

QCD at Colliders

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1 QCD Lagrangian, Feynman rules, Colour algebra

2 Feynman rules & colour algebra

3 Running coupling

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QCD LAGRANGIAN

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$SU(3)$ of colour: evidence

- strongly interacting particles as bound states of spin-1/2 quarks

(evidence for spin from sum rules)

- mesons: $q\bar{q}$, baryons qqq
electric charges: $+2/3$ and $-1/3$

(to produce integer charges $0, \pm 1, \pm 2$)

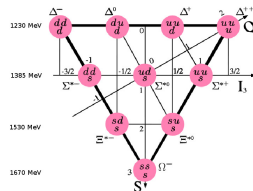
- for Δ^{++} need 3 u quarks, all in spin $\uparrow$
→ need new quantum number

(to avoid Pauli's law)

- introduce colours r, g, b , such that
 $r + g + b = \text{no charge}$

(called colours from first colour TVs)

- simplest group: $SU(3)$



Constructing the Lagrangian

- quarks come in 3 colours i : $\psi_q \equiv q^i$ (fundamental representation of $SU(3)$)
- interact with 8 gluons G^a (adjoint representation of $SU(3)$: colour+anti-colour)
- in gauge-fixing term emergence of Fadeev-popov ghosts (spin-0 fermions with colours like gluons, only as loop-internal particles)

$$\mathcal{L}_{QCD} = \sum_q \bar{q}^i (i \not{D}_{ij} - m_q \delta_{ij}) q^j - \frac{1}{4} G_{\mu\nu}^a G^{a, \mu\nu} + \mathcal{L}_{g.f.}$$

- gauge-covariant derivative through 8 Gell-Mann matrices T_{ij}^a

$$D_{ij}^\mu = \partial^\mu \delta_{ij} - ig_s G^{a, \mu} T_{ij}^a$$

- gauge-kinetic term through structure constants f^{abc} giving tensors

$$G_{\mu\nu}^a = \partial_\mu G_\nu^a - \partial_\nu G_\mu^a + g_s f^{abc} G_\mu^b G_\nu^c$$

Feynman rules

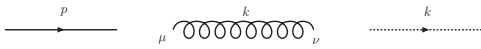


Diagram showing the Feynman rules for quark, gluon, and ghost lines. A quark line is represented by a solid line with an arrow and momentum p . A gluon line is represented by a wavy line with momentum k and indices μ, ν . A ghost line is represented by a dotted line with momentum k .

$$\frac{i}{\not{p} - m + i0} \quad -\frac{i}{k^2 + i0} \left[g^{\mu\nu} + (\zeta - 1) \frac{k^\mu k^\nu}{k^2 + i0} \right] \quad \frac{i}{k^2 + i0}$$

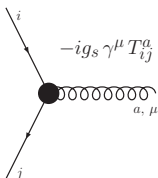


Diagram showing a quark-gluon vertex. Two quark lines with momenta i and j meet at a vertex, and a gluon line with momentum a, μ is attached. The vertex factor is $-ig_s \gamma^\mu T_{ij}^a$.

$$-ig_s \gamma^\mu T_{ij}^a$$

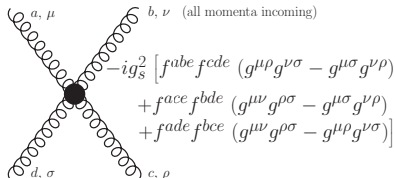


Diagram showing a four-gluon vertex. Four gluon lines with momenta a, μ , b, ν , c, ρ , and d, σ meet at a vertex. The vertex factor is $-ig_s^2$ times the sum of three terms involving structure constants and metric tensors.

$$-ig_s^2 \left[f^{abe} f^{cde} (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) + f^{ace} f^{bde} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\sigma} g^{\nu\rho}) + f^{ade} f^{bce} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma}) \right]$$

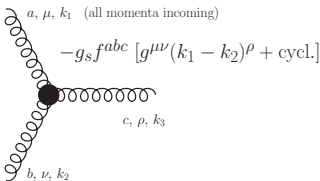


Diagram showing a three-gluon vertex. Three gluon lines with momenta a, μ, k_1 , b, ν, k_2 , and c, ρ, k_3 meet at a vertex. The vertex factor is $-g_s f^{abc} [g^{\mu\nu} (k_1 - k_2)^\rho + \text{cycl.}]$.

$$-g_s f^{abc} [g^{\mu\nu} (k_1 - k_2)^\rho + \text{cycl.}]$$

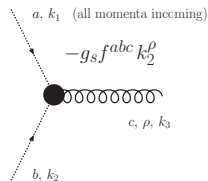


Diagram showing a ghost-gluon vertex. A ghost line with momentum a, k_1 and a gluon line with momentum c, ρ, k_3 meet at a vertex, and another gluon line with momentum b, k_2 is attached. The vertex factor is $-g_s f^{abc} k_2^\rho$.

$$-g_s f^{abc} k_2^\rho$$

Colour algebra: generators

- algebra

$$\left[T^a, T^b \right] = if^{abc} T^c$$

- normalisation of generators

$$\text{Tr} \left[T^a T^b \right] = T_{ij}^a T_{ji}^b = T_R \delta^{ab},$$

where $T_R = \frac{1}{2}$.

- product

$$\sum_a T_{ij}^a T_{kl}^a = \frac{1}{2} \left(\delta_{il} \delta_{kj} - \frac{1}{N} \delta_{ij} \delta_{kl} \right),$$

for $SU(N)$, i.e. $N \equiv 3$ for QCD.

- in particular (for self-energy, etc.):

$$\sum_a T_{ij}^a T_{jl}^a = \frac{1}{2} \frac{N^2 - 1}{N} \delta_{il} = C_F \delta_{il},$$

where Casimir operator of fundamental representation $C_F = \frac{N^2 - 1}{2N}$ is “colour charge” of quark

- similarly, from the expression

$$f^{abc} = -2i \operatorname{Tr} \left[\left[T^a, T^b \right] T^c \right]$$

for the structure constants, one finds

$$\sum_{a,b} f^{abc} f^{abd} = C_A \delta^{cd},$$

where Casimir operator of adjoint representation $C_A = N$ is “colour charge” of gluon

$$\begin{aligned}\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \lambda_2 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \lambda_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} & \lambda_5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} & \lambda_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\ \lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} & \lambda_8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}\end{aligned}$$

Running coupling

- as in QED, coupling changes with renormalisation scale μ_R (running coupling)

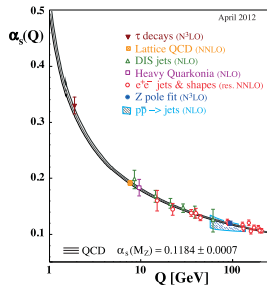
$$\mu_R^2 \frac{\partial \alpha(\mu_R^2)}{\partial \mu_R^2} = \beta(\alpha)$$

with

$$-\beta(\alpha) = \sum_{n=0}^{\infty} b_n \alpha^{2+n} = \frac{\beta_0}{4\pi} \alpha_s^2 + \frac{\beta_1}{(4\pi)^2} \alpha_s^3 + \dots$$

and

$$\beta_0 = \frac{11}{3} C_A - \frac{4}{3} T_R n_f.$$



- compare with QED:

$$\beta_0 = -\frac{2n_f}{3}n_f.$$

- due to gluon self-interaction: different sign in QCD β function
- effective forms

$$\alpha_s(\mu_R^2) \equiv \frac{g_s^2(\mu_R^2)}{4\pi} = \frac{1}{\frac{\beta_0}{4\pi} \log \frac{\mu_R^2}{\Lambda_{\text{QCD}}^2}}.$$

and

$$\alpha_s(\mu_R^2) \equiv \frac{g_s^2(\mu_R^2)}{4\pi} = \frac{1}{\frac{\beta_0}{4\pi} \log \frac{\mu_R^2}{\Lambda_{\text{QCD}}^2}}.$$

- “asymptotic freedom” ($\longrightarrow$ “infrared slavery”):
QCD quanta only in bound states at low energies

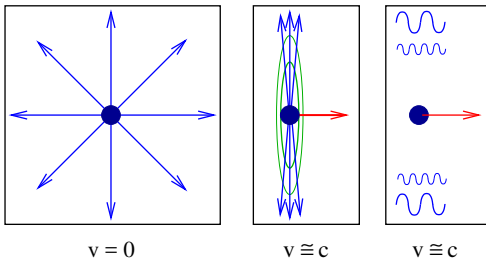
BASIC CONCEPTS

Contents

- 2.a) Electromagnetic Radiation, DGLAP equations in QED
- 2.b) Running of α_s , bound states, factorization and DGLAP for QCD
- 2.f) Parton Distribution Functions
- 2.g) Hadrons in the final state: Hadronization
- 2.h) Summary

An electromagnetic analogy

- consider a charge Z moving at constant velocity v



- at $v = 0$: radial E field only
- at $v = c$: B field emerges $\vec{E} \perp \vec{B}$, $\vec{B} \perp \vec{v}$, $\vec{E} \perp \vec{v}$,
- energy flow $\sim$ Poynting vector $\vec{S} \sim \vec{E} \times \vec{B}$, $\parallel \vec{v}$
- approximate classical fields by “equivalent quanta”: photons

- spectrum of photons:

(in dependence on energy ω and transverse distance $b_\perp$)

$$dn_\gamma = \frac{Z^2\alpha}{\pi} \cdot \frac{d\omega}{\omega} \cdot \frac{db_\perp^2}{b_\perp^2}$$

- Fourier transform to transverse momenta $k_\perp$:

$$dn_\gamma = \frac{Z^2\alpha}{\pi} \cdot \frac{d\omega}{\omega} \cdot \frac{dk_\perp^2}{k_\perp^2}$$

note: divergences for $k_\perp \rightarrow 0$ (collinear) and $\omega \rightarrow 0$ (soft)

- applicable to electron (point charge, $Z = 1$)
- therefore: Fock state for lepton = superposition (coherent):

$$|e\rangle_{\text{phys}} = |e\rangle + |e\gamma\rangle + |e\gamma\gamma\rangle + |e\gamma\gamma\gamma\rangle + \dots$$

photon fluctuations will “recombine”

Radiation in classical electrodynamics

- consider final state radiation in $\gamma^* \rightarrow \ell \bar{\ell}$
(electron velocities/momenta labelled as v and v'/p and p')
- classical electromagnetic spectrum from **radiation function**:

(this is from Jackson or any other reasonable book on ED)

$$\frac{d^2 I}{d\omega d\Omega} = \frac{e^2}{4\pi^2} \left| \vec{\epsilon}^* \cdot \left(\frac{\vec{v}}{1 - \vec{v} \cdot \vec{n}} - \frac{\vec{v}'}{1 - \vec{v}' \cdot \vec{n}} \right) \right|^2,$$

with ϵ the polarisation vector and $\vec{n}(\Omega)$ the direction of the radiation

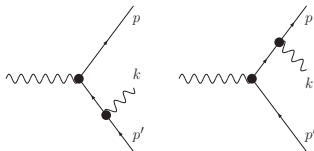
- recast with four-momenta, equivalent photon spectrum:

$$\begin{aligned} dn_\gamma &= \frac{d^3 k}{(2\pi)^3 2k_0} \frac{\alpha}{\pi} \left| \epsilon_\mu^* \left(\frac{p^\mu}{p \cdot k} - \frac{p'^\mu}{p' \cdot k} \right) \right|^2 \\ &= \frac{d^3 k}{(2\pi)^3 2k_0} \frac{\alpha}{\pi} \left| W_{pp';k} \right|^2 \end{aligned}$$

with the **eikonal** $W_{pp';k}$

Radiation in QED

- repeat exercise in QFT, Feynman diagrams:



$$\mathcal{M}_{X \rightarrow e^+ e^- \gamma} = e \bar{u}(p) \left[\Gamma \frac{\not{p}' - \not{k}}{(p' - k)^2} \gamma^\mu - \gamma^\mu \frac{\not{p} + \not{k}}{(p + k)^2} \Gamma \right] u(p') \epsilon_\mu^*(k)$$

$$\xrightarrow{\text{soft}} e \epsilon_\mu^*(k) \left[\frac{p^\mu}{p \cdot k} - \frac{p'^\mu}{p' \cdot k} \right] \bar{u}(p') \Gamma u(p) = e \mathcal{M}_{X \rightarrow e^+ e^- \gamma} \cdot W_{pp';k}$$

- manifestation of [Low's theorem](#):
soft radiation independent of spin ($\rightarrow$ classical)

(radiation decomposes into soft, classical part with logs – i.e. dominant – and hard collinear part)

DGLAP equations for QED

(Dokshitzer–Gribov–Lipatov–Altarelli–Parisi Equations)

- define probability to find electron or photon in electron:

$$\text{at LO in } \alpha(\text{noemission}) : \ell(x, k_{\perp}^2) = \delta(1 - x)$$

$$\text{and } \gamma(x, k_{\perp}^2) = 0$$

(introduced x = energy fraction w.r.t. physical state)

- including emissions:
 - probabilities change
 - energy fraction ξ of **lepton parton** w.r.t. the **physical lepton object** reduced by some fraction $z = x/\xi$
 - reminder: differential of photon number w.r.t. $k_{\perp}^2$:

$$dn_{\gamma} = \frac{\alpha}{\pi} \frac{dk_{\perp}^2}{k_{\perp}^2} \frac{d\omega}{\omega} \leftrightarrow \frac{dn_{\gamma}}{d \log k_{\perp}^2} = \frac{\alpha}{\pi} \frac{dx}{x}$$

- evolution equations (trivialised)

$$\frac{d\ell(x, k_{\perp}^2)}{d \log k_{\perp}^2} = \frac{\alpha(k_{\perp}^2)}{2\pi} \int_x^1 \frac{d\xi}{\xi} \mathcal{P}_{\ell\ell} \left(\frac{x}{\xi}, \alpha(k_{\perp}^2) \right) \ell(\xi, k_{\perp}^2)$$
$$\frac{d\gamma(x, k_{\perp}^2)}{d \log k_{\perp}^2} = \frac{\alpha(k_{\perp}^2)}{2\pi} \int_x^1 \frac{d\xi}{\xi} \mathcal{P}_{\gamma\ell} \left(\frac{x}{\xi}, \alpha(k_{\perp}^2) \right) \ell(\xi, k_{\perp}^2).$$

- $k_{\perp}^2$ plays the role of “resolution parameter”
- the $\mathcal{P}_{ab}(z)$ are the **splitting functions**, encoding quantum mechanics of the “splitting cross section”, for example (at LO)

$$\mathcal{P}_{\ell\ell}(z) = \frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z)$$

- if $\gamma \rightarrow \ell\bar{\ell}$ splittings included, have to add entries/splitting functions into **evolution equations** above

Aside: Lifetime of electron–photon fluctuations ($e(P) \rightarrow e(p) + \gamma(k)$)

- first estimate: use uncertainty relation and Lorentz time dilation

- $P^2 = (p + k)^2 = M_{\text{virt}}^2$ the virtual mass of the incident electron
- life time = life time in rest frame $\cdot$ time dilation

$$\tau \sim \frac{1}{M_{\text{virt}}} \cdot \frac{E}{M_{\text{virt}}} = \frac{E}{(p + k)^2} \sim \frac{E}{2Ek(1 - \cos \theta)} \approx \frac{k}{k^2 \sin^2 \theta / 2} \approx \frac{\omega}{k_{\perp}^2}$$

- second estimate: use uncertainty relation and assume only photon off-shell

- energy balance of photon

$$P^2 = 2p \cdot k + k^2, \quad \text{therefore } k^2 \approx -k_{\perp}^2 \approx -2p \cdot k < 0.$$

- assume photon momentum to be $k^{\mu} = (\omega, \vec{k}_{\perp}, k_{\parallel})$,
shift in energy for photon going on-shell: $\delta\omega \sim k_{\perp}^2 / \omega$, therefore

$$\tau_{\gamma} \sim \frac{1}{\delta\omega} \approx \frac{\omega}{k_{\perp}^2} \approx \frac{\omega}{\omega^2 \sin^2 \theta} \approx \frac{1}{\omega \theta^2}$$

- lifetime larger with smaller transverse momentum

(i.e. with larger transverse distance)

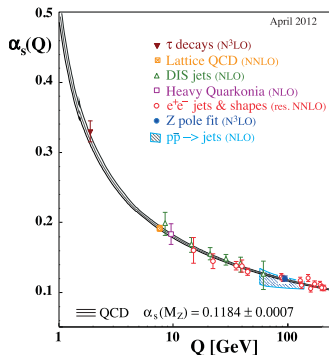
Running of α_s

- quantum effect due to loops:
couplings change with scale
- “running” for QCD
“crawling” for QED
- running driven by β -function

$$\begin{aligned}\beta(\alpha_s) &= \mu_R^2 \frac{\partial \alpha_s(\mu_R^2)}{\partial \mu_R^2} \\ &= \frac{\beta_0}{4\pi} \alpha_s^2 + \frac{\beta_1}{(4\pi)^2} \alpha_s^3 + \dots\end{aligned}$$

with

$$\begin{aligned}\beta_0 &= \frac{11}{3} C_A - \frac{4}{3} T_R n_f \\ \beta_1 &= \frac{34}{3} C_A^2 - \frac{20}{3} C_A T_R n_f - 4 C_F T_R n_f\end{aligned}$$



- explicit expression for strong coupling

$$\alpha_s(\mu_R^2) \equiv \frac{g_s^2(\mu_R^2)}{4\pi} = \frac{1}{\frac{\beta_0}{4\pi} \log \frac{\mu_R^2}{\Lambda_{\text{QCD}}^2}}$$

with Λ_{QCD} the Landau pole of QCD, $\Lambda_{\text{QCD}} \approx 250\text{MeV}$.

