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Quantum Field Theory WS 05/06

Exercise 1

1. (a) Convince yourself that the definition of the Lagrange density

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - j_\kappa A^\kappa$$

is equivalent to

$$\mathcal{L} = \frac{1}{2}\left(\vec{E}^2 - \vec{B}^2\right) - \rho\phi + \vec{j}\cdot\vec{A}.$$

Here $A^\mu = (\phi, \vec{A})$ and $j^\mu = (\rho, \vec{j})$ denote the contravariant vector potential and current, respectively. The field strength tensor is given by $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. Derive the equations of motion of the electromagnetic fields $\vec{E}$ and $\vec{B}$.

(b) The dual field strength tensor can be obtained through

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}\,,$$

with $\epsilon^{\mu\nu\rho\sigma}$ the Levi-Civita tensor. Prove that the homogeneous Maxwell equations can be written as $\partial_\mu \tilde{F}^{\mu\nu} = 0$ and that $F_{\mu\nu}\tilde{F}^{\mu\nu} = -4\vec{E}\cdot\vec{B}$.

2. (a) Consider the Hamilton operator

$$\hat{H}_{\rm F} = i\omega\hat{\psi}\hat{\pi}\,,$$

where the Hermitian operators $\hat{\psi}$ and $\hat{\pi}$ fulfill the anti-commutation relations

$$\{\hat{\psi},\hat{\pi}\} = 0\,,\quad \{\hat{\psi},\hat{\psi}\} = \hat{1}\,,\quad \{\hat{\pi},\hat{\pi}\} = \hat{1}\,.$$

In analogy to the bosonic harmonic oscillator, introduce the creation and annihilation operators

$$\hat{f}^\dagger = \frac{1}{\sqrt{2}}\left(\hat{\psi} - i\hat{\pi}\right) \quad\text{and}\quad \hat{f} = \frac{1}{\sqrt{2}}\left(\hat{\psi} + i\hat{\pi}\right)$$

and solve the eigenvalue problem of $\hat{H}_F$.

(b) Determine the spectrum of the supersymmetric Hamilton operator

$$\hat{H}_{\rm SUSY} = \{\hat{Q}^\dagger, \hat{Q}\}\,,$$

where $\hat{Q}^\dagger = \sqrt{\omega}\hat{b}\hat{f}^\dagger$ and $\hat{Q} = \sqrt{\omega}\hat{b}^\dagger\hat{f}$ and $\hat{b}^{(\dagger)}$ the creation and annihilation operators of the bosonic oscillator.

(turn over)

3. The (anti-) symmetrization-operator is defined by its action on an n-particle wave function

$$
\begin{aligned}
\hat{\mathcal{S}}\Psi(x_1,..,x_N) &\equiv \frac{1}{N!}\sum_{\sigma\epsilon\mathcal{P}_N}\hat{P}_\sigma\Psi(x_1,..,x_N),\\
\hat{\mathcal{A}}\Psi(x_1,..,x_N) &\equiv \frac{1}{N!}\sum_{\sigma\epsilon\mathcal{P}_N}\mathrm{sign}(\sigma)\hat{P}_\sigma\Psi(x_1,..,x_N),
\end{aligned}
$$

where σ is an element of the permutation group $\mathcal{P}_N$ and

$$
\hat{P}_\sigma\Psi(x_1,..,x_N) \equiv \Psi(x_{\sigma(1)},..,x_{\sigma(N)}).
$$

(a) Prove that the relations $\hat{\mathcal{S}}^2 = \hat{\mathcal{S}}$, $\hat{\mathcal{A}}^2 = \hat{\mathcal{A}}$ and $\hat{\mathcal{S}}\hat{\mathcal{A}} = \hat{\mathcal{A}}\hat{\mathcal{S}} = 0$ hold true.

(b) Prove the **symmetry postulate**, stating that the Hilbert space of a system of identical particles contains either symmetric or anti-symmetric states only.

4. Compute the integrals

$$
I_{ee} = \frac{e^2}{2}\frac{1}{V^2}\int_V\int_V \mathrm{d}^3r\,\mathrm{d}^3r'\,\frac{e^{-i\left(\vec{k}_1-\vec{k}_3\right)\cdot\vec{r}}\,e^{-i\left(\vec{k}_2-\vec{k}_4\right)\cdot\vec{r}'}}{|\vec{r}-\vec{r}'|}e^{-\alpha|\vec{r}-\vec{r}'|},
$$

and

$$
I_I = \frac{e^2}{2}\frac{N^2}{V^2}\int_V\int_V \mathrm{d}^3R\,\mathrm{d}^3R'\,\frac{e^{-\alpha|\vec{R}-\vec{R}'|}}{|\vec{R}-\vec{R}'|},
$$

where $\alpha > 0$ and

$$
\delta_{\vec{k}_1,\vec{k}_2} \equiv \frac{1}{V}\int_V \mathrm{d}^3r\, e^{-i\left(\vec{k}_1-\vec{k}_2\right)\cdot\vec{r}}.
$$