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Quantum Field Theory WS 05/06

Exercise 5

1. **Generalized Wick theorem**

   Wick's theorem for unordered products of bosonic operators $A_i = A(x_i)$ can be stated as

$$
\begin{aligned}
A_1 \ldots A_n \;=\;& :A_1A_2..A_n: + :A_1A_2..A_n: + \cdots + :A_1..A_{n-1}A_n: + \cdots + :A_1..A_n: + \\
&+ :A_1A_2A_3A_4..A_n: + \cdots + :A_1A_2A_3A_4..A_n: + \ldots \\
&+ :A_1A_2A_3A_4A_5A_6..A_n: + \cdots + :A_1A_2A_3A_4A_5A_6..A_n: + \ldots \\
&+ \ldots .
\end{aligned}
$$

   In words: The product of linear operators equals the sum of all normal ordered products with all possible pairings. A normal ordered product with pairings is thereby defined as the product of the vacuum expectation values of the contracted field pairs and the normal ordered product of the remaining, unpaired fields, e.g.

$$
\begin{aligned}
:..A_i..A_j..A_k..A_l..: \;=\;& \langle 0|A_iA_k|0\rangle\langle 0|A_jA_l|0\rangle \times \\
& :..A_{i-1}A_{i+1}..A_{j-1}A_{j+1}..A_{k-1}A_{k+1}..A_{l-1}A_{l+1}..: \;.
\end{aligned}
$$

   Use the above version of Wick's theorem to convince yourself that

$$
\begin{aligned}
:A_1..A_{k_1}::A_{k_1+1}..A_{k_2}: .. :A_{k_s+1}..A_n: \;=\;& :A_1..A_n: + :A_1..A_{k_1+1}..A_n: + \ldots \\
&+ :A_1..A_{k_1}..A_n: + \ldots \\
&+ :A_1..A_{k_1+1}..A_{k_2+1}..A_n: + \ldots \\
&+ :A_1A_2..A_{k_1+1}A_{k_1+2}..A_n: + \ldots .
\end{aligned}
$$

   In words: The result is the sum of all pairings with the exception of the pairings between already normal ordered operators.

2. **$\lambda\phi^4$ scalar field theory**

   Consider the interacting scalar field theory

$$
\hat{H}(\hat{\phi},\hat{\pi}) = :\int \mathrm{d}^3x \left[\frac{1}{2}\left(\hat{\pi}^2(x) + |\vec{\nabla}\hat{\phi}(x)|^2 + m_0^2\hat{\phi}^2(x)\right) + \frac{\lambda_0}{4!}\hat{\phi}^4(x)\right]: \;.
$$

   The Hamilton operator of the corresponding free theory is

$$
\hat{H}_{\mathrm{in}}(\hat{\phi}_{\mathrm{in}},\hat{\pi}_{\mathrm{in}}) = :\int \mathrm{d}^3x\,\frac{1}{2}\left(\hat{\pi}_{\mathrm{in}}^2(x) + |\vec{\nabla}\hat{\phi}_{\mathrm{in}}(x)|^2 + m^2\hat{\phi}_{\mathrm{in}}^2(x)\right): \;.
$$

   (a) Show that

$$
\int \mathrm{d}^3x f(x)\,\overleftrightarrow{\partial_0}\,\langle\beta_{\mathrm{out}}|\hat{\phi}_{\mathrm{in}}(x) - \hat{\phi}_{\mathrm{out}}(x)|\alpha_{\mathrm{in}}\rangle\,,
$$

   where $f(x)$ fulfills the Klein-Gordon equation $(\Box_x + m^2)f(x) = 0$, is time independent.

(turn over)

(b) Verify that the two-point Greens function satisfies the relation

$$(\Box_x + m_0^2)\langle 0|T[\hat{\phi}(x)\hat{\phi}(y)]|0\rangle = -\frac{\lambda_0}{3!}\langle 0|T[\hat{\phi}^3(x)\hat{\phi}(y)]|0\rangle - i\delta^4(x-y)\,.$$

(c) Use Wick's theorem to determine the lowest- and first-order contribution to the two- and four-point Greens function. Use the LSZ-reduction formula to calculate the connected S-matrix element $S^c_{\beta\alpha} = \langle p_1, p_2; \mathrm{out}|k_1, k_2; \mathrm{in}\rangle$ up to first order in perturbation theory.

3. **Two- and Three-particle phase space**

Consider the production of two- and three- massless particles with four-momenta k_i.

(a) Prove that the two-body phase space integral yields

$$\rho_2 = \int \frac{\mathrm{d}^3 k_1}{(2\pi)^3 2\omega_1}\frac{\mathrm{d}^3 k_2}{(2\pi)^3 2\omega_2}(2\pi)^4\delta^4(q-k_1-k_2) = \frac{1}{8\pi}.$$

(b) Show that the three-particle phase space integral

$$\rho_3 = \int \frac{\mathrm{d}^3 k_1}{(2\pi)^3 2\omega_1}\frac{\mathrm{d}^3 k_2}{(2\pi)^3 2\omega_2}\frac{\mathrm{d}^3 k_3}{(2\pi)^3 2\omega_3}(2\pi)^4\delta^4(q-k_1-k_2-k_3)$$

can be written as

$$\rho_3 = \frac{q^2}{128\pi^3}\int \mathrm{d}x_1 \mathrm{d}x_2\,,$$

where $x_i = 2k_i \cdot q/q^2$, for $i = 1, 2, 3$. Find the region of integration for x_1 and x_2.

Hint: A convenient frame to evaluate the above integrals is the center-of-mass frame of the momenta, given by $q = (q_0, 0, 0, 0)$.