- infrared slavery & asymptotic freedom
- at low scales: bound states only, no free QCD quanta, **confinement**
- at high scales: free QCD quanta **perturbative expansion in α_s**
- trivial (perturbative!) particle spectrum:

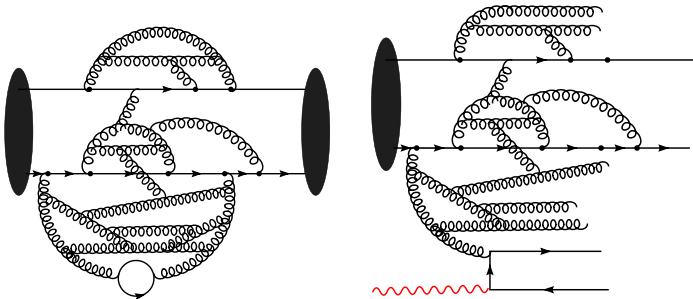
$$dn_g^{q,g} = C_{q,g} \cdot \frac{\alpha_s(k_\perp^2)}{\pi} \cdot \frac{d\omega}{\omega} \cdot \frac{dk_\perp^2}{k_\perp^2}$$

but the gluons carry charge $\implies$ proliferation of soft particles

(from gluons emitted by gluons and from α_s at small $k_\perp$)

Factorization: the Underlying Picture

- picture of hard interaction: coherence broken by **interaction**
- hadron does not recombine but disintegrates



- probing the deep structure of hadrons with photon:
deep-inelastic scattering (DIS)
- condition for DIS on protons $Q \gg m_P \approx 1/R_P$ for scale Q
(typically the virtual mass of the photonic probe)

- using Breit frame:

$$P^\mu = (P_0, \vec{0}, P_z) \text{ with } P_z = \sqrt{P_0^2 - m_P^2} \approx P_0$$

$$q^\mu = (0, \vec{0}, -q_z) \text{ with } Q^2 = -q^2 = q_z^2.$$

- introducing Bjorken variable (“Bjorken- x ”)

$$x_B = -\frac{q^2}{2P \cdot q} = \frac{q_z^2}{2P_z q_z}$$

- interaction time between photon and proton from longitudinal wavelength of photon $\tau_{\text{int}} \sim \lambda_z \sim 1/q_z$
- parton wavelength must be as large as photon wavelength for them to “see” each other: $\implies p_z = q_z$
- lifetime of parton $\tau_{\text{life}} \sim p_z/p_{\perp}^2 \geq \tau_{\text{int}}$
must be larger than interaction time!

let the parton live before it interacts!

- therefore: $p_z \geq p_{\perp}$ for interaction to happen
- if this holds: collision of two quasi-free particles
collinear factorisation
into hard process and **independent** proton $\rightarrow$ parton transitions
- then cross section proportional to probabilities to find partons in proton, given by the **parton distribution functions** $f_{q/p}$

$$\sigma_{\text{DIS}} \sim \sum_q e_q^2 f_{q/p}(x, Q^2),$$

Hadrons in initial state: DGLAP equations of QCD

- similar to QED case:
define **probabilities** (at LO) to find a parton q – quark or gluon – in hadron h at energy fraction x and resolution parameter/scale Q :
parton distribution function (PDF) $f_{q/h}(x, Q^2)$
- scale-evolution of PDFs: DGLAP equations

$$\begin{aligned} & \frac{\partial}{\partial \log Q^2} \begin{pmatrix} f_{q/h}(x, Q^2) \\ f_{g/h}(x, Q^2) \end{pmatrix} \\ &= \frac{\alpha_s(Q^2)}{2\pi} \int_x^1 \frac{dz}{z} \begin{pmatrix} \mathcal{P}_{qq}\left(\frac{x}{z}\right) & \mathcal{P}_{qg}\left(\frac{x}{z}\right) \\ \mathcal{P}_{gq}\left(\frac{x}{z}\right) & \mathcal{P}_{gg}\left(\frac{x}{z}\right) \end{pmatrix} \begin{pmatrix} f_{q/h}(z, Q^2) \\ f_{g/h}(z, Q^2) \end{pmatrix}, \end{aligned}$$

- QCD splitting functions:

$$\mathcal{P}_{qq}^{(1)}(x) = C_F \left[\frac{1+x^2}{(1-x)_+} + \frac{3}{2} \delta(1-x) \right] = \left[P_{qq}^{(1)}(x) \right]_+ + \gamma_q^{(1)} \delta(1-x)$$

$$\mathcal{P}_{qg}^{(1)}(x) = T_R \left[x^2 + (1-x)^2 \right] = P_{qg}^{(1)}(x)$$

$$\mathcal{P}_{gq}^{(1)}(x) = C_F \left[\frac{1+(1-x)^2}{x} \right] = P_{gq}^{(1)}(x)$$

$$\begin{aligned} \mathcal{P}_{gg}^{(1)}(x) = 2C_A \left[\frac{x}{(1-x)_+} + \frac{1-x}{x} + x(1-x) \right] \\ + \frac{11C_A - 4n_f T_R}{6} \delta(1-x) = \left[P_{gg}^{(1)}(x) \right]_+ + \gamma_g^{(1)} \delta(1-x). \end{aligned}$$

- remark: IR regularisation by $+$ -prescription & terms $\sim \delta(1-x)$ from physical conditions on splitting functions

(flavour conservation for $q \rightarrow qg$ and momentum conservation for $g \rightarrow gg, q\bar{q}$)

Parton distribution functions

- simple idea: proton = $|uud\rangle$
(valence quarks) only

- naively: no interactions

$$\longrightarrow f_{u,d/p} \sim \delta\left(x - \frac{1}{3}\right)$$

- elastic interactions

$\longrightarrow$ Gaussian smearing

- strong interactions:

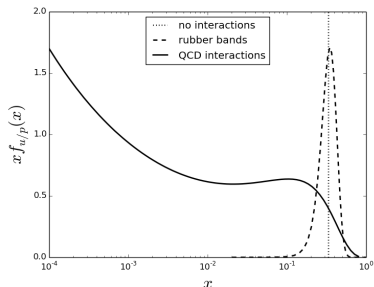
develop “sea” = soft partons

will depend on resolution scale

remember: $dn \propto \log \omega \log k_{\perp}^2$

- in fact, due to $g \rightarrow gg$, sea increases much faster,

$$f_{\text{sea}/p}(x, Q^2) \sim x^{-\lambda}, \quad \lambda \gtrsim 1.$$



- however, **sum rules** put constraints,
example: momentum sum rule

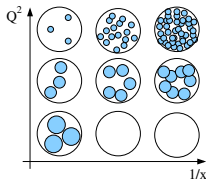
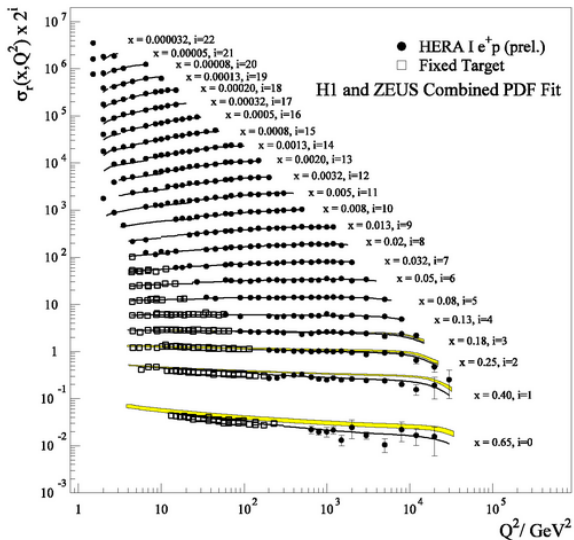
$$\int_0^1 dx x \sum_i f_{i/h}(x, \mu^2) = 1 \quad \forall \mu^2 \text{ and for all hadrons } h,$$

- also flavour sum rules, for protons:

$$\int_0^1 dx \left[f_{u/p}(x, \mu^2) - f_{\bar{u}/p}(x, \mu^2) \right] = 2$$

$$\int_0^1 dx \left[f_{d/p}(x, \mu^2) - f_{\bar{d}/p}(x, \mu^2) \right] = 1$$

$$\int_0^1 dx \left[f_{q/p}(x, \mu^2) - f_{\bar{q}/p}(x, \mu^2) \right] = 0 \quad \text{for } q \in \{s, c, b\},$$



PDFs: inputs

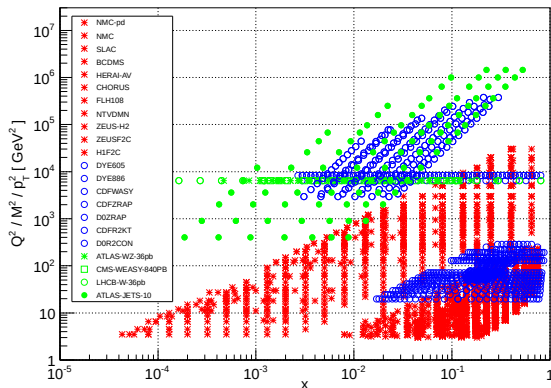
- PDFs are essentially non-perturbative objects

(how many partons in a nucleon?)

- obtained through (global) fits: experimental data + DGLAP

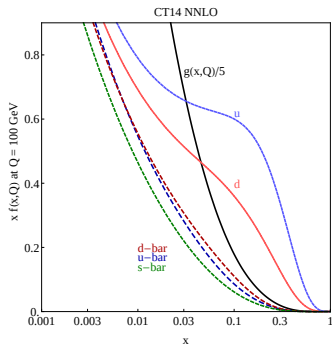
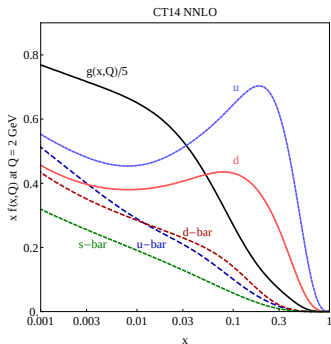
(data: usually total and differential cross sections from different processes)

NNPDF2.3 dataset



PDFs: parametrisations and results

Example below: CT14NNLO, at $Q = 2$ GeV and $Q = 100$ GeV



Hadrons in the final state

- consider QCD final state radiation
- pattern for $q \rightarrow qg$ similar to $\ell \rightarrow \ell\gamma$ in QED:

$$dw^{q \rightarrow qg} = \frac{\alpha_s(k_\perp^2)}{2\pi} C_F \frac{dk_\perp^2}{k_\perp^2} \frac{d\omega}{\omega} \left[1 + \left(1 - \frac{\omega}{E} \right)^2 \right]$$

$$\stackrel{\omega=E(1-z)}{=} \frac{\alpha_s(k_\perp^2)}{2\pi} C_F \frac{dk_\perp^2}{k_\perp^2} dz \frac{1+z^2}{1-z} = \frac{\alpha_s(k_\perp^2)}{2\pi} C_F \frac{dk_\perp^2}{k_\perp^2} dz P_{qg}^{(1)}(z).$$

- divergent structures for:

$z \rightarrow 1$ (soft divergence)	$\longleftrightarrow$	infrared/soft logarithms
$k_\perp^2 \rightarrow 0$ (collinear/mass divergence)	$\longleftrightarrow$	collinear logarithms
- cut regularise with cut-off $k_{\perp,\min} \sim 1\text{GeV} > \Lambda_{\text{QCD}}$

- find two perturbative regimes:
 - a regime of **jet production**, where $k_{\perp} \sim k_{\parallel} \sim \omega \gg k_{\perp, \min}$ and emission probabilities scale like $w \sim \alpha_s(k_{\perp}) \ll 1$; and
 - a regime of **jet evolution**, where $k_{\perp, \min} \leq k_{\perp} \ll k_{\parallel} \leq \omega$ and therefore emission probabilities scale like $w \sim \alpha_s(k_{\perp}) \log^2 k_{\perp}^2 \gtrsim 1$.
- in jet production:
standard fixed-order perturbation theory
- in jet evolution regime,
perturbative parameter **not** α_s any more
but rather **towers** of $\exp [\alpha_s \log k_{\perp}^2 \log k_{\parallel}]$
- induces counting of **leading logarithms** (LL), $\alpha_s L^{2n}$,
next-to leading logarithms (NLL), $\alpha_s L^{2n-1}$, etc.

- consider spatio-temporal structure in classical QED case
 - assume a charge comes into existence at $t = 0$ with $v = 0$:
in its rest frame radial $\vec{E}$ spreads out in sphere $r' \leq t'$
 - assume charge moves with $v \rightarrow 1$ and Lorentz factor E/m :
then in lab frame field at radial distance $r_\perp$ will arrive at $t = \gamma t' = Er_\perp/m$
- translate to classical QCD:
 - light quarks with constituent mass $m \approx \Lambda_{\text{QCD}} \approx 1/R$
or $m = m_Q$ for heavy quarks

(assume here typical hadron radius R)

$$t^{(\text{had})} \approx \begin{cases} ER^2 & \text{for light quarks} \\ \frac{ER}{m_Q} & \text{for heavy quarks.} \end{cases}$$

- repeat exercise in quantum mechanics
- confining forces associated with gluons with $k \approx k_{\perp} \approx k_{\parallel} \approx 1/R \approx m$ in hadronic rest frame
- they emerge at (as before with Lorentz factor E/m)

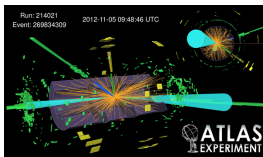
$$t^{(\text{had})} \approx \frac{k_{\parallel}}{k_{\perp}^2} \approx \frac{ER}{m} \approx k_{\parallel} R^2,$$

- demand hadronization time $\geq$ formation time:

$$t^{(\text{form})} \approx \frac{k_{\parallel}}{k_{\perp}^2} \leq k_{\parallel} R^2 \approx t^{(\text{had})}$$

- therefore $k_{\perp} \geq 1/R = \mathcal{O}(\text{few } \Lambda_{\text{QCD}})$
- breakdown of perturbative picture at scales/transverse momenta $\mathcal{O}(\text{few } \Lambda_{\text{QCD}})$
- transition to bound states (phase transition), “gluers” replace gluons
- no first-principle understanding: $\implies$ models

What are jets?



- pictorially: jets are collimated hadrons flying roughly in the same direction
 - may contain hundreds of hadrons (at LHC) = a lot of information
 - you don't always need fancy algorithms to "see" the jets
 - but you do to give them a precise and quantitative meaning

- jets are usually
- jets unavoidable in high-energy parton scattering
- related to the underlying perturbative dynamics
- perturbative picture typically well understood.
example: jet cross sections
- partons fragment through multiple parton emissions

(i.e. the quarks and gluons)

(soft & collinear divergences dominate, large logs overcome "small" coupling)

- fragmentation & hadronisation dominated by low $p_{\perp}$.
- therefore: hard partons result in collimated bunches of softer hadrons

Jet definitions

- purpose of a “jet algorithm” is to reduce the complexity of the final state, simplifying many hadrons to simpler objects that one can hope to calculate.
- jet algorithms map the momenta of the final state particles into the momenta of a certain number of jets:

$$\{p_i\} \xleftrightarrow{\text{jet algo}} \{j_l\}$$

- for this jet definition to be useful,
 - the rules must be the same, independent of the level of application: QCD resilience/robustness;
 - the rules must be complete, with no ambiguities;
 - the rules must be experimental feasible and theoretically sensible.
 $\implies$ **Infrared safety** crucial!
- most algorithms contain a resolution parameter, R , which controls the extension of the jet in phase space

(anti-) k_T jets

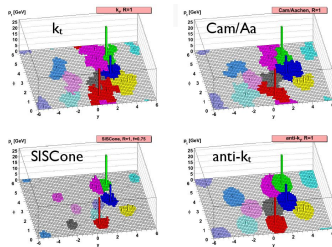
- main idea: sequential recombination
- distance between two objects i and j (and to the beam):

$$d_{ij} = \min\{p_{i,\perp}^{2D}, p_{j,\perp}^{2D}\} \frac{\cosh^2 \Delta\eta_{ij} - \cos^2 \Delta\phi_{ij}}{R^2}$$

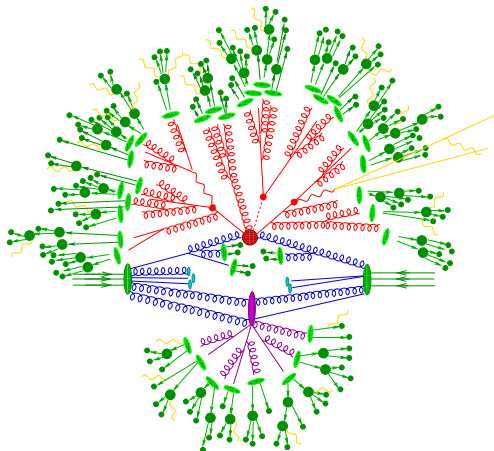
$$d_i = p_{i,\perp}^{2D}$$

- combine two objects with smallest d_{ij} , until smallest $d_{ij} > d_{\text{cut}}$.
- “cone-size” R , usually fixed to values such as $R = 0.4$
- value of D defines jet algorithm:

$$D = \begin{cases} -1 & \text{anti-}k_T \\ 0 & \text{Cambridge-Aachen} \\ 1 & k_T \end{cases}$$



