Kinematics

- Reminder: Lorentz-transformations
  Four-vectors, scalar-products and the metric
- Phase-space integration
- Two-body decays
- Scattering
- The role of the beam-axis in collider experiments
**Units**

- In this lecture, natural units will be used:
  \[ \hbar = c = 1 \]

- This implies that energy, momentum, mass all have the same dimension:
  eV (electron volt) or multiples (keV, MeV, GeV, TeV)

- orientation: \( 1 \text{GeV} \approx m_{\text{proton}}, \ 511 \text{keV} \approx m_{\text{electron}} \)

- Also, time, space etc. have units of inverse energy.

- Translation through: \( \hbar c \approx 0.2 \text{GeV} \cdot \text{fm} \) (\( 1 \text{fm} \approx r_{\text{proton}} \))
  hence \( 1/\text{GeV} \approx 0.2 \text{fm} \), similar for time.

- Also: angular momentum now dimensionless etc.
Reminder: Lorentz-transformations

The speed of light

Michelson-Morley experiment (1887)

Is there a medium (ether), light requires for travel?

No. The speed of light is constant (no addition of velocities with earth’s velocity)
Reminder: Lorentz-transformations

Special relativity

Einstein’s theory of special relativity (1905)

- The laws of physics are independent of the choice of inertial frame.
- The speed of light is the same in all inertial frames.
- Consequences: Time dilation, length contraction, relativity of simultaneity, equivalence of mass and energy, composition of velocities.

(all covered in the lecture on special relativity)
Reminder: Lorentz-transformations

Lorentz-transformations

Consider two inertial systems S and S'.

Two “interesting” types of transformations between the two systems: boosts and rotations.

Boost: System S' moves with relative velocity \( \vec{v} = v \vec{e}_z \) (remember \( c = 1 \) \( \iff \) velocity in units of light-speed)

Rotation: S' rotated with \( \vec{\theta} = \theta \vec{e}_z \) around z-axis

Transformations \( B \) and \( R \) are represented by two matrices, \( \hat{B} \) and \( \hat{R} \), respectively.

They act on four-vectors \( x^\mu = (t, \vec{x})^T \), composed of time and spatial three-vector.
Reminder: Lorentz-transformations

**Boosts**

\[ x'^\mu = \hat{B}(\vec{v})x^\nu = \hat{B}_\nu^\mu(\vec{v})x^\nu \]

\[
\hat{B}_\nu^\mu(\vec{v} = v\vec{e}_z) = \begin{pmatrix}
\frac{1}{\sqrt{1-v^2}} & 0 & 0 & -\frac{v}{\sqrt{1-v^2}} \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-\frac{v}{\sqrt{1-v^2}} & 0 & 0 & \frac{1}{\sqrt{1-v^2}}
\end{pmatrix}
\]

and similarly for the other axes.

(The indices will be explained in a few slides)

*Note: Boosts mix temporal and spatial components.*
**Reminder: Lorentz-transformations**

**Rotations**

\[ x'^\mu = \hat{R}(\vec{\theta}) x^\nu = \hat{R}^\mu_\nu(\vec{\theta}) x^\nu \]

\[ \hat{R}^\mu_\nu(\vec{\theta} = \theta \vec{e}_z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta & 0 \\ 0 & \sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

and similarly for the other axes.

(The indices will be explained in a few slides)

**Note:** Rotations mix the spatial components, but leave their overall length invariant.
Reminder: Lorentz-transformations

**Alternative form for boosts**

**Boosts, once more:**

- Would be nice to have a simple “geometric” interpretation for boosts, like angles in rotations (e.g. additivity etc.).
- Use hyperbolic functions sinh and cosh.

Then (with $\cosh \eta = 1 / \sqrt{1 - \nu^2}$)

\[
\hat{B}^\mu_\nu (\nu \vec{e}_z) = \begin{pmatrix}
\cosh \eta & 0 & 0 & -\sinh \eta \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-\sinh \eta & 0 & 0 & \cosh \eta 
\end{pmatrix}
\]
Reminder: Lorentz-transformations

The structure of space-time

Distances in space-time:

Would be nice to have invariance of length of four-vectors under boosts and rotations.

\[ t^2 + x^2 + y^2 + z^2 \] doesn’t work.

But: \[ t^2 - x^2 - y^2 - z^2 = t'^2 - x'^2 \] does work !!!

