Exercise 9

1. **Parity I**

   Prove that the free Dirac equation is invariant under the transformation

   $$x = (t, \vec{x}) \to x' = (t, -\vec{x})\,, \quad \psi(x) \to \psi'(x') = \gamma^0 \psi(t, \vec{x})\,.$$

2. **Parity II**

   The unitary parity transformation of the Dirac field operator is given by

   $$\hat{\mathcal{P}}\hat{\psi}(t,\vec{x})\hat{\mathcal{P}}^{-1} = \gamma^0\hat{\psi}(t,-\vec{x}) \quad \text{and} \quad \hat{\mathcal{P}}\hat{\bar{\psi}}(t,\vec{x})\hat{\mathcal{P}}^{-1} = \hat{\bar{\psi}}(t,-\vec{x})\gamma^0\,.$$

   (a) Use the expansion of $\hat{\psi}$ in terms of creation- and annihilation-operators to determine

   $$\hat{\mathcal{P}}\hat{b}(\vec{p},s)\hat{\mathcal{P}}^{-1}\,, \quad \hat{\mathcal{P}}\hat{b}^\dagger(\vec{p},s)\hat{\mathcal{P}}^{-1}\,, \quad \hat{\mathcal{P}}\hat{d}(\vec{p},s)\hat{\mathcal{P}}^{-1} \quad \text{and} \quad \hat{\mathcal{P}}\hat{d}^\dagger(\vec{p},s)\hat{\mathcal{P}}^{-1}\,.$$

   Do particles and anti-particles have the same parity?

   (b) Determine the parity transformation properties of the field bilinears

   $$\hat{\bar{\psi}}\hat{\psi}\,, \quad \hat{\bar{\psi}}\gamma^\mu\hat{\psi}\,, \quad \hat{\bar{\psi}}\gamma^\mu\gamma^5\hat{\psi}\,, \quad i\hat{\bar{\psi}}\gamma^5\hat{\psi}\,.$$

   (c) How do momentum $\hat{P}^\mu$ and angular-momentum $\hat{\vec{J}}$ behave under parity transformation? What results for the helicity of the particle?

3. **Feynman propagator for the free Dirac field**

   Show that the Feynman propagator of the second-quantized Dirac field yields

   $$S_{F\alpha\beta}(x-y) \equiv \langle 0|T\left[\hat{\psi}_\alpha(x)\hat{\bar{\psi}}_\beta(y)\right]|0\rangle = (i\partial\!\!\!/ + m)_{\alpha\beta}\, i\Delta_F(x-y)\,,$$

   with Δ_F the scalar Feynman propagator.

4. **Generating functional for the free Dirac field**

   The generating functional for the free Dirac field, using four-component spinors $\bar{\chi}$ and χ as external sources, reads

   $$\begin{aligned} Z_0[\bar{\chi},\chi] &= \mathcal{N}\int \mathcal{D}\bar{\psi}\mathcal{D}\psi\,\exp\left(i\int \mathrm{d}^4x\,\left(\bar{\psi}(i\partial\!\!\!/ - m)\psi + \bar{\chi}\psi + \bar{\psi}\chi\right)\right) \\ &= Z_0[0,0]\exp\left(-\int \mathrm{d}^4x\mathrm{d}^4y\,\bar{\chi}(x)S_F(x-y)\chi(y)\right)\,. \end{aligned} \tag{1}$$

   From Eq. (1) derive the free two- and four-point function.