Exercise 3

1. Assume Θ_i with $i = 1, .., N$ to be N anti-commuting, Grassmann, numbers. Then

$$\{\Theta_i, \Theta_j\} = 0 \quad \text{and} \quad \left\{\frac{\mathrm{d}}{\mathrm{d}\Theta_i}, \Theta_j\right\} = \delta_{ij}\,, \quad \left\{\frac{\mathrm{d}}{\mathrm{d}\Theta_i}, \frac{\mathrm{d}}{\mathrm{d}\Theta_j}\right\} = 0$$

hold true. As rules for differentiation we use

$$\frac{\mathrm{d}}{\mathrm{d}\Theta_i} 1 = 0\,, \quad \frac{\mathrm{d}}{\mathrm{d}\Theta_i} \Theta_j = \delta_{ij} \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}\Theta_i} \Theta_j\Theta_k = \delta_{ij}\,\Theta_k - \delta_{ik}\,\Theta_j\,.$$

For the integration involving Grassmann numbers we require

$$\int \mathrm{d}\Theta_i\, 1 = 0 \quad \text{and} \quad \int \mathrm{d}\Theta_i\, \Theta_i = 1\,, \quad \text{with} \quad \{\mathrm{d}\Theta_i, \mathrm{d}\Theta_j\} = 0 \text{ and } \{\mathrm{d}\Theta_i, \Theta_j\} = \delta_{ij}\,.$$

As for the case of differentiation the order of differentials and variables is important,

$$\int \mathrm{d}\Theta_1\mathrm{d}\Theta_2\Theta_1\Theta_2 = -\int \mathrm{d}\Theta_1\mathrm{d}\Theta_2\Theta_2\Theta_1 = -\int \mathrm{d}\Theta_2\mathrm{d}\Theta_1\Theta_1\Theta_2 = -1\,.$$

With the rules given, calculate the integrals

$$\int \mathrm{d}\Theta_1\mathrm{d}\Theta_2\, e^{-\frac{1}{2}\vec{\Theta}^T A\vec{\Theta}} \quad \text{and} \quad \int \mathrm{d}\Theta_1\mathrm{d}\Theta_2\, e^{-\frac{1}{2}\vec{\Theta}^T A\vec{\Theta}+\vec{\rho}^T\vec{\Theta}}\,,$$

with A an anti-symmetric 2×2 matrix, $\vec{\Theta}^T = (\Theta_1, \Theta_2)$ and $\vec{\rho}$ a two-component vector of Grassmann variables. Compare your result to the case when using commuting variables and a symmetric matrix instead.

2. Show that

$$K(\vec{x}, t; \vec{x}', t_0) = \left(\frac{m}{2\pi i(t-t_0)}\right)^{3/2} \exp\left(\frac{im(\vec{x}-\vec{x}')^2}{2(t-t_0)}\right)\Theta(t-t_0)$$

is a Greens function of the free Schrödinger equation.

3. Proof that the charged Klein-Gordon field

$$\hat{\phi}(x) = \int \frac{\mathrm{d}^3k}{\sqrt{(2\pi)^3 2k_0}} \left(\hat{a}(\vec{k})e^{-ikx} + \hat{b}^\dagger(\vec{k})e^{ikx}\right)$$

when quantized using the anti-commutation relations

$$\{\hat{a}(\vec{k}), \hat{a}^\dagger(\vec{k}')\} = \{\hat{b}(\vec{k}), \hat{b}^\dagger(\vec{k}')\} = \delta(\vec{k}-\vec{k}')\,,$$
$$\{\hat{a}(\vec{k}), \hat{a}(\vec{k}')\} = \{\hat{a}^\dagger(\vec{k}), \hat{a}^\dagger(\vec{k}')\} = \{\hat{b}(\vec{k}), \hat{b}(\vec{k}')\} = \{\hat{b}^\dagger(\vec{k}), \hat{b}^\dagger(\vec{k}')\} = 0\,,$$

violates micro-causality.

(turn over)

Remarks:

- To proof that micro-causality is violated, one has to show that the commutator of two observables does not vanish for space-like separations $(x-y)^2 < 0$. Therefore, express $[\hat{\phi}^\dagger(x)\hat{\phi}(x), \hat{\phi}^\dagger(y)\hat{\phi}(y)]$ in terms of

$$i\Delta_1(x-y) \equiv \{\hat{\phi}(x), \hat{\phi}^\dagger(y)\}\,,$$

  and then evaluate Δ_1. Note that it is sufficient to consider the case $x_0 = y_0$.

- When applying the transformation $\tilde{q} = m \sinh t$ to the integral

$$\int_0^\infty \mathrm{d}\tilde{q}\, \frac{\tilde{q}\,\sin(\tilde{q}\tilde{x})}{\sqrt{\tilde{q}^2 + m^2}}$$

  it arises to be equal to the MacDonald function $K_1(m\tilde{x})$ (a modified Bessel function of the second type). In the limit of large $m\tilde{x}$ the function becomes $K_1(m\tilde{x}) \sim \sqrt{\frac{\pi}{2m\tilde{x}}}e^{-m\tilde{x}}$.