Summary



QCD @ FIXED ORDER

Contents

- 3.a) Master formula for cross sections at hadron colliders
- 3.b) Example: W production at LO and NLO
- 3.c) Subtleties: scaling, giant K -factors and all that

Master Formula for Cross Sections at Hadron Colliders

$$\begin{aligned}
 \sigma_{2 \rightarrow n} &= \sum_{a,b} \int_0^1 dx_a dx_b \underbrace{f_{a/h_1}(x_a, \mu_F) f_{b/h_2}(x_b, \mu_F)}_{\text{parton distribution functions}} \hat{\sigma}_{ab \rightarrow n}(\mu_F, \mu_R) \\
 &= \sum_{a,b} \int_0^1 \underbrace{dx_a dx_b}_{\text{IS phase space}} \underbrace{f_{a/h_1}(x_a, \mu_F) f_{b/h_2}(x_b, \mu_F)}_{\text{parton distribution functions}} \\
 &\quad \times \underbrace{\frac{1}{2\hat{s}}}_{\text{incoming flux}} \times \underbrace{\int d\Phi_n}_{\text{FS phase space}} \underbrace{|\mathcal{M}_{ab \rightarrow n}|^2(\Phi_n; \mu_F, \mu_R)}_{\text{amplitude squared}}.
 \end{aligned}$$

- inspect N -particle final state phase space integral

$$d\Phi_N = \left[\prod_{i=1}^N \underbrace{\frac{d^4 p_i}{(2\pi)^4}}_{\text{Lorentz-invariant volume}} \underbrace{(2\pi)\delta(p_i^2 - m_i^2)}_{\text{particle on-shell}} \underbrace{\Theta(E_i)}_{\text{with positive energy}} \right] \\ \times \underbrace{(2\pi)^4 \delta^4 \left(x_a P_{h_1} + x_b P_{h_2} - \sum_{i=1}^N p_i \right)}_{\text{total 4-momentum conservation}}$$

- note: incoming parton momenta are given by Bjorken- x times hadron momenta:

$$p_{a,b} = x_{a,b} P_{h_{1,2}}$$

Example: W Production at Leading Order

- test case for the development of ideas: W production
- matrix element/amplitude for real W production:

$$\mathcal{M}_{u\bar{d} \rightarrow W^+} = -\frac{iV_{ud}g_W\delta_{ij}}{\sqrt{2}}\bar{d}_i(p_2)\gamma^\mu\frac{1-\gamma^5}{2}u_j(p_1)\epsilon_\mu^{(W)},$$

- CKM matrix element V_{ud} , weak coupling g_W (from Fermi constant), subscripts i, j indicate quark colours
- squared amplitude (summing/averaging over FS/IS internal d.o.f.)
internal d.o.f. = **polarisations**, **colours**, ...

$$\begin{aligned}\sum |\mathcal{M}|^2 &= \frac{3}{4 \cdot 9} \frac{|V_{ud}|^2 g_W^2}{2} \text{Tr} \left[\not{p}_2 \gamma^\mu \not{p}_1 \gamma^\nu \frac{1-\gamma^5}{2} \right] \left[-g_{\mu\nu} + \frac{Q_\mu Q_\nu}{m_W^2} \right] \\ &= \frac{|V_{ud}|^2 g_W^2}{12} Q^2 = \frac{|V_{ud}|^2 g_W^2}{12} m_W^2.\end{aligned}$$

- add in one-particle final state phase space

$$\begin{aligned}\hat{\sigma}_{u\bar{d} \rightarrow W^+}^{(\text{LO})} &= \frac{1}{2\hat{s}} \int \frac{d^4 p_W}{(2\pi)^4} (2\pi)^4 \delta^4(p_u + p_{\bar{d}} - p_W) (2\pi) \delta(p_W^2 - m_W^2) |\mathcal{M}|^2 \\ &= \frac{\pi \delta(\hat{s} - m_W^2)}{\hat{s}} |\mathcal{M}|^2 = \frac{\pi \delta(\hat{s} - m_W^2)}{\hat{s}} \frac{g_W^2 |V_{ud}|^2 m_W^2}{12}.\end{aligned}$$

- therefore, hadronic cross section:

$$\sigma^{(\text{LO})} = \int_0^1 dx_u dx_{\bar{d}} \sum_{u, \bar{d}} f_{u/h_1}(x_u, \mu_F) f_{\bar{d}/h_2}(x_{\bar{d}}, \mu_F) \hat{\sigma}_{u\bar{d} \rightarrow W^+}^{(\text{LO})}$$

- with the integral over the energy fractions rewritten as

$$dx_u dx_{\bar{d}} = \frac{d\hat{s}}{s} dy_W.$$

from c.m.-energy squared $\hat{s}$ and rapidity y

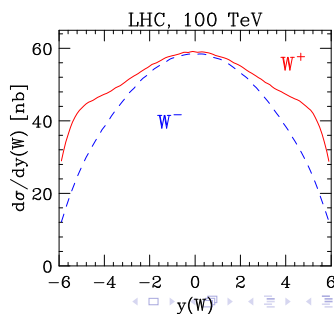
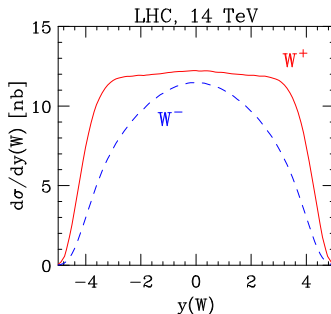
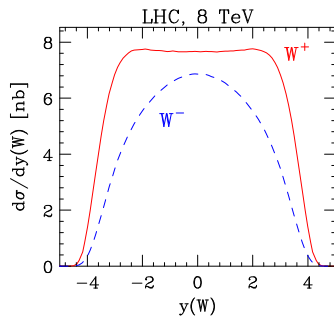
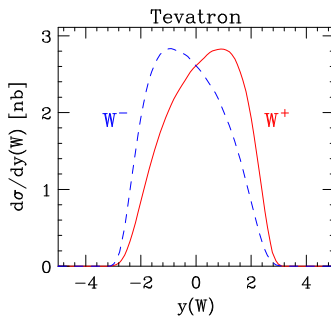
$$\hat{s} = x_u x_{\bar{d}} s \quad \text{and} \quad y_W = y_{\text{c.m.}} = \frac{1}{2} \log \frac{x_u}{x_{\bar{d}}}.$$

- limits on rapidity from $x_u = m_W^2/(sx_{\bar{d}}) \geq m_W^2/s$ since $x_{\bar{d}} \leq 1$:

$$|y_W| \leq y_{\text{max}} = \frac{1}{2} \log \frac{s}{m_W^2}.$$

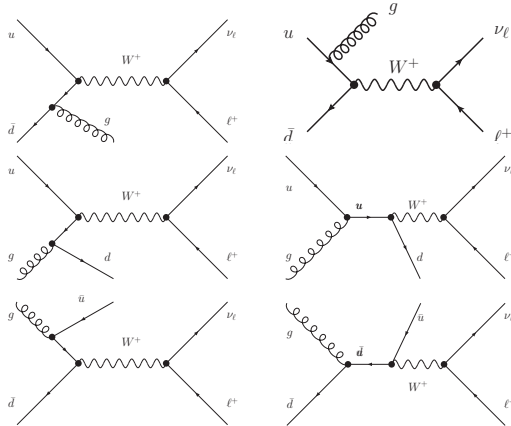
- therefore cross section given by

$$\sigma^{(\text{LO})} = \frac{\pi g_W^2 |V_{ud}|^2}{12s} \int_{-y_{\text{max}}}^{y_{\text{max}}} dy_W \sum_{u, \bar{d}} f_{u/h_1}(x_u, \mu_F) f_{\bar{d}/h_2}(x_{\bar{d}}, \mu_F).$$



Example: $W + j$ Production at Leading Order

- add additional quark or gluon to final state; Feynman diagrams:



- amplitudes (ignoring W -decays):

$$\mathcal{M}_{u\bar{d} \rightarrow gW^+} = \frac{ig_s g_W V_{ud}}{\sqrt{2}} \bar{v}_{d,i} \left[\gamma_\nu T_{ij}^a \frac{\not{p}_{\bar{d}} - \not{p}_g}{(p_{\bar{d}} - p_g)^2} \gamma_{\mu L} \right. \\ \left. + \gamma_{\mu L} \frac{\not{p}_u - \not{p}_g}{(p_u - p_g)^2} \gamma_\nu T_{ij}^a \right] u_{u,j} \epsilon_W^\mu \epsilon_g^{\nu,a},$$

$$\mathcal{M}_{ug \rightarrow dW^+} = \frac{ig_s g_W V_{ud}}{\sqrt{2}} \bar{u}_{d,i} \left[\gamma_\nu T_{ij}^a \frac{\not{p}_g - \not{p}_d}{(p_g - p_d)^2} \gamma_{\mu L} \right. \\ \left. + \gamma_{\mu L} \frac{\not{p}_u + \not{p}_g}{(p_u + p_g)^2} \gamma_\nu T_{ij}^a \right] u_{u,j} \epsilon_W^\mu \epsilon_g^{*\nu,a}$$

$$\mathcal{M}_{\bar{d}g \rightarrow \bar{u}W^+} = \frac{ig_s g_W V_{ud}}{\sqrt{2}} \bar{v}_{d,i} \left[\gamma_{\mu L} \frac{\not{p}_g - \not{p}_{\bar{u}}}{(p_g - p_{\bar{u}})^2} \gamma_\nu T_{ij}^a \right. \\ \left. + \gamma_\nu T_{ij}^a \frac{\not{p}_g + \not{p}_{\bar{d}}}{(p_g + p_{\bar{d}})^2} \gamma_{\mu L} \right] v_{u,j} \epsilon_W^\mu \epsilon_g^{*\nu,a}$$

- squaring them and using Mandelstam variables:

$$|\mathcal{M}|_{u\bar{d} \rightarrow gW^+}^2 = \frac{4\pi\alpha_s C_F g_W^2 |V_{ud}|^2}{12} \frac{\hat{t}^2 + \hat{u}^2 + 2m_W^2 \hat{s}}{\hat{t}\hat{u}}$$

and

$$|\mathcal{M}|_{ug \rightarrow dW^+}^2 = |\mathcal{M}|_{\bar{d}g \rightarrow \bar{u}W^+}^2 = \frac{4\pi\alpha_s T_R g_W^2 |V_{ud}|^2}{12} \frac{\hat{s}^2 + \hat{u}^2 + 2m_W^2 \hat{t}}{-\hat{s}\hat{u}}.$$

- used identity below for compactness ($ab \rightarrow 12$):

$$\hat{s} + \hat{t} + \hat{u} = m_a^2 + m_b^2 + m_1^2 + m_2^2,$$

- rewrite as **LO matrix element** times “**radiation part**”:

$$|\mathcal{M}|_{u\bar{d} \rightarrow gW^+}^2 = \frac{\hat{\sigma}_{u\bar{d} \rightarrow W^+}^{(\text{LO})}}{\pi} \cdot (4\pi\alpha_s C_F) \frac{\hat{t}^2 + \hat{u}^2 + 2m_W^2 \hat{s}}{\hat{t}\hat{u}}$$

$$|\mathcal{M}|_{ug \rightarrow dW^+}^2 = \frac{\hat{\sigma}_{u\bar{d} \rightarrow W^+}^{(\text{LO})}}{\pi} \cdot (4\pi\alpha_s T_R) \frac{\hat{s}^2 + \hat{u}^2 + 2m_W^2 \hat{t}}{-\hat{s}\hat{u}}.$$

- identifying, with c.m. angles θ^* , and for $u\bar{d} \rightarrow gW^+$

$$\hat{t} = -2p_u p_g = -2E_u E_g (1 - \cos \theta_{ug}^*)$$

$$\hat{u} = -2p_{\bar{d}} p_g = -2E_{\bar{d}} E_g (1 - \cos \theta_{\bar{d}g}^*).$$

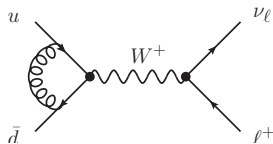
shows two possible divergences:

soft ($E_g \rightarrow 0$) and **collinear** ($\theta_g^* \rightarrow 0$)

- must regularise with cuts (**jet algorithm**)

Example: W Production at Next-To-Leading Order

- diagrams for $W + j$ above can act as
 - **LO** contributions to $W + j$, or as
 - **real corrections at NLO to inclusive W production**
- must add virtual corrections (loops)



$$\mathcal{M}_{u\bar{d} \rightarrow W^+}^{(1)} = \frac{g_W}{\sqrt{2}} g_s^2 \bar{v}_i(\bar{d}) \left\{ \mu^{4-D} \int \frac{d^D k}{(2\pi)^D} \frac{g^{\nu\rho} \delta^{ab}}{k^2} \right. \\ \left. \times \left[\gamma_\nu T_{ik}^a \frac{\not{p}_d + \not{k}}{(p_d + k)^2} \gamma^{\mu L} \frac{\not{p}_u - \not{k}}{(p_u - k)^2} \gamma_\rho T_{kj}^b \right] \right\} u_j(u) \epsilon_\mu(W^+)$$

- UV and IR divergent

- matrix element above to be multiplied with LO matrix element:

$$\hat{\sigma}^{(\text{NLO})} \sim \left| \mathcal{M}^{(0)} + \mathcal{M}^{(1)} \right|_{\mathcal{O}(\alpha_s)}^2 = \left| \mathcal{M}^{(0)} \right|^2 + 2 \left| \mathcal{M}^{(0*)} \mathcal{M}^{(1)} \right| + \mathcal{O}(\alpha_s^2)$$

- technology:
 - analytically continue to $D = 4 - 2\varepsilon$ dimensions
 - evaluate numerator (traces, contractions, etc.)
 - use Feynman trick(s) to map onto master integrals of the form

$$\mu^{4-D} \int \frac{d^D k}{(2\pi)^D} \frac{k^{2m}}{(k^2 + p^2)^n}$$

- evaluate integrals over Feynman parameters

- therefore, result for relevant contribution

$$2 \left| \mathcal{M}_{u\bar{d} \rightarrow W^+}^{(1*)} \mathcal{M}_{u\bar{d} \rightarrow W^+}^{(0)} \right| = \left| \mathcal{M}_{u\bar{d} \rightarrow W^+}^{(0)} \right|^2 \frac{\alpha_s}{2\pi} C_F \left(\frac{\mu^2}{Q^2} \right)^\varepsilon c_\Gamma \left(-\frac{2}{\varepsilon^2} - \frac{3}{\varepsilon} - 8 + \pi^2 \right)$$

with $Q^2 = (p_u + p_d)^2 = m_W^2$ and $c_\Gamma = 4\pi^\varepsilon / \Gamma(1 - \varepsilon)$

- note: finite term “8” is scheme-dependent

(scheme for evaluation of Dirac algebra/ numerator in D dimensions)

- note: UV divergences already renormalised

(this is explicit, when self-energies/wave function renormalisation is added, gauge invariance at work!)

