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Quantum Field Theory WS 05/06

Exercise 7

1. **Harmonic oscillator**

   Use the explicit expression for the Greens function of the harmonic oscillator,

$$G(x_1,t;x_0,0) = \left(\frac{m}{2\pi i}\frac{\omega}{\sin\omega t}\right)^{1/2} \times \exp\left\{\frac{im\,\omega}{2\,\sin\omega t}\left[(x_1^2+x_0^2)\cos\omega t - 2x_1x_0\right]\right\}\,,$$

   to derive the energy eigenvalues and eigenfunctions of the ground state and the first excitation level.

2. **Time-ordered products and n-point functions**

   (a) Convince yourself that, using the quantum mechanical path integral, matrix elements of time-ordered products of operators $\hat{x}(t_i)$ ($t \le t_i \le t'$; $\forall\, i = 1,..,n$) can be written as

$$\langle x',t'|T\left[\hat{x}(t_1)..\hat{x}(t_n)\right]|x,t\rangle = \int \mathcal{D}x\mathcal{D}p\, x(t_1)..x(t_n)\,\exp\left\{i\int\limits_t^{t'} \mathrm{d}\tau\,(p\dot{x} - H(x,p))\right\}.$$

   (b) Use the above result to show that the corresponding ground-state expectation value can be obtained from

$$\langle 0|T\left[\hat{x}(t_1)..\hat{x}(t_n)\right]|0\rangle = \lim_{\substack{t'\to\infty\\ t\to-\infty}} \frac{\int \mathcal{D}x\mathcal{D}p\, x(t_1)..x(t_n)\,\exp\left\{i\int\limits_t^{t'} \mathrm{d}\tau\,(p\dot{x} - H(x,p))\right\}}{\int \mathcal{D}x\mathcal{D}p\,\exp\left\{i\int\limits_t^{t'} \mathrm{d}\tau\,(p\dot{x} - H(x,p))\right\}}.$$

   Hint: To do the limit $t' \to \infty$ and $t \to -\infty$, analyticly continue the time-coordinate to the complex axis by $t' \to -i\infty$ and $t \to i\infty$.

   (c) Express the n-point function in terms of the generating functional

$$Z[F] = \mathcal{N}\int \mathcal{D}x\mathcal{D}p\,\exp\left\{i\int\limits_{-\infty}^{\infty} \mathrm{d}\tau\,(p\dot{x} - H(x,p) + F(\tau)x(\tau))\right\}\,,$$

   with $F(t)$ a classical external force.

3. **Free scalar field with source term**

   The generating functional for the free scalar field $\phi(x)$ has the form

$$Z_0[J] = \mathcal{N}\int \mathcal{D}\phi\,\exp\left\{i\int \mathrm{d}^4x\,\mathcal{L}_J\right\}\,,$$

   where the Lagrange density with an external c-number source $J(x)$ is given by

$$\mathcal{L}_J = \frac{1}{2}\left(\partial^\mu\phi(x)\partial_\mu\phi(x) - m^2\phi^2(x)\right) + J(x)\phi(x)\,.$$

(a) Show that $\mathcal{L}_J$ leads to the equation of motion $(\Box_x + m^2)\phi(x) = J(x)$ with a classical solution that can be obtained by the usual Greens function method

$$\phi_{cl}(x) = -\int \mathrm{d}^4 y \Delta_F(x-y) J(y)\,.$$

The corresponding Greens function $\Delta_F(x-y)$ is the Feynman propagator for the scalar field in position space

$$\Delta_F(x-y) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{e^{-ik(x-y)}}{k^2 - m^2 + i\varepsilon}\,.$$

(b) Show that by a change of variable $\phi(x) = \phi_{cl}(x) + \eta(x)$ in the Lagrange density the generating functional can be expressed in the form

$$Z_0[J] = Z_0[0] \exp\left\{-\frac{i}{2}\int \mathrm{d}^4 x \mathrm{d}^4 y\, J(x)\Delta_F(x-y)J(y)\right\}\,.$$