Remember types of distances: time-like \((x^2 > 0)\), light-like \((x^2 = 0)\), and space-like \((x^2 < 0)\) and consequences for causality structure.
Reminder: Lorentz-transformations

Mathematical formulation

Four-vectors, co-and contra-variant indices:

- How to realize this? \(\Leftrightarrow\) introduce a “metric” \(g^{\mu\nu}\)

\[
g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
\end{pmatrix}
\]

- Metric raises and lowers indices:
  - Take \(x^\mu = (t, \vec{x})\) then: \(x_\mu = g_{\mu\nu}x^\nu = (t, -\vec{x})\)
  - and \(x^\mu = g^{\mu\nu}x_\nu\)

- Then \(x^2 = x^\mu x_\mu = x^\mu g_{\mu\nu}x^\nu = t^2 - \vec{x}^2\)
Reminder: Lorentz-transformations

Properties

Further properties:

- Inverse transformation: \( \hat{B}^\mu_\nu \neq \hat{B}^\mu_\nu = \left[ \hat{B}^\mu_\nu \right]^T = \left[ \hat{B}^\mu_\nu \right]^{-1} \)

- General Lorentz-transformations \( \Lambda \) (boosts and rotations) leave metric invariant:

\[
g^{\mu\nu} = \Lambda^\mu_\rho (\vec{v}, \vec{\theta}) g^{\rho\kappa} \Lambda^\nu_\kappa (\vec{v}, \vec{\theta}) \quad \Rightarrow \quad g = \Lambda^{-1} g \Lambda
\]

- Apply Einstein convention
  (summing over repeated indices only for pairs of lower & upper ones)

- Apply it to ALL quantities
  especially energy + momentum \( \Leftrightarrow \) four-momentum

\[
p^\mu = (E, \vec{p})
\]
Reminder: Lorentz-transformations

**Four-momenta**

*Mass-shell and four-momentum conservation:*

- **Four-momenta** $p^\mu = (E, \vec{p})$ of particles and other objects must satisfy relativistic energy-momentum relation (aka “on-shell” or “mass-shell” condition)

  $$E^2 = m^2 + \vec{p}^2 \implies p^2 = p^\mu p_\mu = m^2$$

- **In rest-frame of particle** ($\vec{p} = 0$), trivially $E^2 = m^2$.

- **Conservation of total energy and momentum:**

  $$P^\mu_{\text{before}} = P^\mu_{\text{after}}, \quad \text{with} \quad P^\mu = \sum_{i \in \text{particles}} p^\mu_i$$
Phase space integration

One-particle phase space

Why is this important?

- For scattering amplitudes in QM (or cross-section—see later for example in classical mechanics) must sum/integrate over all allowed states, including different momenta.

- Should be Lorentz-invariant $\Leftrightarrow$ need such integral over all possible momenta of one particle:

$$dPS_{1\text{particle}} = \frac{d^4p}{(2\pi)^4} (2\pi)\delta(p^2 - m^2)\Theta(E) = \frac{d^3\vec{p}}{(2\pi)^3 2E}$$

- $\text{inv.volume}$
- on-shell
- only positive energies
Phase space integration

Two-particle phase space

A simple particle decay: \( P \rightarrow p_1 + p_2 \)

- Add in total four momentum conservation, yields

\[
\begin{align*}
\text{d}PS^{(2)} &= \int \frac{\text{d}^4 p_1}{(2\pi)^4} \frac{\text{d}^4 p_2}{(2\pi)^4} (2\pi)^4 \delta^4(P - p_1 - p_2) \\
&\quad \times (2\pi) \delta(p_1^2 - m_1^2) \Theta(E_1) (2\pi) \delta(p_2^2 - m_2^2) \Theta(E_2)
\end{align*}
\]

- Use the delta-function for \( p_2 \):

\[
\begin{align*}
\text{d}PS^{(2)} &= \int \frac{\text{d}^4 p_1}{(2\pi)^2} \delta(p_1^2 - m_1^2) \delta((P - p_1)^2 - m_2^2) \\
&\quad \times \Theta(E_1) \Theta(E - E_1)
\end{align*}
\]
Phase space integration

Two-particle phase space

A simple particle decay: $P \rightarrow p_1 + p_2$ (cont’d)

Go to rest-frame of $P = (M,0,0,0)$, transform on polar coordinates and do the angle integrals.