- IR divergences in virtual part = poles in $1/\varepsilon$
- but also: IR divergences in real part, if $p_{\perp} \rightarrow 0$
(or, equivalently, if $\hat{t} \rightarrow 0$ or $\hat{u} \rightarrow 0$)
- Kinoshita–Lee–Nauenberg theorem guarantees that
IR divergences cancel for physically sensible observables
- examples for such observables:
total cross section σ , differential cross sections $d\sigma/d\mathcal{O}$ for observables $\mathcal{O}$
based on physical particles (jets, pions, ...)
- counter-examples:
number of emitted gluons, or their individual transverse momenta

- direct integration of two-body phase space in D dimensions:

$$\begin{aligned}
 \mu^{4-D} \int \frac{d^D k}{(2\pi)^D} \left| \mathcal{M}_{u\bar{d} \rightarrow W^+ g}^{(0)} \right|^2 &= \left| \mathcal{M}_{u\bar{d} \rightarrow W^+}^{(0)} \right|^2 \frac{\alpha_s}{2\pi} C_F \left(\frac{\mu^2}{Q^2} \right)^\epsilon \sigma \\
 &\times \left[\overbrace{\left(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{\pi^2}{3} \right) \delta(1-z)}^{\text{cancelled with virtual}} \right. \\
 &+ \left(\frac{4}{1-z} \log \frac{(1-z)^2}{z} \right)_+ - 2(1+z) \log \frac{(1-z)^2}{z} \\
 &\left. - \underbrace{\frac{2}{\epsilon} \frac{\mathcal{P}_{qq}^{(1)}(z)}{C_F}}_{\text{absorbed into PDF}} \right]
 \end{aligned}$$

- total result for gluon emission:

$$\begin{aligned}\hat{\sigma}_{u\bar{d}\rightarrow W^+}^{(\text{NLO})} &= \hat{\sigma}_{u\bar{d}\rightarrow W^+}^{(\text{LO})} \left\{ 1 + \frac{\alpha_s(\mu_R)}{2\pi} C_F \left[\left(\frac{4\pi^2}{3} - 8 \right) \delta(1-z) \right. \right. \\ &\quad + \left(\frac{4}{1-z} \log \frac{(1-z)^2}{z} \right)_+ - 2(1+z) \log \frac{(1-z)^2}{z} \\ &\quad \left. \left. - 2 \frac{\mathcal{P}_{qq}^{(1)}(z)}{C_F} \log \frac{\mu_F^2}{Q^2} \right] \right\}\end{aligned}$$

- add in gluon initial states:

$$\begin{aligned}\hat{\sigma}_{ug\rightarrow dW^+}^{(\text{NLO})} &= \hat{\sigma}_{u\bar{d}\rightarrow W^+}^{(\text{LO})} \\ &= \frac{\alpha_s(\mu_R)}{2\pi} T_R \left\{ -\frac{\mathcal{P}_{qg}^{(1)}(z)}{T_R} \log \frac{\mu_F^2}{m_W^2} \right. \\ &\quad \left. + \left[z^2 + (1-z)^2 \right] \log \frac{(1-z)^2}{z} + \frac{1}{2}(1-z)(1+7z) \right\}\end{aligned}$$

- in general, cannot perform phase space integral for real emission contribution $\mathcal{R}_N$ analytically
- must resort to Monte Carlo integration for N -particle Born phase space Φ_B and $(N + 1)$ -particle real emission phase space $\Phi_{\mathcal{R}}$
(note: have absorbed PDFs etc. into phase space)
- but: cannot MC integrate in D dimensions
- solution: identify singularities in real contribution
- subtract them from real contribution with term $\mathcal{S}$
- have to add it back to virtual $\mathcal{V}$

- therefore structure of NLO calculation for N -body production

$$\begin{aligned} d\sigma &= d\Phi_B \mathcal{B}_N(\Phi_B) + d\Phi_B \mathcal{V}_N(\Phi_B) + d\Phi_{\mathcal{R}} \mathcal{R}_N(\Phi_{\mathcal{R}}) \\ &= d\Phi_B \left(\mathcal{B}_N + \mathcal{V}_N + \mathcal{I}_N^{(S)} \right) + d\Phi_{\mathcal{R}} (\mathcal{R}_N - \mathcal{S}_N) \end{aligned}$$

- assumed here that $\Phi_{\mathcal{R}} = \Phi_B \otimes \Phi_1$

$$\int d\Phi_1 \mathcal{S}_N(\Phi_B \otimes \Phi_1) = \mathcal{I}_N^{(S)}(\Phi_B)$$

- nice feature: **universal IR behaviour of gauge theories**
 → can write process independent subtraction kernels as

$$\begin{aligned} \mathcal{S}_N(\Phi_B \otimes \Phi_1) &= \mathcal{B}_N(\Phi_B) \otimes \mathcal{S}_1(\Phi_B \otimes \Phi_1) \\ \mathcal{I}_N^{(S)}(\Phi_B \otimes \Phi_1) &= \mathcal{B}_N(\Phi_B) \otimes \mathcal{I}_1^{(S)}(\Phi_B) \end{aligned}$$

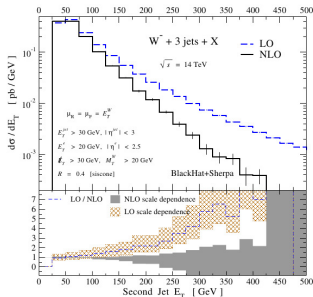
with **universal** $\mathcal{S}_1(\Phi_B \otimes \Phi_1)$ and $\mathcal{I}_1^{(S)}(\Phi_B)$

Subtleties: scaling, giant K -factors and all that

- which order is which? can be tricky ...
- higher-order reduce scale dependence
however: effect still may be large!
⇒ how to choose the “right” scale?
- generic idea (borrowed from later):
analyse process in terms of “parton splittings”
 - choose renormalisation scale as geometric mean of scales
 - choose factorisation scale from largest deflection of colour flow
- K -factor as ratio to LO cross section
of course, just a number for total cross section
but not necessarily flat for distributions!
- occurrence of “giant” K -factors when new channels open

Choices, choices, choices

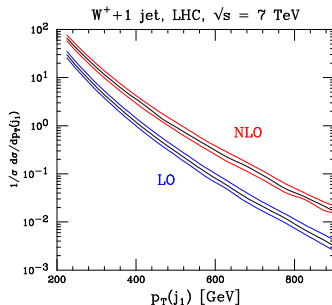
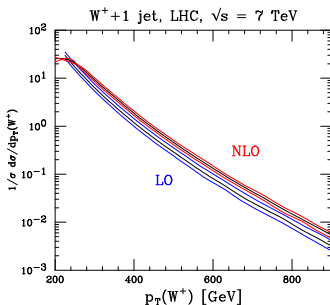
- common lore: NLO calculations reduce scale uncertainties
- this is, in general, true. however:
unphysical scale choices will yield unphysical results



- more ways of botching it at higher orders with more legs

Example: giant K -factor in $W + j$

- manifestation in differential jet rates



- $p_{\perp}^{(W)}$ relatively stable, 1st jet not
- hints at large impact of 2-parton configurations

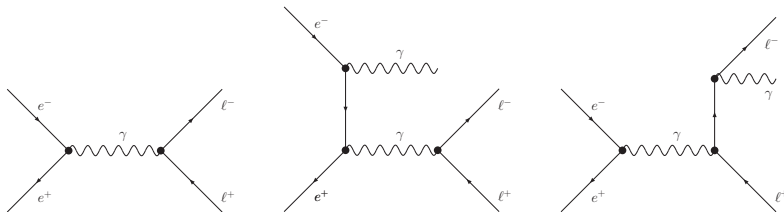
QCD TO ALL ORDERS

Contents

- 4.a) $p_{\perp}$ -spectrum of emissions
- 4.b) Q_T & threshold resummation in QCD
- 4.c) Jet rates in e^-e^+ annihilations
- 4.d) Parton showers
- 4.e) Summary

A QED example: $p_{\perp}^{\mu\mu}$ in $e^-e^+ \rightarrow \mu^-\mu^+$

- Feynman diagrams:



- ignore emission of FS leptons
- treat problem as $e^-e^+ \rightarrow \gamma^*\gamma$ with virtual mass Q

- matrix element squared for $e^-e^+ \rightarrow \gamma^*\gamma$ with photon virtual mass Q

$$|\mathcal{M}|_{e^-e^+ \rightarrow \gamma\gamma^*}^2 = 32\pi^2\alpha^2 \left[\frac{\hat{t}}{\hat{u}} + \frac{\hat{u}}{\hat{t}} + \frac{2Q^2(Q^2 - \hat{t} - \hat{u})}{\hat{t}\hat{u}} \right],$$

- LO cross section for $e^-e^+ \rightarrow \gamma^*$: $\hat{\sigma}_0 = 4\pi\alpha^2/(3\hat{s})$, therefore

$$\frac{d\hat{\sigma}_R}{d\hat{t}dQ^2} = \frac{\alpha}{2\pi} \frac{\hat{\sigma}_0}{Q^2} \left[\frac{\hat{t}}{\hat{u}} + \frac{\hat{u}}{\hat{t}} + \frac{2Q^2(Q^2 - \hat{t} - \hat{u})}{\hat{t}\hat{u}} \right]$$

- note symmetry under $\hat{t} \longleftrightarrow \hat{u}$
- remember Mandelstam identity, here $\hat{s} + \hat{t} + \hat{u} = Q^2$

$p_{\perp}$ spectrum in single- and all-emission approximation

- introduce $Q_{\perp}$ the transverse momentum of the virtual photon and assume low- $Q_{\perp}$ limit, $Q^2 \gg Q_{\perp}^2 \rightarrow 0$
- symmetry under $\hat{t} \longleftrightarrow \hat{u}$, consider $\hat{t} \rightarrow 0$ only:

$$Q_{\perp}^2 \approx -\hat{t} \rightarrow 0 \quad \text{and} \quad \hat{u} \xrightarrow{\hat{t} \rightarrow 0} Q^2 - \hat{s}$$

- integrate over all masses Q^2 , with $4m_{\ell}^2 \leq Q^2 \leq \hat{s} - 2\sqrt{\hat{s}} Q_{\perp}$ results in (approximatively)

$$\begin{aligned} \frac{d\hat{\sigma}_R}{dQ_{\perp}^2} &= \hat{\sigma}_0 \frac{\alpha}{\pi} \frac{1}{Q_{\perp}^2} \left[\log \frac{\hat{s}}{Q_{\perp}^2} + \mathcal{O}(1) \right] \\ \longrightarrow d\hat{\sigma}_R &\approx \hat{\sigma}_0 \frac{\alpha}{\pi} \frac{dQ_{\perp}^2}{Q_{\perp}^2} \log \frac{\hat{s}}{Q_{\perp}^2} \end{aligned}$$

(this is a double-logarithmic spectrum $\rightarrow$ DL approximation, DLLA)

- calculate cross section for $Q_{\perp} > Q_{\perp,\min}$:

$$d\hat{\sigma}_R(> Q_{\perp,\min}) \approx \hat{\sigma}_0 \frac{\alpha}{\pi} \int_{Q_{\perp,\min}^2}^{\hat{s}} \frac{dQ_{\perp}^2}{Q_{\perp}^2} \log \frac{\hat{s}}{Q_{\perp}^2} \approx \hat{\sigma}_0 \frac{\alpha}{2\pi} \log^2 \frac{\hat{s}}{Q_{\perp,\min}^2}$$

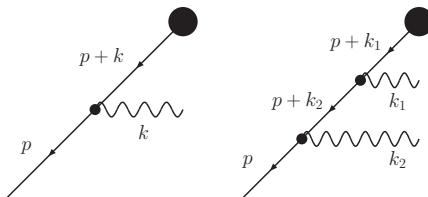
- but total cross section for γ^* production finite:

$$\begin{aligned} \sigma_{e^-e^+ \rightarrow \gamma^*} &= \hat{\sigma}_0 \left(1 + \mathcal{O}(\alpha) \cdot \text{non-logarithmic} \right) \\ &= \int_0^{Q_{\perp,\min}^2} dQ_{\perp}^2 \frac{d\hat{\sigma}_R + d\hat{\sigma}_V}{dQ_{\perp}^2} + \int_{Q_{\perp,\min}^2}^{\hat{s}} dQ_{\perp}^2 \frac{d\hat{\sigma}_R}{dQ_{\perp}^2} \end{aligned}$$

- this implies that

$$\int_0^{Q_{\perp,\min}^2} dQ_{\perp}^2 \frac{d\hat{\sigma}_R + d\hat{\sigma}_V}{dQ_{\perp}^2} \approx \hat{\sigma}_0 \left(1 - \frac{\alpha}{2\pi} \log^2 \frac{\hat{s}}{Q_{\perp,\min}^2} \right) = \hat{\sigma}_0 \left(1 + \Sigma^{(1)} \right)$$

- consider many emissions



- expression for single emission in soft limit

$$e \epsilon_\mu(k) \bar{u}(p) \gamma^\mu \frac{\not{p} + \not{k}}{(p+k)^2} \dots \approx e \bar{u}(p) \frac{p \cdot \epsilon}{p \cdot k} \dots,$$

becomes for n emissions

$$\frac{e^n}{n!} \frac{p \cdot \epsilon_1}{p \cdot k_1} \frac{p \cdot \epsilon_2}{p \cdot k_2} \dots \frac{p \cdot \epsilon_n}{p \cdot k_n} \dots$$

- therefore, n -photon contribution $\Sigma^{(n)}$

$$\Sigma^{(n)}(p_{\perp}^2) = \frac{1}{n!} \left[\frac{\alpha}{\pi} \int_0^{p_{\perp}^2} d \log Q_{\perp}^2 \log \frac{\hat{s}}{Q_{\perp}^2} \right]^n = \frac{1}{n!} \left[-\frac{\alpha}{2\pi} \log^2 \frac{\hat{s}}{p_{\perp}^2} \right]^n,$$

- summing all orders:

$$\Sigma(Q_{\perp}^2) = \sum_{n=0}^{\infty} \frac{1}{n!} \left[-\frac{\alpha}{2\pi} \log^2 \frac{\hat{s}}{Q_{\perp}^2} \right]^n = \exp \left[-\frac{\alpha}{2\pi} \log^2 \frac{\hat{s}}{Q_{\perp}^2} \right]$$

- normalised differential cross section with respect to the transverse momentum squared in DLLA

$$\frac{1}{\hat{\sigma}_0} \frac{d\hat{\sigma}}{dQ_{\perp}^2} = -\frac{d}{dQ_{\perp}^2} \Sigma(Q_{\perp}^2) = \frac{\alpha}{\pi} \frac{1}{Q_{\perp}^2} \log \frac{\hat{s}}{Q_{\perp}^2} \exp \left[-\frac{\alpha}{2\pi} \log^2 \frac{\hat{s}}{Q_{\perp}^2} \right].$$

- Σ is the **Sudakov form factor**

(a no-emission probability, we will see much more of it!)

Impact parameter space

- problem in reasoning so far:
every emitted photon gives a transverse momentum $k_{\perp}$, total transverse momentum comes as the sum over all of them:

$$\vec{Q}_{\perp} = - \sum_i \vec{k}_{\perp,i}$$

- rewrite as δ function and Fourier transform:

$$\delta^2 \left(\vec{Q}_{\perp} + \sum_{i=0}^n \vec{k}_{\perp,i} \right) = \frac{1}{2\pi} \int d^2 b_{\perp} \exp \left[i \vec{b}_{\perp} \cdot \left(\vec{Q}_{\perp} + \sum_{i=0}^n \vec{k}_{\perp,i} \right) \right].$$

- rewrite single emission terms in impact parameter space:

$$\begin{aligned}\nu(k_{\perp,i}) &= \frac{\alpha}{\pi} \frac{1}{k_{\perp,i}^2} \log \frac{\hat{s}}{k_{\perp,i}^2} \\ &\longleftrightarrow \frac{1}{2\pi} \int d^2 k_{\perp,i} \exp \left[-i \vec{b}_{\perp,i} \cdot \vec{k}_{\perp,i} \right] \nu(k_{\perp,i}) = \nu(b_{\perp,i})\end{aligned}$$

- summing over all emissions

$$\frac{d\hat{\sigma}^{(\text{all})}}{dQ_{\perp}^2} = \frac{\hat{\sigma}_0}{\pi} \int db_{\perp}^2 J_0(b_{\perp} Q_{\perp}) \exp \left[-\frac{1}{2\pi} \frac{dk_{\perp}^2}{k_{\perp}^2} \frac{\alpha(k_{\perp}^2)}{\pi} \log \frac{\hat{s}}{k_{\perp,i}^2} \right]$$

after δ -function has been taken care of

Q_T resummation in QCD

- apply the same reasoning on $p_{\perp}^{(W)}$:

$$\frac{d\sigma_{AB \rightarrow W^+}}{dy dQ_{\perp}^2} = \sum_{ij} \frac{\hat{\sigma}_{ij \rightarrow W^+}^{(LO)}}{2\pi} \left\{ \int \frac{db_{\perp}^2}{\pi} \left[J_0(\vec{b}_{\perp} \cdot \vec{Q}_{\perp}) \tilde{W}_{ij}(b; Q, x_A, x_B) \right] \right\}$$

with

$$\hat{\sigma}_{ij \rightarrow W^+}^{(LO)} = \frac{4\pi^2 \alpha |V_{ij}|^2}{12 \sin^2 \theta_W m_W^2}$$

- function $\tilde{W}_{ij}(b; Q, x_A, x_B)$ incorporates resummation with:
Sudakov form factor, PDFs, potential higher orders
- may have to supplement (hard) real emission corrections

- at LL accuracy (and with $k_{\perp, \min} = 1/b_{\perp}$)

$$\tilde{W}_{ij}(b; Q, x_A, x_B) =$$

$$f_{i/A}\left(x_A, \frac{1}{b_{\perp}}\right) f_{j/B}\left(x_B, \frac{1}{b_{\perp}}\right) \exp\left[-\int_{1/b_{\perp}^2}^{Q^2} \frac{dk_{\perp}^2}{k_{\perp}^2} \left(A(k_{\perp}^2) \log \frac{Q^2}{k_{\perp}^2}\right)\right]$$

(remember $x_{A,B} = M_X / \sqrt{s} e^{\pm y}$)

- must regularise large- $b_{\perp}$ /low- $k_{\perp}$ regime:
add non-perturbative dampening function (such as $\exp(-b_{\perp}^2)$)

(will ignore this added term)

- full beauty result:

$$\frac{d\sigma_{AB \rightarrow W^+}}{dy dQ_{\perp}^2} = \sum_{ij} \frac{\hat{\sigma}_{ij \rightarrow W^+}^{(LO)}}{2\pi} \left\{ \int \frac{db_{\perp}^2}{\pi} \left[J_0(\vec{b}_{\perp} \cdot \vec{Q}_{\perp}) \tilde{W}_{ij}(b; Q, x_A, x_B) \right] \right. \\ \left. + Y_{ij \rightarrow W^+}(Q_{\perp}; Q, x_A, x_B) \right\}$$

- function Y as difference of full real correction and real emission parts coming from $\tilde{W}_{ij}$ (“matching”)

