infinity.

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[1] In four-dimensional Euclidean space we can introduce spherical coordinates as

$$k_1 = \bar{k}\sin\vartheta\sin\varphi\sin\chi\,, \quad k_2 = \bar{k}\sin\vartheta\cos\varphi\sin\chi\,, \quad k_3 = \bar{k}\cos\vartheta\sin\chi\,, \quad k_4 = \bar{k}\cos\chi.$$

Accordingly the four-dimensional volume integral reads

$$\int \mathrm{d}^4k = \int_0^\infty \mathrm{d}\bar{k}\bar{k}^3 \int_0^{2\pi} \mathrm{d}\varphi \int_0^\pi \mathrm{d}\vartheta \sin\vartheta \int_0^\pi \mathrm{d}\chi \sin^2\chi\,.$$

When integrating functions, which depend on $\bar{k}^2$ only, this can be replaced by

$$\int \mathrm{d}^4k = \pi^2 \int_0^\infty \mathrm{d}\bar{k}^2\bar{k}^2\,.$$