$$dPS^{(2)} = \int_0^M dE_1 \int \frac{\rho_1^2 d\rho_1 d^2\Omega_1}{4\pi^2} \delta(E_1^2 - \rho_1^2 - m_1^2)$$

$$\times \delta((M - E_1)^2 - \rho_1^2 - m_2^2)$$

$$= \frac{1}{\pi} \int_0^M dE_1 \frac{\rho_1^2}{2\rho_1} \delta(M^2 - 2ME_1 - m_2^2) \left|_{\rho_1^2 = E_1^2 - m_1^2}^{\rho_1^2} \right.$$
Phase space integration

Two-particle phase space

A simple particle decay: $P \rightarrow p_1 + p_2$ (cont’d)

- Do the energy integral:

$$dPS^{(2)} = \frac{\sqrt{(M^2 - m_1^2 - m_2^2)^2 - 4m_1^2m_2^2}}{8\pi M^2} = \frac{\rho_1}{4\pi M}$$

and decay-kinematics (up to the angles) fixed by

$$\vec{p}_1 = -\vec{p}_2 = \rho \vec{e}_r$$

$$E_{1,2} = \frac{M^2 \pm (m_1^2 - m_2^2)}{2M}$$
Scattering processes

Cross sections

**Definition (classical mechanics)**

Consider a beam of particles, incident on a target with impact parameter $b$. In dependence on $b$, $dN(\theta)$ particles are scattered into a polar angle region $[\theta, \theta+d\theta]$. The differential cross section $d\sigma$ then reads

$$
d\sigma(\theta) = \frac{dN(\theta)}{n}
$$

where $n$ denotes the number of particles passing a unit area orthogonal to the beam per unit time.

The cross section has dimension of an area.
Scattering processes

Cross sections

Rewriting the definition:

- Assume that the relation between $b$ and $\theta$ is uniquely defined $b \rightarrow b(\theta)$.

- Use that the number of particles in a “ring” around the target with radius $b$ and size $db$ ends in a well-defined angle region, then $dN = 2\pi b db \cdot n$

- Therefore, the cross section is given by

$$d\sigma = 2\pi b(\theta) \frac{db}{d\theta} d\theta$$
Scattering processes

Cross sections

Example: Hard sphere in classical mechanics

Consider scattering off a hard sphere (radius $a$)

$$V(r) = \begin{cases} +\infty & r \leq a \\ 0 & \text{else} \end{cases}$$

Sketch: pictorially clear that the incident angle on the sphere equals the reflected angle.
Maths: next slide.
Scattering processes

Cross sections

Example: Hard sphere (cont’d)

Connection angle-impact parameter:

\[ b = a \sin \theta_0 = a \sin \frac{\pi - \theta}{2} = a \cos \frac{\theta}{2} \]

Plug this into the defining equation:

\[ d\sigma = \frac{\pi a^2}{2} \sin \frac{\theta}{2} d\theta \]

Total cross section from integration, yields \( \sigma_{\text{tot}} = \pi a^2 \)
Scattering processes

Cross sections

Cross section in quantum mechanics:

- **Problem:** Classical mechanics is deterministic, quantum mechanics is probabilistic. To live up to this,

  \[ \sigma = \frac{\text{transition rate}}{\text{(\# of targets)(incoming flux)}} \]

- **In rest frame of target:** \( \text{flux} = \frac{\text{(\# of particles)} \cdot v}{V} \)

  if particles have velocity \( v \). \( V \) is a “reaction volume”.

- **Taken together:** \( \sigma = \frac{\text{transition prob.}}{\text{unit time}} \cdot \frac{v}{V} \)
Scattering processes

Cross sections

Cross section in quantum mechanics (cont’d)

- Taken together: \( \sigma = \frac{\text{transition prob.}}{\text{unit time}} \cdot \frac{v}{V} \)

- Inspection shows: Units of cross section still = area.

- But now: Reaction volume? Transition probability? What’s the connection to quantum mechanics?
Scattering processes

Cross sections

Cross section in quantum mechanics (cont’d)

Fermi’s golden rule suggest that the transition probability between two states, $\alpha \rightarrow \beta$ per unit volume and unit time is given by

$$\frac{P_{\alpha \rightarrow \beta}}{V} = (2\pi)^4 \delta^4(P_\alpha - P_\beta) \left| \langle \beta | \hat{S} | \alpha \rangle \right|^2$$

where total four-momentum conservation is realised through the delta-function and $\langle \beta | \hat{S} | \alpha \rangle$ is the corresponding matrix element, calculable from first principles (pert.expansion $\Leftrightarrow$ Feynman diagrams).
Lorentz-invariance would be nice ⇔ want suitable expression for the incoming flux (the velocity) not only in target-frame ⇔ see below.