- anatomy of resummation part

$$\begin{aligned}
 & \tilde{W}_{ij}(b; Q, x_A, x_B) \\
 = & \sum_{ab} \left\{ \int_{x_A}^1 \frac{d\xi_A}{\xi_A} \int_{x_B}^1 \frac{d\xi_B}{\xi_B} \left[f_{a/A} \left(\xi_A, \frac{1}{b_\perp} \right) f_{b/B} \left(\xi_B, \frac{1}{b_\perp} \right) \right. \right. \\
 & \quad \underbrace{\times C_{ia} \left(\frac{x_A}{\xi_A}, b_\perp; \mu \right) C_{jb} \left(\frac{x_B}{\xi_B}, b_\perp; \mu \right)}_{\text{corrections to factorisation}} \underbrace{H_{ab} \left(\frac{x_A}{\xi_A}, \frac{x_B}{\xi_B}; \mu \right)}_{\text{genuine loop corrections}} \\
 & \quad \left. \times \exp \left[- \underbrace{\int_{1/b_\perp}^{Q^2} \frac{dk_\perp^2}{k_\perp^2} \left(A(k_\perp^2) \log \frac{Q^2}{k_\perp^2} + B(k_\perp^2) \right)}_{\text{Sudakov form factor}} \right] \right\}
 \end{aligned}$$

- terms A , B , C , and H all with expansion in α_s , e.g.

$$A(\mu) = \sum_N \left(\frac{\alpha_s(\mu)}{2\pi} \right)^N A^{(N)} \dots$$

- process-independent are the $A^{(i)}$ and

$$A_{q,g}^{(1)} = 2C_{F,A}$$

$$A_{q,g}^{(2)} = 2C_{F,A}K = 2C_{q,g} \left[C_A \left(\frac{67}{18} - \frac{\pi^2}{6} \right) - \frac{10}{9} T_R n_f \right]$$

$$B_{q,g}^{(1)} = -3C_F, \quad -2\beta_0$$

$$C_{ia}^{(0)} \left(z, \frac{1}{b_{\perp}}, \mu_F^2 \right) = \delta_{ia} \delta(1-z)$$

$$C_{ia}^{(1)} \left(z, \frac{1}{b_{\perp}}, \mu_F^2 \right) = P_{ia}^{(1)} \log \frac{b_0^2}{b_{\perp}^2 \mu_F^2} - P_{ia}^{\epsilon}(z) + \delta_{ia} \delta(1-z) C_a \frac{\pi^2}{6}.$$

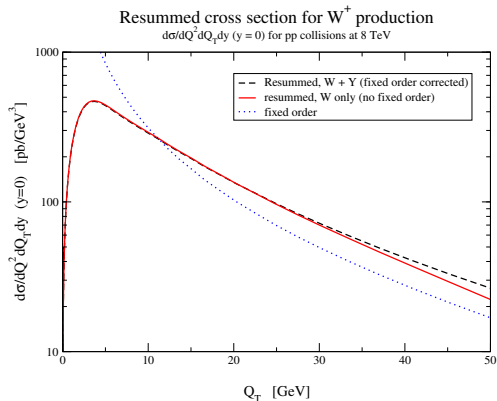
- logarithmic accuracy:

accuracy	terms included
LL	$A^{(1)}$
NLL	$A^{(2)}, B^{(1)}, C^{(1)}$
NNLL	$A^{(3)}, B^{(2)}, C^{(2)}$

- note: log-counting differs by community

(I adopt counting of Collins–Soper–Sterman and Catani–deFlorian–Grazzini)

- example result: interplay of fixed order and resummation



- note: parton shower will act similar to Q_T resummation

Threshold resummation in QCD

- another source of large logarithms: **production thresholds**
- produce heavy system (singlet) with mass Q , add multiple soft gluon emissions to “hit” threshold Q from above,
 $\sim 1/(1-z)_+$ from splitting function
- integration over splitting yields logarithms of the form

$$\alpha_s^n \left[\frac{\log^k(1-z)}{1-z} \right]_+.$$

- interesting limit: large logs from “squeezed” phase space, gluon emissions do not change kinematics of system

- log-enhancement of cross section encoded in $W_{ij}^{(\text{thres})}$:

$$\frac{d\sigma_{AB \rightarrow X}}{dQ^2} = \int_0^1 d\tau \int_0^1 dx_i dx_j f_{i/A}(x_i, \mu_F) f_{j/B}(x_j, \mu_F) \delta\left(\tau - \frac{Q^2}{S x_i x_j}\right) \cdot W_{ij}^{(\text{thres})}(\tau, Q, \mu_R, \mu_F),$$

- relevant limit $\tau = Q^2/\hat{s} \rightarrow 1$
- usually discussed after Mellin transform, therefore only qualitatively
 - important: no phase space for hard collinear emissions,
 - therefore completely driven by soft eikonal, identical A's!!!!

$$\mathbf{M}_N \left[W_{ij}^{(\text{thres})} \right]^{(1)} = \frac{4C_F}{2\pi} \int_0^1 dz \frac{z^N - 1}{1 - z} \int_{Q^2}^{(1-z)Q^2} \frac{dq^2}{q^2} \alpha_s((1-z)q^2),$$

- subleading terms start at $\mathcal{O}(\alpha_s^2)$
- same choice of scale, $k_{\perp}^2 = (1-z)q^2$

Resummed jet rates in e^+e^- -annihilations

- use Durham jet measure ($k_{\perp}$ -type):

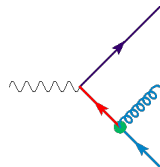
$$k_{\perp,ij}^2 = 2\min(E_i^2, E_j^2)(1 - \cos\theta_{ij}) > Q_{\text{jet}}^2$$

- jets given by $k_{\perp} \geq Q_{\text{jet}}$
- remember probabilistic interpretation of Sudakov form factor

$$\begin{aligned}\mathcal{P}_{\text{no emission}} &= \Delta_{q,g}(Q, Q_0) \\ &= \exp \left[- \int_{Q_0^2}^{Q^2} \frac{dq_{\perp}^2}{q_{\perp}^2} \frac{C_{F,A}\alpha_s(q_{\perp}^2)}{2\pi} \left(A \log \frac{Q^2}{q_{\perp}^2} + B \right) \right]\end{aligned}$$

- jet rates from no-emission probabilities

- evaluate 2- and 3-jet rate
- for example: 2-jet rate given by probs of no emission above Q_{jet} for either quark



$$\mathfrak{Re}_2(Q_{\text{jet}}) = \left[\Delta_q(E_{\text{c.m.}}^2, Q_{\text{jet}}^2) \right]^2$$

$$\mathfrak{Re}_3(Q_{\text{jet}}) = 2\Delta_q(E_{\text{c.m.}}^2, Q_{\text{jet}}^2) \cdot \int \frac{dq_\perp^2}{q_\perp^2} \left[\Gamma_q(q_\perp^2) \Delta_q(E_{\text{c.m.}}^2, q_\perp^2) \Delta_q(q_\perp^2, Q_{\text{jet}}^2) \Delta_g(q_\perp^2, Q_{\text{jet}}^2) \right]$$

- used “integrated splitting function”

$$\Gamma_{q,g}(q_\perp^2) = \frac{C_{F,A}\alpha_s(q^2)}{2\pi} \left(A \log \frac{Q^2}{q^2} + B \right)$$

An analogy: Radioactive decays

- consider radioactive decay of an unstable isotope with half-life τ .

(and ignore factors of $\ln 2$.)

- “survival” probability after time t is given by

$$\mathcal{S}(t) = \mathcal{P}_{\text{nodec}}(t) = \exp[-t/\tau]$$

(note “unitarity relation”: $\mathcal{P}_{\text{dec}}(t) = 1 - \mathcal{P}_{\text{nodec}}(t)$.)

- probability for an isotope to decay at time t :

$$\frac{d\mathcal{P}_{\text{dec}}(t)}{dt} = -\frac{d\mathcal{P}_{\text{nodec}}(t)}{dt} = \frac{1}{\tau} \exp(-t/\tau)$$

- now: connect half-life with width $\Gamma = 1/\tau$.
- probability for isotope decay at any fixed time t determined by Γ .

- spice things up now: add time-dependence, $\Gamma = \Gamma(t')$
- rewrite

$$\Gamma t \longrightarrow \int_0^t dt' \Gamma$$

- decay-probability at a given time t is given by

$$\frac{d\mathcal{P}_{\text{dec}}(t)}{dt} = \Gamma(t) \exp \left[- \int_0^t dt' \Gamma(t') \right] = \Gamma(t) \mathcal{P}_{\text{nodec}}(t)$$

(unitarity strikes again: $d\mathcal{P}_{\text{dec}}(t)/dt = -d\mathcal{P}_{\text{nodec}}(t)/dt$.)

- interpretation of l.h.s.:
 - first term is for the actual decay to happen.
 - second term is to ensure that no decay before t
 $\implies$ conservation of probabilities.
the exponential is - of course - called the **Sudakov form factor**.

The pattern of QCD radiation

- a detour: Altarelli-Parisi equation, once more
- AP describes the scaling behaviour of the parton distribution function

(which depends on Bjorken-parameter and scale Q^2)

$$\frac{dq(x, Q^2)}{d \ln Q^2} = \int_x^1 \frac{dy}{y} \left[\alpha_s(Q^2) P_q(x/y) \right] q(y, Q^2)$$

- term in square brackets determines the probability that the parton emits another parton at scale Q^2 and Bjorken-parameter y

(after the splitting, $x \rightarrow yx + (1 - y)x$.)

- driving term: Splitting function $P_q(x)$
important property: universal, process independent

Rederiving the splitting functions

- differential cross section for gluon emission in $e^+e^- \rightarrow \text{jets}$

$$\frac{d\sigma_{ee \rightarrow 3j}}{dx_1 dx_2} = \sigma_{ee \rightarrow 2j} \frac{C_F \alpha_s}{\pi} \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$$

singular for $x_{1,2} \rightarrow 1$.

- rewrite with opening angle θ_{qg} and gluon energy fraction $x_3 = 2E_g/E_{\text{c.m.}}$:

$$\frac{d\sigma_{ee \rightarrow 3j}}{d \cos \theta_{qg} dx_3} = \sigma_{ee \rightarrow 2j} \frac{C_F \alpha_s}{\pi} \left[\frac{2}{\sin^2 \theta_{qg}} \frac{1 + (1-x_3)^2}{x_3} - x_3 \right]$$

singular for $x_3 \rightarrow 0$ ("soft"), $\sin \theta_{qg} \rightarrow 0$ ("collinear").

- re-express collinear singularities

$$\begin{aligned}\frac{2d\cos\theta_{qg}}{\sin^2\theta_{qg}} &= \frac{d\cos\theta_{qg}}{1-\cos\theta_{qg}} + \frac{d\cos\theta_{qg}}{1+\cos\theta_{qg}} \\ &= \frac{d\cos\theta_{qg}}{1-\cos\theta_{qg}} + \frac{d\cos\theta_{\bar{q}g}}{1-\cos\theta_{\bar{q}g}} \approx \frac{d\theta_{qg}^2}{\theta_{qg}^2} + \frac{d\theta_{\bar{q}g}^2}{\theta_{\bar{q}g}^2}\end{aligned}$$

- independent evolution of two jets (q and $\bar{q}$)

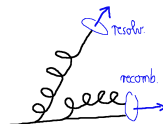
$$d\sigma_{ee\rightarrow 3j} \approx \sigma_{ee\rightarrow 2j} \sum_{j\in\{q,\bar{q}\}} \frac{C_F\alpha_s}{2\pi} \frac{d\theta_{jg}^2}{\theta_{jg}^2} P(z) ,$$

- note: same form for any $t \propto \theta^2$:
- transverse momentum $k_{\perp}^2 \approx z^2(1-z)^2 E^2 \theta^2$
- invariant mass $q^2 \approx z(1-z) E^2 \theta^2$

$$\frac{d\theta^2}{\theta^2} \approx \frac{dk_{\perp}^2}{k_{\perp}^2} \approx \frac{dq^2}{q^2}$$

- parametrisation-independent observation:
(logarithmically) divergent expression for $t \rightarrow 0$.
- practical solution: cut-off Q_0^2 .
 $\implies$ divergence will manifest itself as $\log Q_0^2$.
- similar for $P(z)$: divergence for $z \rightarrow 0$ cured by cut-off.

- what is a parton?
collinear pair/soft parton recombine!
- introduce resolution criterion $k_{\perp} > Q_0$.



- combine virtual contributions with unresolvable emissions:
cancels infrared divergences $\implies$ finite at $\mathcal{O}(\alpha_s)$

(Kinoshita-Lee-Nauenberg, Bloch-Nordsieck theorems)

- unitarity: probabilities add up to one
 $\mathcal{P}(\text{resolved}) + \mathcal{P}(\text{unresolved}) = 1$.



- the Sudakov form factor, once more
- differential probability for emission between q^2 and $q^2 + dq^2$:

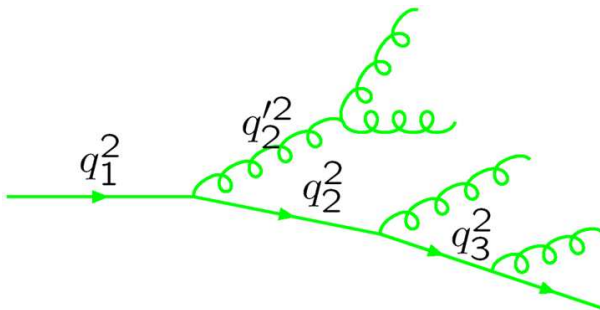
$$d\mathcal{P} = \frac{\alpha_s}{2\pi} \frac{dq^2}{q^2} \int_{z_{\min}}^{z_{\max}} dz P(z) =: dq^2 \Gamma(q^2)$$

- from radioactive example: evolution equation for Δ

$$-\frac{d\Delta(Q^2, q^2)}{dq^2} = \Delta(Q^2, q^2) \frac{d\mathcal{P}}{dq^2} = \Delta(Q^2, q^2) \Gamma(q^2)$$

$$\Rightarrow \Delta(Q^2, q^2) = \exp \left[- \int_{q^2}^{Q^2} dk^2 \Gamma(k^2) \right]$$

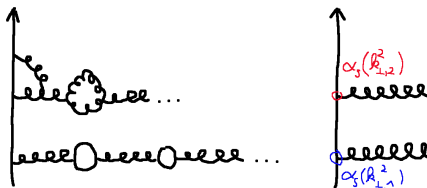
- maximal logs if emissions ordered
- impacts on radiation pattern: in each emission t becomes smaller



$$q_1^2 > q_2^2 > q_3^2, q_1^2 > q_2'^2$$

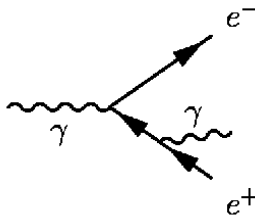
Quantum improvements

- improvement: inclusion of various quantum effects
- trivial: effect of summing up higher orders (loops) $\alpha_s \rightarrow \alpha_s(k_{\perp}^2)$



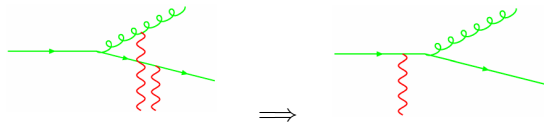
- much faster parton proliferation, especially for small $k_{\perp}^2$.
- avoid Landau pole: $k_{\perp}^2 > Q_0^2 \gg \Lambda_{\text{QCD}}^2 \implies Q_0^2 = \text{physical parameter.}$

- soft limit for single emission also universal
- problem: soft gluons come from all over (not collinear!)
quantum interference? still independent evolution?
- answer: not quite independent.
- consider case in QED



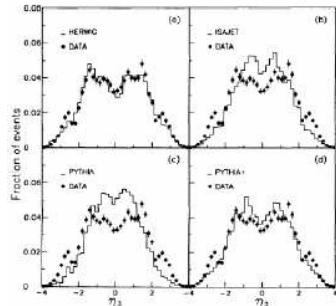
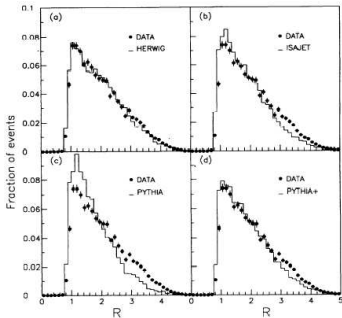
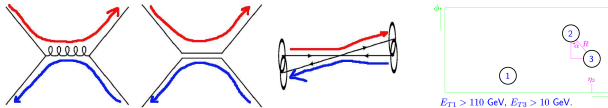
- assume photon into e^+e^- at θ_{ee} and photon off electron at θ
photon momentum denoted as k
- energy imbalance at vertex: $k_{\perp}^{\gamma} \sim k_{\parallel} \theta$, hence $\Delta E \sim k_{\perp}^2/k_{\parallel} \sim k_{\parallel} \theta^2$.
- formation time for photon emission: $\Delta t \sim 1/\Delta E \sim k_{\parallel}/k_{\perp}^2 \sim 1/(k_{\parallel} \theta^2)$.
- ee-separation: $\Delta b \sim \theta_{ee} \Delta t$
- must be larger than transverse wavelength of photon:
 $\theta_{ee}/(k_{\parallel} \theta^2) > 1/k_{\perp} = 1/(k_{\parallel} \theta)$
- thus: $\theta_{ee} > \theta$ must be satisfied for photon to form
- angular ordering as manifestation of quantum coherence

- pictorially:



gluons at large angle from combined colour charge!

