Quantum Field Theory WS 05/06

Exercise 8

1. **Chirality**
   Consider the Dirac γ^μ-matrices with the properties
   $$(\gamma^0)^\dagger = \gamma^0\,, \quad (\gamma^i)^\dagger = -\gamma^i\,, \quad \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\,.$$
   (a) Prove that the chirality $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ fulfills
   $$(\gamma_5)^\dagger = \gamma_5\,, \quad (\gamma_5)^2 = 1\,, \quad \{\gamma_5, \gamma^\mu\} = 0\,.$$
   (b) For the projectors $P_L = \frac{1}{2}(1-\gamma_5)$ and $P_R = \frac{1}{2}(1+\gamma_5)$ show that
   $$P_L^2 = P_L\,, \quad P_R^2 = P_R\,, \quad P_L P_R = P_R P_L = 0\,.$$
   (c) The left- and right handed parts of a Dirac spinor ψ are defined by $\psi_L = P_L\psi$ and $\psi_R = P_R\psi$. What is the result of applying γ_5 on $\psi_{L/R}$? Show that
   $$\bar\psi_L = \frac{1}{2}\bar\psi(1+\gamma_5)\,, \quad \bar\psi_R = \frac{1}{2}\bar\psi(1-\gamma_5)\,,$$
   and
   $$\bar\psi_L\gamma^\mu\psi_L + \bar\psi_R\gamma^\mu\psi_R = \bar\psi\gamma^\mu\psi\,, \quad \bar\psi_L\psi_R + \bar\psi_R\psi_L = \bar\psi\psi\,.$$
   (d) Starting from the Dirac equation $(i\partial\!\!\!/ - m)\,\psi = 0$, derive two coupled equations for $\psi_{L/R}$. Under which circumstances chirality is a conserved quantity?

2. **Chirality vs. Helicity**
   The free Dirac equation in natural units reads
   $$(i\partial\!\!\!/ - m)\,\psi = 0\,.$$
   (a) Using the ansatz
   $$\psi = \begin{pmatrix} \phi_0 \\ \chi_0 \end{pmatrix} e^{-ip\cdot x}\,,$$
   with ϕ_0, χ_0 being two-component spinors, deduce the relation
   $$\chi_0 = \frac{\vec\sigma\cdot\vec p}{E+m}\phi_0\,.$$
   How does this simplify for helicity eigenstates? Determine the relation of χ_0 and ϕ_0 for eigenstates of chirality and compare the two results.
   (b) For arbitrary E and m, decompose the positive helicity state into pieces with left and right chirality. What is the probability to measure left chirality? Especially consider the cases $E/m \to \infty$ and $E/m \to 1$.

(turn over)

3. **Charge conjugation**

In presence of an electromagnetic field the Dirac equation is modified by the minimal replacement $i\partial_\mu \to iD_\mu = i\partial_\mu - eA_\mu$, using the real electromagnetic four-potential A_μ. The factor $e = \pm|e|$ represents the charge of the particle in question. We define the charge-conjugated spinor ψ_c by

$$\psi_c = \hat{C}\bar{\psi}^T = \hat{C}\gamma^0\psi^* = i\gamma^2\psi^* \,.$$

(a) Determine the equation of motion for ψ_c and interprete the result.

(b) Confirm the basic properties of the charge conjugation operator $\hat{C}$

$$\hat{C} = -\hat{C}^{-1} = -\hat{C}^\dagger = -\hat{C}^T \,, \quad \hat{C}^{-1}\gamma^\mu\hat{C} = -\gamma^{\mu T}$$

and compute the following expressions

$$(\psi_c)_c \,, \quad \bar{\psi}_c\psi_c \,, \quad \bar{\psi}_c\gamma^\mu\psi_c \,.$$

(c) Determine the expectations values

$$\langle\hat{\vec{x}}\,\rangle_c \,, \quad \langle\hat{\vec{p}}\,\rangle_c \,, \quad \langle\hat{\vec{\Sigma}}\,\rangle_c \,, \quad \langle\hat{\vec{L}}\,\rangle_c \,, \quad \langle\hat{\vec{J}}\,\rangle_c \,,$$

with $\hat{\vec{\Sigma}} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$, $\hat{\vec{L}} = \hat{\vec{x}} \times \hat{\vec{p}}$, $\hat{\vec{J}} = \hat{\vec{L}} + \frac{1}{2}\hat{\vec{\Sigma}}$ and

$$\langle\hat{\mathcal{O}}\rangle_c = \langle\psi_c|\hat{\mathcal{O}}|\psi_c\rangle = \int \mathrm{d}^3x\, \psi_c^\dagger(x)\hat{\mathcal{O}}\psi_c(x) \,,$$

in terms of $\langle\hat{\mathcal{O}}\rangle$.