Assume initial state $\alpha$ consists of particles 1 and 2 and final state $\beta$ of particles 3...n, then, taking everything together:

$$d\sigma = \frac{(2\pi)^4 \delta^4(p_1 + p_2 - p_3 - \cdots - p_n)}{4 \sqrt{(p_1p_2)^2 - m_1^2 m_2^2}} \left| \langle 12 | \hat{S} | 3 \cdots n \rangle \right|^2 \times \prod_{i=3}^{n} \frac{d^3 p_i}{(2\pi)^3 2E_i}$$
Scattering processes

**Pair production**

**Pair production processes**

- Often pair-production processes relevant:
  \[ e^+e^- \rightarrow q\bar{q}, \quad e^+e^- \rightarrow W^+W^- \text{ etc.} \]

- Especially for production of heavy objects (which may decay further)

- Typically evaluated in centre-of-mass frame of incoming particles:
  \[ p_1 \text{ c.m.s} = -p_2, \quad E_{\text{c.m.}} = E_1 + E_2 \]
**Scattering processes**

**Pair production**

**Pair production processes**

- Phase space looks similar to two-body decay (above) of a mass $E$:

\[
\frac{\left| \langle 12 | \hat{S} | 34 \rangle \right|^2}{4 \sqrt{(p_1 p_2)^2 - m_1^2 m_2^2}} \cdot \frac{\rho_3}{E} \cdot \frac{d^2 \Omega_3}{4\pi}
\]

- Here: Spatial orientation may be relevant (compare with classical examples), $\rho_3$ is three-momentum of particle 3, and $\Omega_3$ its solid angle, typically w.r.t. the axis of the incoming (beam-) particles.
Scattering processes

Connection to observables

Luminosity and event rates

- An important quantity at particle colliders is its “luminosity“, defined by $\sigma \cdot \mathcal{L} = \text{Rate}$

- **Fixed target experiments:**

  Flux: $\phi_A = n_A \cdot S \cdot v$

  Scatters per unit area: $N_B = n_B \cdot l$

$\Rightarrow$ with previous definitions:

$\mathcal{L} = \phi_A \cdot N_B = n_A \cdot n_n \cdot v \cdot V$
Scattering processes

Connection to observables

Luminosity and event rates

Scattering experiments:

Here density \( n_B = \frac{\Phi_B}{h^2 \nu_B} \)

Area density \( N_B = \frac{h}{\sin \theta} n_B \)

\( \Rightarrow \) with previous definitions: \( \mathcal{L} = \frac{\Phi_A \Phi_B}{h \nu_B \sin \theta} \)

Note: Units in both cases relate xsecs with rates:

\( [\mathcal{L}] = m^{-2} s^{-1} \)
Scattering processes

Connection to observables

Luminosity and event rates

- If a collider has a luminosity of $1 \text{ pb}^{-1} \text{s}^{-1}$, then processes with a cross section of $1 \text{ pb}$ happen in average once per second.

- Units of cross sections: $1 \text{ barn} = 10^{-24} \text{cm}^2 = 100 \text{ fm}^2$

- Typical cross sections are in the nanobarn-femtobarn range.

- Typically a year of collider time $= 10^7 \text{ s beam-time}$. 
Aside: Particle decays

Width and lifetime

Master formula for a decay $P \rightarrow p_1 + p_2 + \cdots + p_n$

- Similar to scattering processes: partial width $\Gamma$

$$d\Gamma = \frac{(2\pi)^4 \delta^4(P - p_1 - p_2 - \cdots - p_n)}{2M}$$

$$\times \left| \langle p_1 p_2 \cdots p_n | \hat{S} | P \rangle \right|^2 \prod_{i=1}^{n} \frac{\mathrm{d}^3 p_i}{(2\pi)^3 2E_i}$$

- Total width (sum of all partial widths) connected to lifetime by $\tau = 1/\Gamma_{\text{tot}}$.