- experimental manifestation:
 ΔR of 2nd & 3rd jet in multi-jet events in pp-collisions



Summary

- different resummation techniques for different situations
- common denominator: going to soft/collinear limits of parton emission phase space
- replacing perturbative parameter α_s with $\alpha_s L^2$ or similar
- important object: **Sudakov form factor**

(identified as no-emission probability)

- will put perturbative technology into Monte Carlo tomorrow

PRECISION MC

Contents

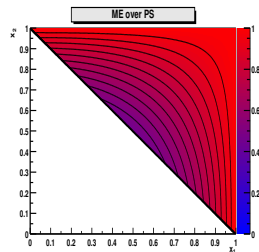
- 5.a) Matrix element corrections
- 5.b) Matching
- 5.c) Merging
- 5.d) Summary

Matrix element corrections

- parton shower ignores interferences typically present in matrix elements
- pictorially

$$\begin{aligned}
 \text{ME} &: \left| \text{diagram 1} + \text{diagram 2} \right|^2 \\
 \text{PS} &: \left| \text{diagram 1} \right|^2 + \left| \text{diagram 2} \right|^2
 \end{aligned}$$

The diagrams show a green wavy line (photon) interacting with a blue line (quark) via a red wavy line (gluon). In the ME case, the two diagrams are summed before squaring. In the PS case, each diagram is squared separately and then summed.



- form many processes $\mathcal{R}_N < \mathcal{B}_N \times \mathcal{K}_N$
- typical processes: $q\bar{q}' \rightarrow V$, $e^-e^+ \rightarrow q\bar{q}$, $t \rightarrow bW$
- practical implementation: shower with usual algorithm, but reject first/hardest emissions with probability $\mathcal{P} = \mathcal{R}_N / (\mathcal{B}_N \times \mathcal{K}_N)$

- analyse **first** emission, given by

$$d\sigma_B = d\Phi_N \mathcal{B}_N(\Phi_N)$$

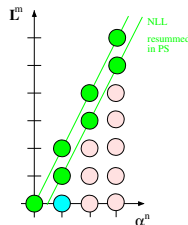
$$\cdot \left\{ \Delta_N^{(\mathcal{R}/\mathcal{B})}(\mu_N^2, t_0) + \int_{t_0}^{\mu_N^2} d\Phi_1 \left[\frac{\mathcal{R}_N(\Phi_N \times \Phi_1)}{\mathcal{B}_N(\Phi_N)} \Delta_N^{(\mathcal{R}/\mathcal{B})}(\mu_N^2, t(\Phi_1)) \right] \right\}$$

once more: integrates to unity $\longrightarrow$ "unitarity" of parton shower

- radiation given by $\mathcal{R}_N$ (correct at $\mathcal{O}(\alpha_s)$)

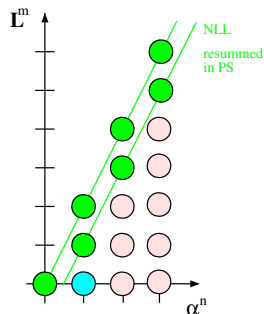
(but modified by logs of higher order in α_s from $\Delta_N^{(\mathcal{R}/\mathcal{B})}$)

- emission phase space constrained by μ_N
- also known as "soft ME correction"
hard ME correction fills missing phase space
- used for "power shower":
 $\mu_N \rightarrow E_{pp}$ and apply ME correction



NLO matching: Basic idea

- parton shower resums logarithms
fair description of collinear/soft emissions
jet evolution (where the logs are large)
- matrix elements exact at given order
fair description of hard/large-angle emissions
jet production (where the logs are small)
- adjust (“match”) terms:
 - cross section at **NLO accuracy** & correct hardest emission in PS to exactly reproduce ME at order α_s ($\mathcal{R}$ -part of the NLO calculation)
(this is relatively trivial)
 - maintain **(N)LL-accuracy** of parton shower
(this is not so simple to see)



POWHEG

- reminder: $\mathcal{K}_{ij,k}$ reproduces process-independent behaviour of $\mathcal{R}_N/\mathcal{B}_N$ in soft/collinear regions of phase space

$$d\Phi_1 \frac{\mathcal{R}_N(\Phi_{N+1})}{\mathcal{B}_N(\Phi_N)} \xrightarrow{\text{IR}} d\Phi_1 \frac{\alpha_s}{2\pi} \mathcal{K}_{ij,k}(\Phi_1)$$

- define **modified Sudakov form factor** (as in ME correction)

$$\Delta_N^{(\mathcal{R}/\mathcal{B})}(\mu_N^2, t_0) = \exp \left[- \int_{t_0}^{\mu_N^2} d\Phi_1 \frac{\mathcal{R}_N(\Phi_{N+1})}{\mathcal{B}_N(\Phi_N)} \right],$$

- assumes factorisation of phase space: $\Phi_{N+1} = \Phi_N \otimes \Phi_1$
- typically will adjust scale of α_s to parton shower scale

- define local K -factors
- start from Born configuration Φ_N with NLO weight:

("local K -factor")

$$\begin{aligned}
 d\sigma_N^{(\text{NLO})} &= d\Phi_N \bar{\mathcal{B}}(\Phi_N) \\
 &= d\Phi_N \left\{ \mathcal{B}_N(\Phi_N) + \underbrace{\mathcal{V}_N(\Phi_N) + \mathcal{B}_N(\Phi_N) \otimes \mathcal{S}}_{\check{\mathcal{V}}_N(\Phi_N)} \right. \\
 &\quad \left. + \int d\Phi_1 [\mathcal{R}_N(\Phi_N \otimes \Phi_1) - \mathcal{B}_N(\Phi_N) \otimes d\mathcal{S}(\Phi_1)] \right\}
 \end{aligned}$$

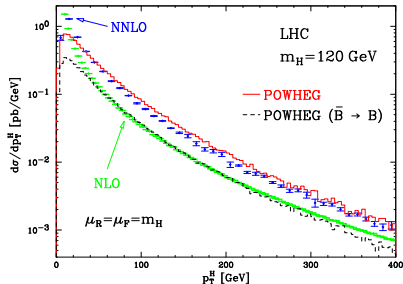
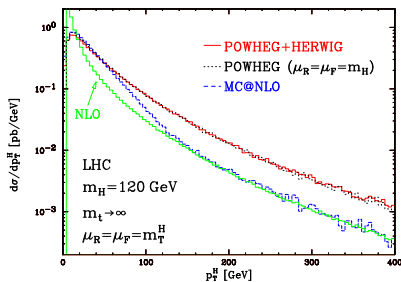
- by construction: exactly reproduce cross section at NLO accuracy
- note: second term vanishes if $\mathcal{R}_N \equiv \mathcal{B}_N \otimes d\mathcal{S}$

(relevant for MC@NLO)

- analyse accuracy of radiation pattern
- generate emissions with $\Delta_N^{(\mathcal{R}/\mathcal{B})}(\mu_N^2, t_0)$:

$$d\sigma_N^{(\text{NLO})} = d\Phi_N \tilde{\mathcal{B}}(\Phi_N) \times \underbrace{\left\{ \Delta_N^{(\mathcal{R}/\mathcal{B})}(\mu_N^2, t_0) + \int_{t_0}^{\mu_N^2} d\Phi_1 \frac{\mathcal{R}_N(\Phi_N \otimes \Phi_1)}{\mathcal{B}_N(\Phi_N)} \Delta_N^{(\mathcal{R}/\mathcal{B})}(\mu_N^2, k_\perp^2(\Phi_1)) \right\}}_{\text{integrating to yield 1 - "unitarity of parton shower"}}$$

- radiation pattern like in ME correction
- pitfall, again: choice of upper scale μ_N^2 (this is vanilla POWHEG!)
- apart from logs: which configurations enhanced by local K -factor
(K -factor for inclusive production of X adequate for X + jet at large $p_\perp$?)



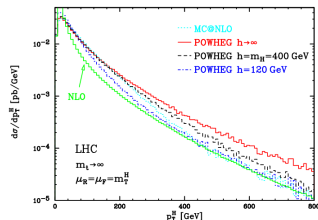
- large enhancement at high $p_{T,h}$
- can be traced back to large NLO correction
- fortunately, NNLO correction is also large $\rightarrow \sim$ agreement

- improving POWHEG
- split real-emission ME as

$$\mathcal{R} = \mathcal{R} \left(\underbrace{\frac{h^2}{p_\perp^2 + h^2}}_{\mathcal{R}^{(S)}} + \underbrace{\frac{p_\perp^2}{p_\perp^2 + h^2}}_{\mathcal{R}^{(F)}} \right)$$

- can “tune” h to mimic NNLO - or other (resummation) result
- differential event rate up to first emission

$$\begin{aligned} d\sigma = & d\Phi_B \bar{\mathcal{B}}^{(R(S))} \left[\Delta^{(R(S)/B)}(s, t_0) + \int_{t_0}^s d\Phi_1 \frac{\mathcal{R}^{(S)}}{\mathcal{B}} \Delta^{(R(S)/B)}(s, k_\perp^2) \right] \\ & + d\Phi_R \mathcal{R}^{(F)}(\Phi_R) \end{aligned}$$



MC@NLO

- MC@NLO paradigm: divide $\mathcal{R}_N$ in soft (“S”) and hard (“H”) part:

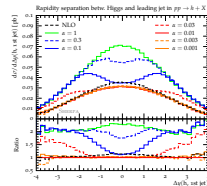
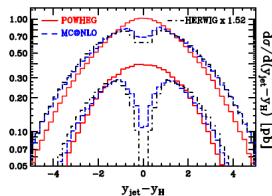
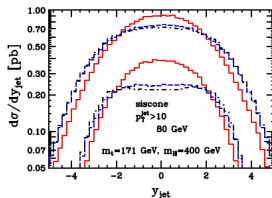
$$\mathcal{R}_N = \mathcal{R}_N^{(S)} + \mathcal{R}_N^{(H)} = \mathcal{B}_N \otimes d\mathcal{S}_1 + \mathcal{H}_N$$

- identify subtraction terms and shower kernels $d\mathcal{S}_1 \equiv \sum_{\{ij,k\}} \mathcal{K}_{ij,k}$
(modify $\mathcal{K}$ in 1st emission to account for colour)

$$d\sigma_N = d\Phi_N \underbrace{\tilde{\mathcal{B}}_N(\Phi_N)}_{\mathcal{B}+\tilde{\mathcal{V}}} \left[\Delta_N^{(\mathcal{K})}(\mu_N^2, t_0) + \int_{t_0}^{\mu_N^2} d\Phi_1 \mathcal{K}_{ij,k}(\Phi_1) \Delta_N^{(\mathcal{K})}(\mu_N^2, k_\perp^2) \right] \\ + d\Phi_{N+1} \mathcal{H}_N$$

- effect: only resummed parts modified with local K -factor

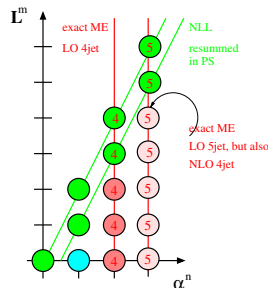
- phase space effects: shower vs. fixed order



- problem: impact of subtraction terms on local K -factor (filling of phase space by parton shower)
- studied in case of $gg \rightarrow H$ above
- proper filling of available phase space by parton shower paramount

Multijet merging: basic idea

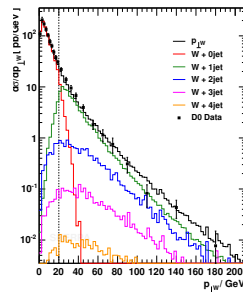
- parton shower resums logarithms
fair description of collinear/soft emissions
jet evolution (where the logs are large)
- matrix elements exact at given order
fair description of hard/large-angle emissions
jet production (where the logs are small)
- combine (“merge”) both:
result: “towers” of MEs with increasing number of jets evolved with PS
 - multijet cross sections at **Born accuracy**
 - maintain **(N)LL accuracy** of parton shower



- separate regions of jet production and jet evolution with jet measure Q_J

("truncated showering" if not identical with evolution parameter)

- matrix elements populate hard regime
- parton showers populate soft domain



Why it works: jet rates with the parton shower

- consider jet production in $e^+e^- \rightarrow \text{hadrons}$
Durham jet definition: relative transverse momentum $k_\perp > Q_J$
- fixed order: one factor α_s and up to $\log^2 \frac{E_{\text{c.m.}}}{Q_J}$ per jet
- use **Sudakov form factor** for resummation &
replace **approximate fixed order** by exact expression:



$$\mathcal{R}_2(Q_J) = \left[\Delta_q(E_{\text{c.m.}}^2, Q_J^2) \right]^2$$



$$\mathcal{R}_3(Q_J) = 2\Delta_q(E_{\text{c.m.}}^2, Q_J^2) \int_{Q_J^2}^{E_{\text{c.m.}}^2} \frac{dk_\perp^2}{k_\perp^2} \left[\frac{\alpha_s(k_\perp^2)}{2\pi} dz \mathcal{K}_q(k_\perp^2, z) \right. \\ \left. \times \Delta_q(E_{\text{c.m.}}^2, k_\perp^2) \Delta_q(k_\perp^2, Q_J^2) \Delta_g(k_\perp^2, Q_J^2) \right]$$

Multijet merging at LO

- expression for first emission

$$\begin{aligned}
 d\sigma = & \quad d\Phi_N \mathcal{B}_N \left[\Delta_N^{(\mathcal{K})}(\mu_N^2, t_0) \right. \\
 & \quad \left. + \int_{t_0}^{\mu_N^2} d\Phi_1 \mathcal{K}_N \Delta_N^{(\mathcal{K})}(\mu_N^2, t_{N+1}) \Theta(Q_J - Q_{N+1}) \right] \\
 & + d\Phi_{N+1} \mathcal{B}_{N+1} \Delta_N^{(\mathcal{K})}(\mu_{N+1}^2, t_{N+1}) \Theta(Q_{N+1} - Q_J)
 \end{aligned}$$

- note: $N + 1$ -contribution includes also $N + 2, N + 3, \dots$

(no Sudakov suppression below t_{n+1} , see further slides for iterated expression)

- potential occurrence of different shower start scales: $\mu_{N,N+1}, \dots$
- “unitarity violation” in square bracket: $\mathcal{B}_N \mathcal{K}_N \longrightarrow \mathcal{B}_{N+1}$

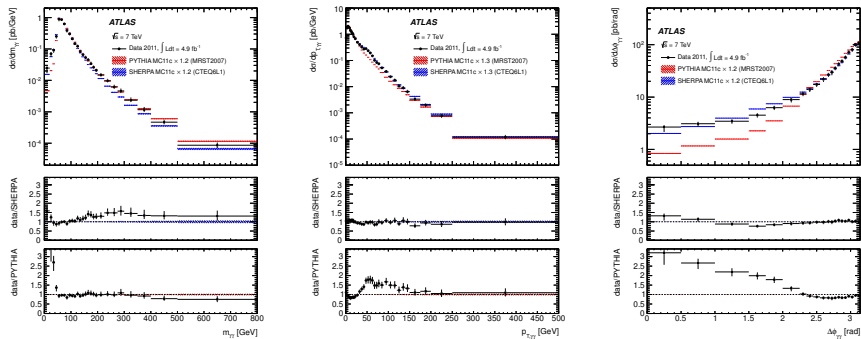
(cured with UMEPS formalism, L. Lönnblad & S. Prestel, JHEP 1302 (2013) 094 &

S. Platzer, arXiv:1211.5467 [hep-ph] & arXiv:1307.0774 [hep-ph])

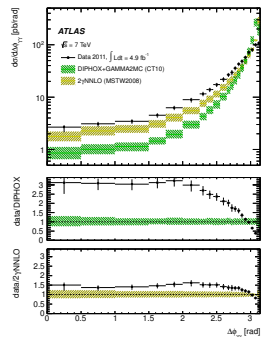
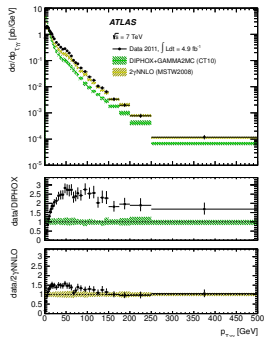
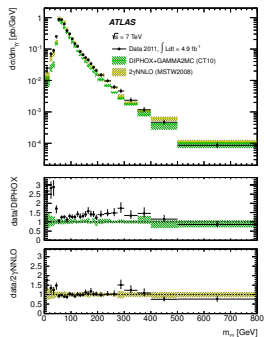
$$\begin{aligned}
d\sigma = & \sum_{n=N}^{n_{\max}-1} \left\{ d\Phi_n \mathcal{B}_n \left[\overbrace{\left[\prod_{j=N}^{n-1} \Theta(Q_{j+1} - Q_J) \right]}^{(n-N) \text{ extra jets}} \left[\overbrace{\prod_{j=N}^{n-1} \Delta_j^{(\mathcal{K})}(t_j, t_{j+1})}^{\text{no emissions off internal lines}} \right] \right. \\
& \times \left. \left[\underbrace{\Delta_n^{(\mathcal{K})}(t_n, t_0)}_{\text{no emission}} + \underbrace{\int_{t_0}^{t_n} d\Phi_1 \mathcal{K}_n \Delta_n^{(\mathcal{K})}(t_n, t_{n+1}) \Theta(Q_J - Q_{n+1})}_{\text{next emission no jet \& below last ME emission}} \right] \right\} \\
& + d\Phi_{n_{\max}} \mathcal{B}_{n_{\max}} \left[\prod_{j=N}^{n_{\max}-1} \Theta(Q_{j+1} - Q_J) \right] \left[\prod_{j=N}^{n_{\max}-1} \Delta_j^{(\mathcal{K})}(t_j, t_{j+1}) \right] \\
& \times \left[\Delta_{n_{\max}}^{(\mathcal{K})}(t_{n_{\max}}, t_0) + \int_{t_0}^{t_{n_{\max}}} d\Phi_1 \mathcal{K}_{n_{\max}} \Delta_{n_{\max}}^{(\mathcal{K})}(t_{n_{\max}}, t_{n_{\max}+1}) \right]
\end{aligned}$$