- Branching ratio for a specific decay: $BR_i = \Gamma_i / \Gamma_{\text{tot}}$
Scattering kinematics, once again

**Orientation w.r.t. the beam axis**

*Already seen: The beam axis is special*

- **Remember pair-production cross section**

\[
\sigma = \frac{\left| \langle 12|\hat{S}|34 \rangle \right|^2}{4\sqrt{(p_1p_2)^2 - m_1^2m_2^2}} \cdot \frac{\rho_3}{E} \cdot \frac{d^2\Omega_3}{4\pi}
\]

- **Clearly, the transition matrix element does not need to be independent of the four-momenta and their orientation (typically, the size is fixed)**

⇒ **The angular integration matters !!!**

Only process-independent axis = beam axis.
Scattering kinematics, once again

**Orientation w.r.t. the beam axis**

*Already seen: The beam axis is special*

- So, naively, \( d^2\Omega_3 \rightarrow 2\pi d \cos \theta_{3,\text{beam}} \)
- Alternatives (maybe boost independent)?
  - Rapidity additive w.r.t. boosts along beam axis
  - Convenient for hadron colliders

*Definition:*

\[
y = \frac{1}{2} \ln \frac{E - p_z}{E + p_z}
\]
Scattering kinematics, once again

Orientation w.r.t. the beam axis

Rapidities and boosts

- **Check behaviour boosts along z-axis:** Try boost $\gamma$

\[
E' = E \cosh \gamma - p_z \sinh \gamma
\]
\[
p'_z = p_z \cosh \gamma - E \sinh \gamma
\]

\[\Leftrightarrow \quad y' = \frac{1}{2} \ln \frac{E' + p'_z}{E' - p'_z} = \frac{1}{2} \ln \frac{(E + p_z)(\cosh \gamma - \sinh \gamma)}{(E - p_z)(\cosh \gamma + \sinh \gamma)} = y + \frac{1}{2} \ln \frac{e^{-\gamma}}{e^{\gamma}} = y - \gamma
\]

- **Rapidities characterise boosts** (equivalence).
Scattering kinematics, once again

Orientation w.r.t. the beam axis

Pseudorapidity

- Problem with rapidity: Not very geometric.
- Introduce a new quantity: pseudorapidity $\eta$

$$\eta = \ln \tan \frac{\theta}{2}$$

Looks a bit cumbersome, but

rapidity\(\text{(massless particle)}\) = pseudorapidity !

Remember: Particles in collider experiments are typically at velocities of roughly 1 (ultrarelativistic)
- Identical to massless limit
Scattering kinematics, once again

Orientation w.r.t. the beam axis

Transverse momentum

- Therefore: Longitudinal direction sorted out
  \[ \Rightarrow \text{for the transverse components directly take transverse momentum} \]
  (obviously invariant w.r.t. longitudinal boosts)

- One-particle phase-space then reads:

\[
\frac{d^3p}{(2\pi)^3 2E} \rightarrow \frac{p^3 dp_\perp dy d\phi}{2E^2 (2\pi)^3}
\]

- At LHC: Detector characteristics described in terms of transverse momentum and pseudorapidity.
Scattering kinematics, once again

**Mandelstam variables**

Pair production processes $p_1 + p_2 \rightarrow p_3 + p_4$

- There are three types of diagrams:

- They can be related to three independent Lorentz-scalars, the Mandelstam variables $s$, $t$, and $u$:

  $$s = (p_1 + p_2)^2 = (p_3 + p_4)^2,$$
  $$t = (p_1 - p_3)^2 = (p_2 - p_4)^2,$$
  $$u = (p_1 - p_4)^2 = (p_2 - p_3)^2$$
Scattering kinematics, once again

**Mandelstam variables**

Pair production processes $p_1 + p_2 \rightarrow p_3 + p_4$

- If a process is dominated by one of those channels, one therefore calls it an s-channel process (or t- or u-channel)

- Note: typically, for massless particles, s, t, and u are connected to the c.m. energy $E$ and the scattering angle $\theta$ through:

$$s = E^2, \quad t = -E^2 \frac{1 - \cos \theta}{2}, \quad u = -E^2 \frac{1 + \cos \theta}{2}$$

and, obviously, $s + t + u = 0$. 
Scattering kinematics, once again

**Mandelstam variables**

Pair production processes $p_1 + p_2 \rightarrow p_3 + p_4$

- *s-channel processes are particularly interesting, because they may exhibit resonances in the process.*

- To understand this, notice that typically, internal lines are related to propagators, which behave like

\[
\text{Prop} \sim \frac{1}{p^2 - m^2 + im\Gamma}
\]

if a particle with mass $m$ and width $\Gamma$ is propagating with momentum $p$. 
Scattering kinematics, once again

Mandelstam variables

Example $e^+ e^- \rightarrow q\bar{q}$