Di-photons @ ATLAS: $m_{\gamma\gamma}$, $p_{\perp,\gamma\gamma}$, and $\Delta\phi_{\gamma\gamma}$ in showers

(arXiv:1211.1913 [hep-ex])



Aside: Comparison with higher order calculations



Multijet-merging at NLO: MEPS@NLO

- basic idea like at LO: towers of MEs with increasing jet multiplicity (but this time at NLO)
- combine them into one sample, remove overlap/double-counting

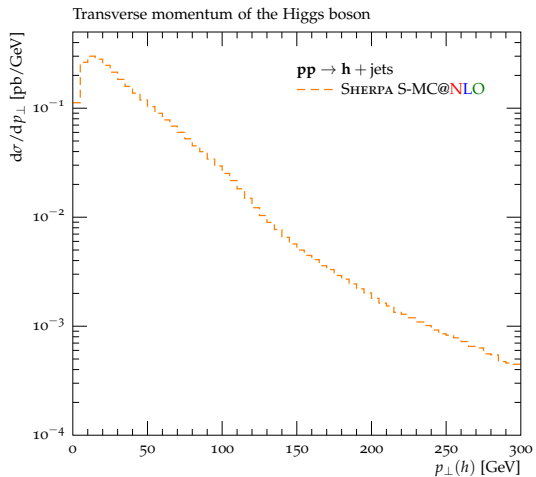
maintain NLO and (N)LL accuracy of ME and PS

- this effectively translates into a merging of MC@NLO simulations and can be further supplemented with LO simulations for even higher final state multiplicities

First emission(s), once more

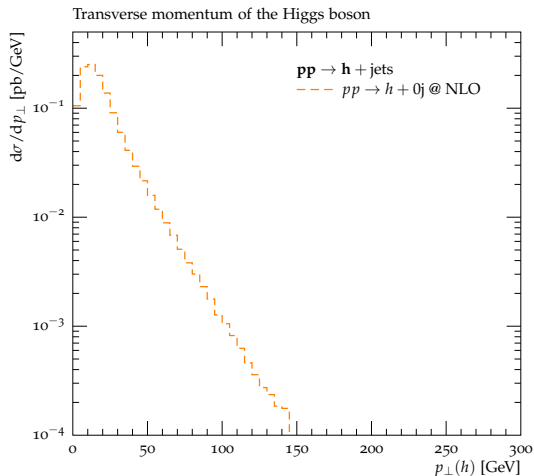
$$\begin{aligned}
d\sigma = & d\Phi_N \tilde{\mathcal{B}}_N \left[\Delta_N^{(\mathcal{K})}(\mu_N^2, t_0) + \int_{t_0}^{\mu_N^2} d\Phi_1 \mathcal{K}_N \Delta_N^{(\mathcal{K})}(\mu_N^2, t_{N+1}) \Theta(Q_J - Q_{N+1}) \right] \\
& + d\Phi_{N+1} \mathcal{H}_N \Delta_N^{(\mathcal{K})}(\mu_N^2, t_{N+1}) \Theta(Q_J - Q_{N+1}) \\
& + d\Phi_{N+1} \tilde{\mathcal{B}}_{N+1} \left(1 + \frac{\mathcal{B}_{N+1}}{\tilde{\mathcal{B}}_{N+1}} \int_{t_{N+1}}^{\mu_N^2} d\Phi_1 \mathcal{K}_N \right) \Theta(Q_{N+1} - Q_J) \\
& \cdot \left[\Delta_{N+1}^{(\mathcal{K})}(t_{N+1}, t_0) + \int_{t_0}^{t_{N+1}} d\Phi_1 \mathcal{K}_{N+1} \Delta_{N+1}^{(\mathcal{K})}(t_{N+1}, t_{N+2}) \right] \\
& + d\Phi_{N+2} \mathcal{H}_{N+1} \Delta_N^{(\mathcal{K})}(\mu_N^2, t_{N+1}) \Delta_{N+1}^{(\mathcal{K})}(t_{N+1}, t_{N+2}) \Theta(Q_{N+1} - Q_J) + \dots
\end{aligned}$$

$p_{\perp}^H$ in MEPS@NLO



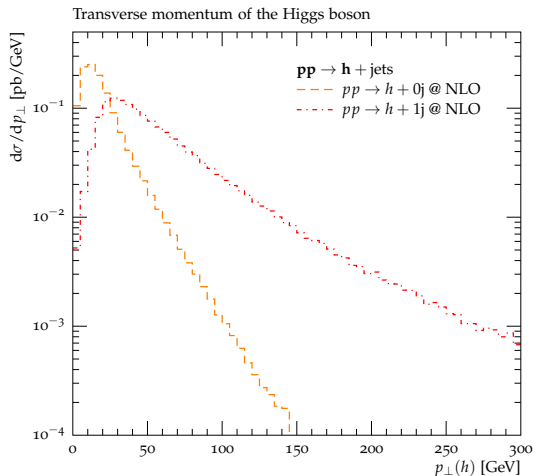
- first emission by MC@NLO

$p_{\perp}^H$ in MEPS@NLO



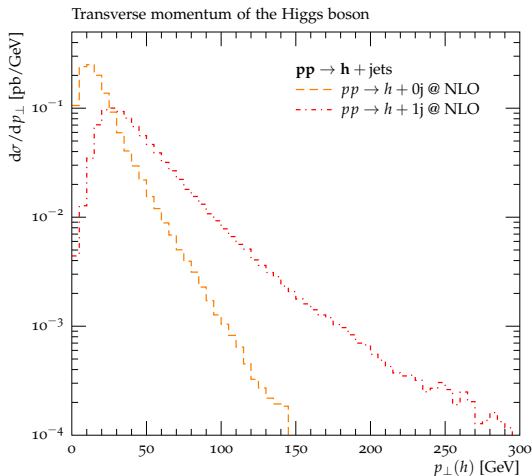
- first emission by MC@NLO
 , restrict to $Q_{n+1} < Q_{\text{cut}}$

$p_{\perp}^H$ in MEPS@NLO



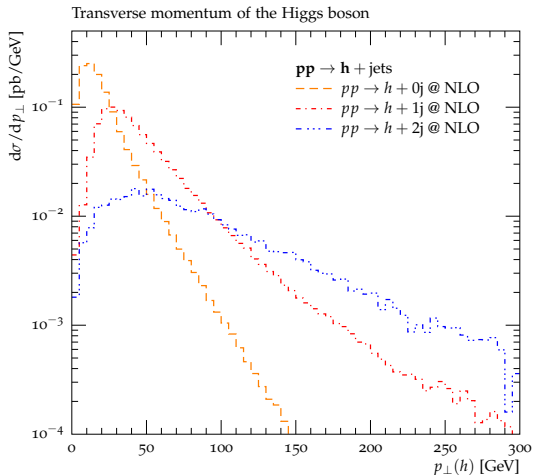
- first emission by MC@NLO
, restrict to $Q_{n+1} < Q_{\text{cut}}$
- MC@NLO $pp \rightarrow h + \text{jet}$ for
 $Q_{n+1} > Q_{\text{cut}}$

$p_{\perp}^H$ in MEPS@NLO



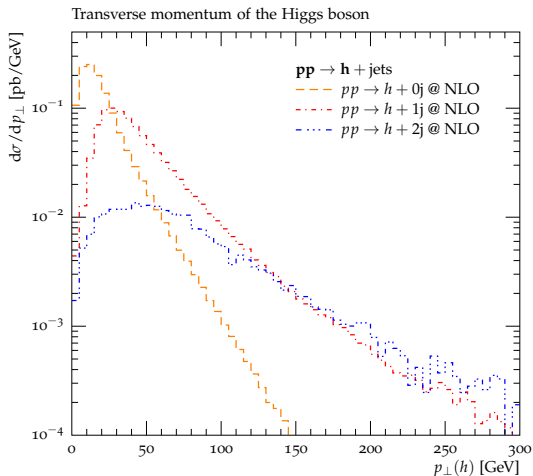
- first emission by MC@NLO
, restrict to $Q_{n+1} < Q_{\text{cut}}$
- MC@NLO $pp \rightarrow h + \text{jet}$ for
 $Q_{n+1} > Q_{\text{cut}}$
- restrict emission off
 $pp \rightarrow h + \text{jet}$ to
 $Q_{n+2} < Q_{\text{cut}}$

$p_{\perp}^H$ in MEPS@NLO



- first emission by MC@NLO
, restrict to $Q_{n+1} < Q_{\text{cut}}$
- MC@NLO $pp \rightarrow h + \text{jet}$ for
 $Q_{n+1} > Q_{\text{cut}}$
- restrict emission off
 $pp \rightarrow h + \text{jet}$ to
 $Q_{n+2} < Q_{\text{cut}}$
- MC@NLO $pp \rightarrow h + 2\text{jets}$
for $Q_{n+2} > Q_{\text{cut}}$

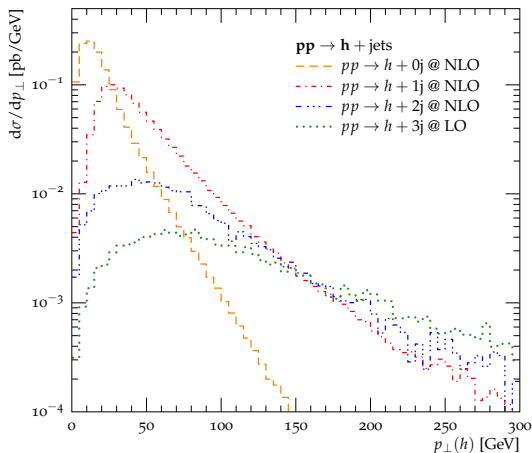
$p_{\perp}^H$ in MEPS@NLO



- first emission by MC@NLO, restrict to $Q_{n+1} < Q_{\text{cut}}$
- MC@NLO $pp \rightarrow h + \text{jet}$ for $Q_{n+1} > Q_{\text{cut}}$
- restrict emission off $pp \rightarrow h + \text{jet}$ to $Q_{n+2} < Q_{\text{cut}}$
- MC@NLO $pp \rightarrow h + 2\text{jets}$ for $Q_{n+2} > Q_{\text{cut}}$
- iterate

$p_{\perp}^H$ in MEPS@NLO

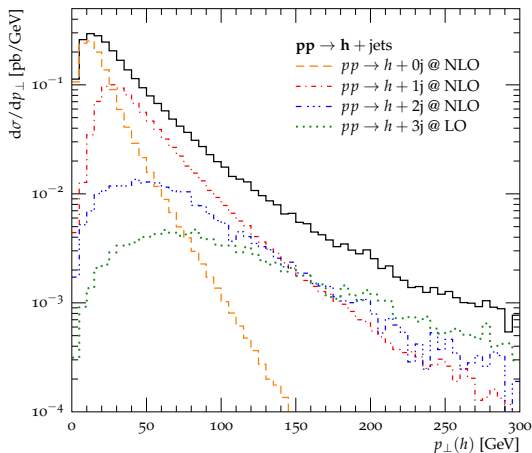
Transverse momentum of the Higgs boson



- first emission by MC@NLO , restrict to $Q_{n+1} < Q_{\text{cut}}$
- MC@NLO $pp \rightarrow h + \text{jet}$ for $Q_{n+1} > Q_{\text{cut}}$
- restrict emission off $pp \rightarrow h + \text{jet}$ to $Q_{n+2} < Q_{\text{cut}}$
- MC@NLO $pp \rightarrow h + 2\text{jets}$ for $Q_{n+2} > Q_{\text{cut}}$
- iterate

$p_{\perp}^H$ in MEPS@NLO

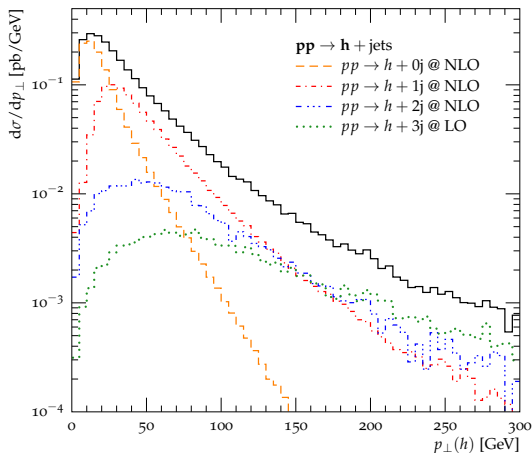
Transverse momentum of the Higgs boson



- first emission by MC@NLO , restrict to $Q_{n+1} < Q_{\text{cut}}$
- MC@NLO $pp \rightarrow h + \text{jet}$ for $Q_{n+1} > Q_{\text{cut}}$
- restrict emission off $pp \rightarrow h + \text{jet}$ to $Q_{n+2} < Q_{\text{cut}}$
- MC@NLO $pp \rightarrow h + 2\text{jets}$ for $Q_{n+2} > Q_{\text{cut}}$
- iterate
- sum all contributions

$p_{\perp}^H$ in MEPS@NLO

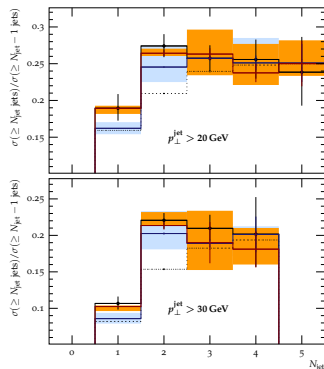
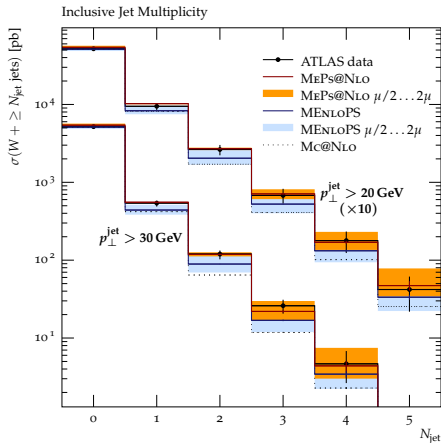
Transverse momentum of the Higgs boson

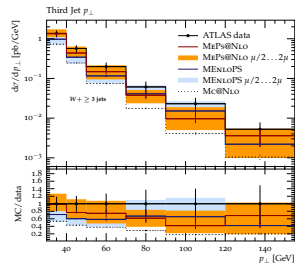
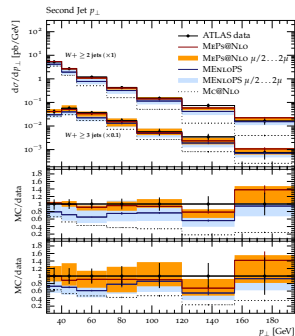
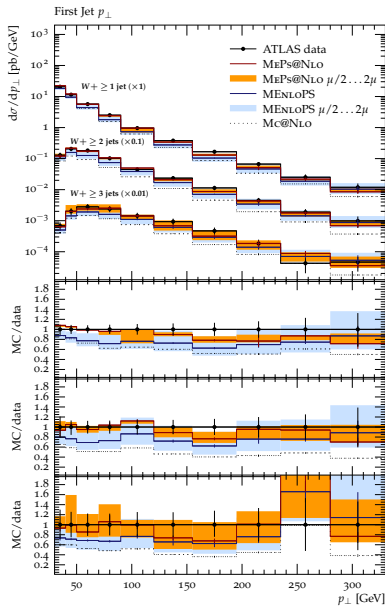


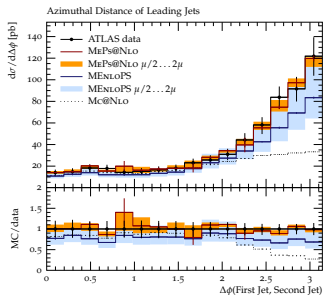
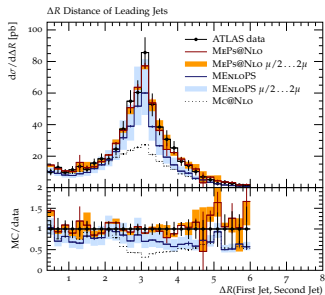
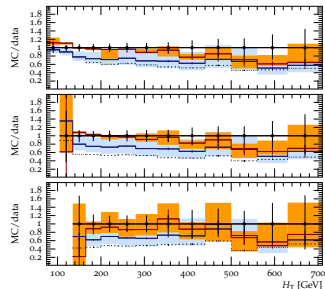
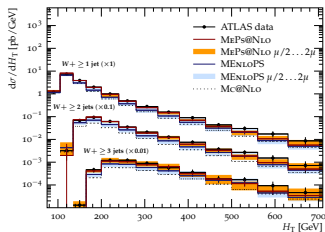
- first emission by MC@NLO , restrict to $Q_{n+1} < Q_{\text{cut}}$
- MC@NLO $pp \rightarrow h + \text{jet}$ for $Q_{n+1} > Q_{\text{cut}}$
- restrict emission off $pp \rightarrow h + \text{jet}$ to $Q_{n+2} < Q_{\text{cut}}$
- MC@NLO $pp \rightarrow h + 2\text{jets}$ for $Q_{n+2} > Q_{\text{cut}}$
- iterate
- sum all contributions
- eg. $p_{\perp}(h) > 200$ GeV has contributions fr. multiple topologies

Example: MEPS@NLO for W +jets

(up to two jets @ NLO, from BLACKHAT, see arXiv: 1207.5031 [hep-ex])







Non-Perturbative Aspects

Contents

- 6.a) Hadronisation: General thoughts
- 6.c) Models: strings & clusters
- 6.d) Underlying event

Local Parton-Hadron Duality (LPHD)

- remember QCD emission pattern

$$dw^{q \rightarrow qg} = \frac{\alpha_s(k_\perp^2)}{2\pi} C_F \frac{dk_\perp^2}{k_\perp^2} \frac{d\omega}{\omega} \left[1 + \left(1 - \frac{\omega}{E} \right) \right] .$$

- spectrum cut-off at small transverse momenta and energies by onset of hadronization, at scales $R \approx 1 \text{ fm}/\Lambda_{\text{QCD}}$
- two (extreme) classes of emissions: gluons and gluers determined by relation of formation and hadronization times

- gluons formed at times R , with momenta $k_{\parallel} \sim k_{\perp} \sim \omega \gtrsim 1/R$
- assuming that hadrons follow partons,

$$\begin{aligned}
 dN_{(\text{hadrons})} &\sim \int_{k_{\perp} > 1/R}^Q \frac{dk_{\perp}^2}{k_{\perp}^2} \frac{C_F \alpha_s(k_{\perp}^2)}{2\pi} \left[1 + \left(1 - \frac{\omega}{E} \right) \right] \frac{d\omega}{\omega} \\
 &\sim \frac{C_F \alpha_s(1/R^2)}{\pi} \log(Q^2 R^2) \frac{d\omega}{\omega}
 \end{aligned}$$

or - relating their energy with that of the gluons -

$$dN_{(\text{hadrons})}/d \log \epsilon = \text{const.},$$

a plateau in log of energy (or in rapidity)

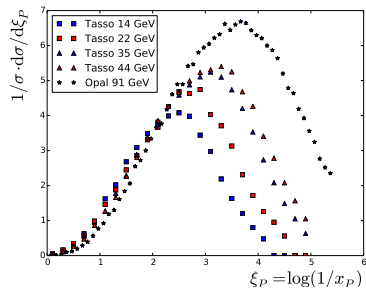
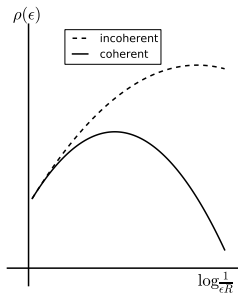
- add **extra gluon radiation** and **encode angular ordering**

$$\begin{aligned}
 N &\propto \int^E \frac{d\epsilon}{\epsilon} \int^1 \frac{d\theta}{\theta} \delta(\epsilon\theta - 1/R) \\
 &\quad + \alpha_s \int^E \frac{d\omega}{\omega} \int^1 \frac{d\theta_0}{\theta_0} \Theta(\omega\theta_0 - 1/R) \int^\omega \frac{d\epsilon}{\epsilon} \int^{\theta_{\max}} \frac{d\theta}{\theta} \delta(\epsilon\theta - 1/R) \\
 &= \begin{cases} 1 + \frac{\alpha_s}{2} \left[\log^2(ER) - \log^2(\epsilon R) \right] & \text{for incoherent sum, } \theta_{\max} = 1 \\ 1 + \alpha_s \log \frac{E}{\epsilon} \log \epsilon R & \text{for coherent sum, } \theta_{\max} = \theta_0 \end{cases}
 \end{aligned}$$

- overall effect: peak of energies not $\approx m_{\text{hadrons}}$, but

$$\langle \epsilon \rangle = \langle E_{\text{had}} \rangle \sim \sqrt{E}.$$

Hump-backed plateau



- impact of additional radiation
- new partons must separate before they can hadronize independently
- therefore, one more time

$$\begin{aligned}
 t^{\text{form}} &\sim \frac{k_{\parallel}}{k_{\perp}^2} \\
 t^{\text{sep}} &\sim R\theta \quad \sim t^{\text{form}}(Rk_{\perp}) \\
 t^{\text{had}} &\sim k_{\parallel}R^2 \quad \sim t^{\text{form}}(Rk_{\perp})^2.
 \end{aligned}$$

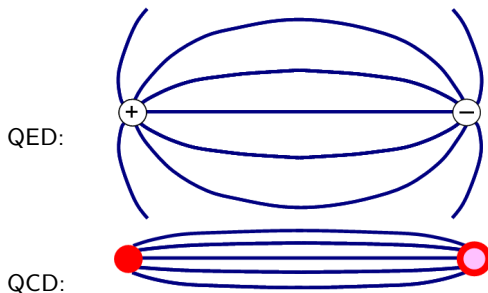
- for gluers $Rk_{\perp} \approx 1$: all times the same
- naively; new & more hadrons following new partons
- but: colour coherence
primary and secondary partons not separated enough in

$$1/R \lesssim \omega_{(\text{hadron})} \lesssim 1/(R\theta)$$

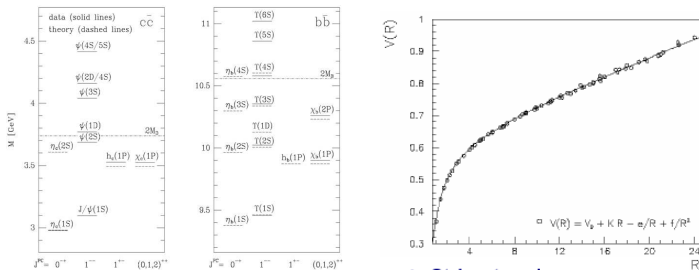
and therefore no independent radiation

Hadronisation: General thoughts

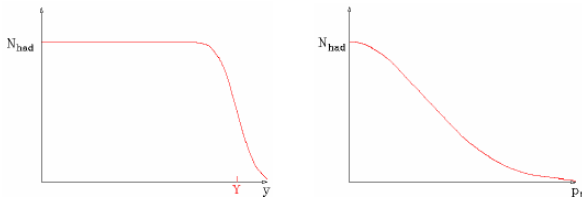
- confinement the striking feature of low-scale strong interactions
- transition from partons to their bound states, the hadrons
- the Meissner effect in QCD



- linear QCD potential in Quarkonia – like a string



- combine some experimental facts into a naive parameterisation
- in $e^+e^- \rightarrow \text{hadrons}$: exponentially decreasing $p_\perp$, flat plateau in y for hadrons



- try “smearing”: $\rho(p_\perp^2) \sim \exp(-p_\perp^2/\sigma^2)$

- use parameterisation to “guesstimate” hadronisation effects:

$$E = \int_0^Y dy d p_{\perp}^2 \rho(p_{\perp}^2) p_{\perp} \cosh y = \lambda \sinh Y$$

$$P = \int_0^Y dy d p_{\perp}^2 \rho(p_{\perp}^2) p_{\perp} \sinh y = \lambda (\cosh Y - 1) \approx E - \lambda$$

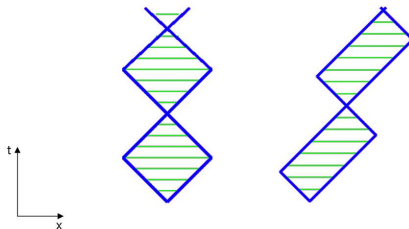
$$\lambda = \int d p_{\perp}^2 \rho(p_{\perp}^2) p_{\perp} = \langle p_{\perp} \rangle.$$

- estimate $\lambda \sim 1/R_{\text{had}} \approx m_{\text{had}}$, with m_{had} 0.1-1 GeV.
- effect: jet acquire non-perturbative mass $\sim 2\lambda E$ ($\mathcal{O}(10\text{GeV})$ for jets with energy $\mathcal{O}(100\text{GeV})$).

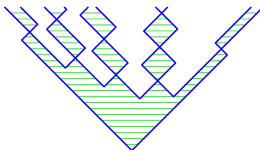
- similar parametrization underlying Feynman-Field model for independent fragmentation
- recursively fragment $q \rightarrow q' + \text{had}$, where
 - transverse momentum from (fitted) Gaussian;
 - longitudinal momentum arbitrary (hence from measurements);
 - flavour from symmetry arguments + measurements.
- problems: frame dependent, “last quark”, infrared safety, no direct link to perturbation theory,

The string model

- a simple model of mesons: yoyo strings
 - light quarks ($m_q = 0$) connected by string, form a meson
 - area law: $m_{\text{had}}^2 \propto \text{area of string motion}$
 - $L=0$ mesons only have 'yo-yo' modes:



- turn this into hadronisation model $e^+e^- \rightarrow q\bar{q}$ as test case
- ignore gluon radiation: $q\bar{q}$ move away from each other, act as point-like source of string
- intense chromomagnetic field within string:
more $q\bar{q}$ pairs created by tunnelling and string break-up
- analogy with QED (Schwinger mechanism):
 $d\mathcal{P} \sim dxdt \exp(-\pi m_q^2/\kappa)$, $\kappa = \text{"string tension"}$.



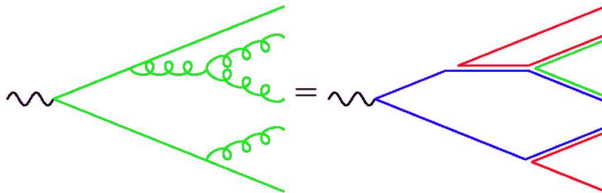
- string model = well motivated model, constraints on fragmentation
(Lorentz-invariance, left-right symmetry, ...)
- how to deal with gluons?
- interpret them as kinks on the string $\implies$ the string effect



- infrared-safe, advantage: smooth matching with PS.

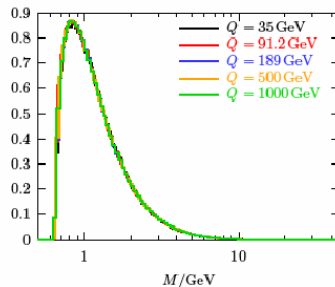
The cluster model

- underlying idea: preconfinement/LPHD
 - typically, neighbouring colours will end in same hadron
 - hadron flows follow parton flows $\rightarrow$ don't produce any hadrons at places where you don't have partons
 - works well in large- N_c limit with planar graphs
- follow evolution of colour in parton showers



- paradigm of cluster model: clusters as continuum of hadron resonances
- trace colour through shower in $N_c \rightarrow \infty$ limit
- force decay of gluons into $q\bar{q}$ or $\bar{d}d$ pairs, form colour singlets from neighbouring colours, usually close in phase space
- mass of singlets: peaked at low scales $\approx Q_0^2$
- decay heavy clusters into lighter ones or into hadrons
(here, many improvements to ensure leading hadron spectrum hard enough, overall effect: cluster model becomes more string-like)
- if light enough, clusters will decay into hadrons
- naively: spin information washed out, decay determined through phase space only $\rightarrow$ heavy hadrons suppressed (baryon/strangeness suppression)

- self-similarity of parton shower will end with roughly the same **local** distribution of partons, with roughly the same invariant mass for colour singlets
- adjacent pairs colour connected, form colourless (white) clusters.
- clusters (“ $\approx$ excited hadrons”) decay into hadrons

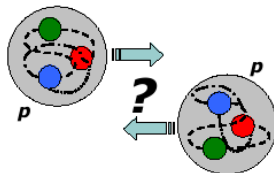


Practicalities

- practicalities of hadronisation models: parameters
 - kinematics of string or cluster decay: 2-5 parameters
 - must “pop” quark or diquark flavours in string or cluster decay — cannot be completely democratic or driven by masses alone
→ suppression factors for strangeness, diquarks 2-10 parameters
 - transition to hadrons, cannot be democratic over multiplets
→ adjustment factors for vectors/tensors etc. 2-6 parameters
- tuned to LEP data, overall agreement satisfying
- validity for hadron collisions not quite clear
(beam remnant fragmentation not in LEP.)
- there are some issues with inclusive strangeness/baryon production

Multiple parton scattering

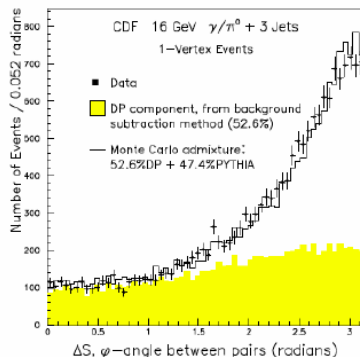
- hadrons = extended objects!
- no guarantee for one scattering only.
- running of α_S
⇒ preference for soft scattering.



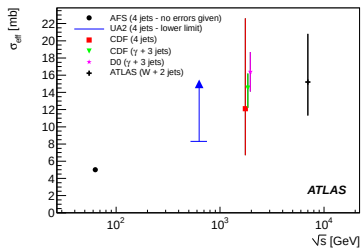
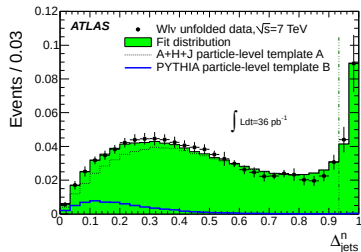
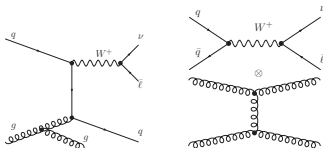
- first experimental evidence for double-parton scattering: events with $\gamma + 3$ jets:
 - cone jets, $R = 0.7$, $E_T > 5$ GeV; $|\eta_j| < 1.3$;
 - “clean sample”: two softest jets with $E_T < 7$ GeV;
- cross section for DPS

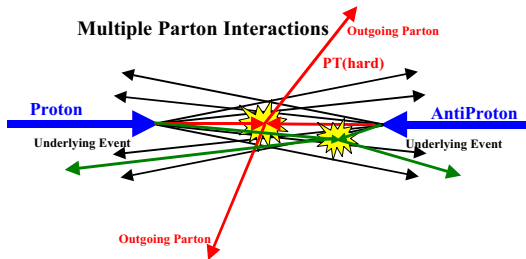
$$\sigma_{\text{DPS}} = \frac{\sigma_{\gamma j} \sigma_{jj}}{\sigma_{\text{eff}}}$$

$$\sigma_{\text{eff}} \approx 14 \pm 4 \text{ mb.}$$



- more measurements, also at LHC
- ATLAS results from $W + 2$ jets





but: how to define the underlying event?

- ① everything apart from the hard interaction, but including IS showers, FS showers, remnant hadronisation.
- ② remnant-remnant interactions, soft and/or hard.
- ③ lesson: **hard to define**

- origin of MPS: parton–parton scattering cross section exceeds hadron–hadron total cross section

$$\sigma_{\text{hard}}(p_{\perp,\text{min}}) = \int_{p_{\perp,\text{min}}^2}^{s/4} dp_{\perp}^2 \frac{d\sigma(p_{\perp}^2)}{dp_{\perp}^2} > \sigma_{pp,\text{total}}$$

for low $p_{\perp,\text{min}}$

- remember

$$\frac{d\sigma(p_{\perp}^2)}{dp_{\perp}^2} = \int_0^1 dx_1 dx_2 f(x_1, q^2) f(x_2, q^2) \frac{d\hat{\sigma}_{2 \rightarrow 2}}{dp_{\perp}^2}$$

- $\langle \sigma_{\text{hard}}(p_{\perp,\text{min}}) / \sigma_{pp,\text{total}} \rangle \geq 1$
- depends strongly on cut-off $p_{\perp,\text{min}}$ (energy-dependent)!

Modelling the underlying event

- take the old PYTHIA model as example:
 - start with hard interaction, at scale Q_{hard}^2 .
 - select a new scale $p_{\perp}^2$ from

$$\exp \left[-\frac{1}{\sigma_{\text{norm}}} \int_{p_{\perp}^2}^{Q_{\text{hard}}^2} dp_{\perp}^2 \frac{d\sigma(p_{\perp}^2)}{dp_{\perp}^2} \right]$$

with constraint $p_{\perp}^2 > p_{\perp,\text{min}}^2$

- rescale proton momentum ("proton-parton = proton with reduced energy").
- repeat until no more allowed $2 \rightarrow 2$ scatter

Modelling the underlying event

- possible refinements:
 - may add impact-parameter dependence $\longrightarrow$ more fluctuations
 - add parton showers to UE
 - “regularisation” to dampen sharp dependence on $p_{\perp,\min}$: replace $1/\hat{t}$ in MEs by $1/(t + t_0)$, also in α_s .
 - treat intrinsic $k_{\perp}$ of partons ($\rightarrow$ parameter)
 - model proton remnants ($\rightarrow$ parameter)

SUMMARY

