Some Examples

- Summary: The structure of Feynman rules and how to build Feynman diagrams
- Summary: Four-vectors and two-particle phase space
- Example 1: Higgs decay into fermions.
- Example 2: Electron-positron annihilation into fermions

The structure of Feynman rules

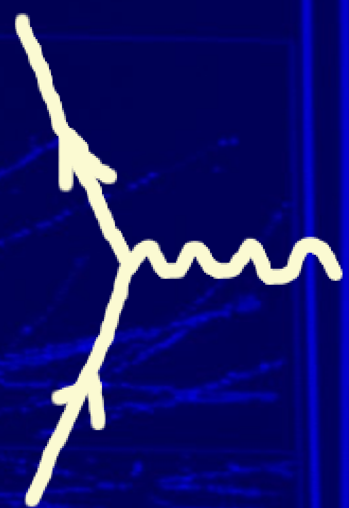
General structure

- In all interactions, charges are conserved.
- In all interactions, total four-momentum is conserved.
- In all interactions, baryon and lepton number are conserved (quarks and leptons must come in pairs).
- Lepton number is conserved per family in the Standard Model.
- There are never more than four particles entering a fundamental vertex (funny enough, two fermions count as three particles)

The structure of Feynman rules

- Interactions of the fermions:
  On the level of fundamental particles (quarks, leptons), there are only two types of interactions or vertices that are allowed for fermions:

Gauge interactions



The specific interaction occurs only, when the fermions have a corresponding charge:

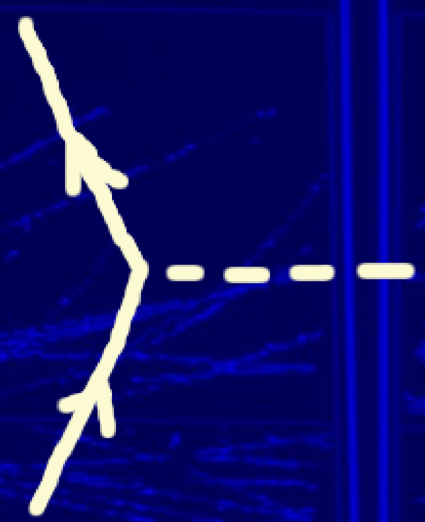
- Electromagnetic charge for photons
- Colour charge for gluons
- Weak charge (isospin) for W bosons.
- The Z boson couples to both electric charge and weak isospin.

All interactions are proportional to the corresponding coupling and the "charge".

The structure of Feynman rules

- Interactions of the fermions:
  On the level of fundamental particles (quarks, leptons), there are only two types of interactions or vertices that are allowed for fermions:

  Interaction with the Higgs boson

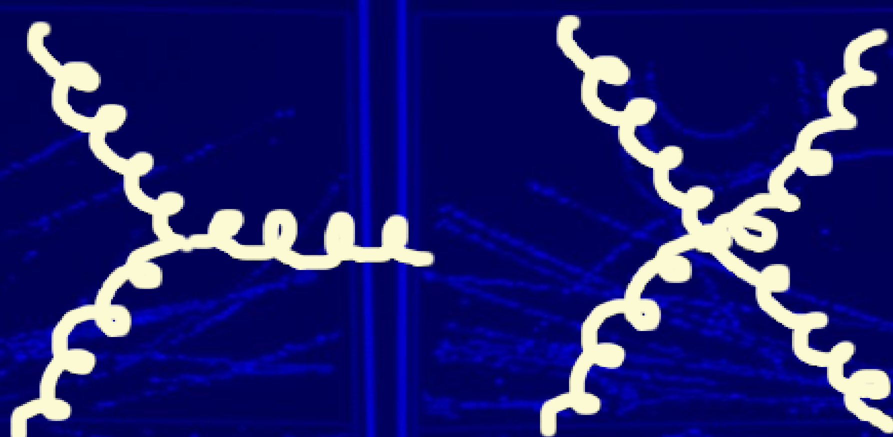


This interaction is proportional to the mass of the fermion – it therefore vanishes for neutrinos and it is important only for the members of the third family: the top and bottom quarks and the tau lepton.

The structure of Feynman rules

- Self-Interactions of the gauge bosons:
  apart from the interaction with the fermions, the gauge bosons may interact among themselves, but only, if they carry corresponding charges.

The self-interactions of the gluons

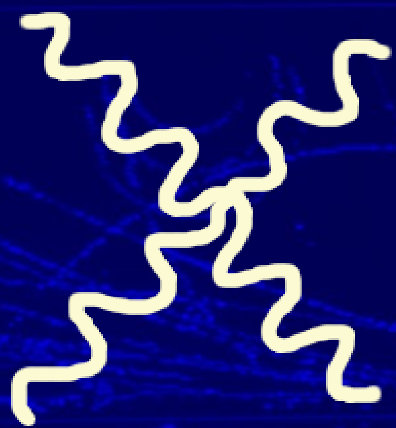


The gluons carry a colour charge – therefore they “see” each other. However, there can be maximally four gluons in one fundamental vertex. This is true for all gauge bosons.

The structure of Feynman rules

- Self-Interactions of the gauge bosons: apart from the interaction with the fermions, the gauge bosons may interact among themselves, but only, if they carry corresponding charges.

The interactions of the W bosons



The photon and Z boson "see" the weak and electromagnetic charge of the W bosons. For the 4-boson vertex the combinations with 4 and 2 W bosons exist. There are no fundamental self-interactions without the W-bosons!

The structure of Feynman rules

- Interactions of the gauge with the Higgs bosons: Leaving aside self-interactions of the Higgs boson, there is only one more type of interaction, namely between gauge and Higgs bosons. They reflect the mechanism that gives mass to the gauge bosons

The WWH and ZZH interactions are proprtional to the mass of the gauge boson. The WWHH and ZZHH vertices are "gauge" interactions, proportional to a weak coupling.

How to build Feynman diagrams

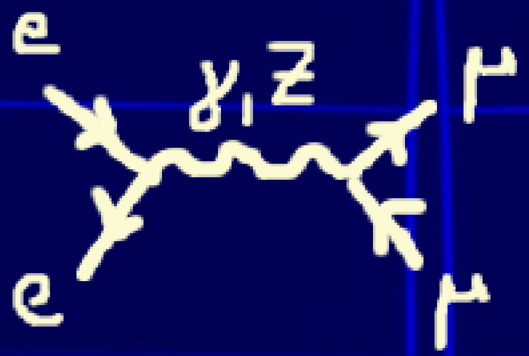
"Cookbook" recipe for tree-level

- Label the external particles (legs) and their momenta in a unique way (e.g. p_1 to p_N) and group them in all permutations of N
- Take pairs of neighbouring particles and check whether there is any vertex mapping joining them. If so, draw the line of the particle and repeat this step until all particles are connected.
- Make sure the arrows on the fermions are correct (direction for particles/anti-particles). In the Standard Model, there are no "clashing" arrows: Any fermion-line can have arrows only pointing along one direction of the line.
- Note: For meson decays it is sometimes useful to decompose the meson into its quark content – one of the quarks may be a non-interacting spectator.

How to build Feynman diagrams

Examples:

- $e^+e^- \longrightarrow \mu^+\mu^-$



There are two diagrams, both in the s-channel, where the electron-positron pair fuses into a gauge boson – either the photon or the Z-boson. The Higgs boson has been omitted here, due to the small electron and muon mass. Lepton flavour conservation disallows vertices where an electron and a muon are joined by a gauge boson.

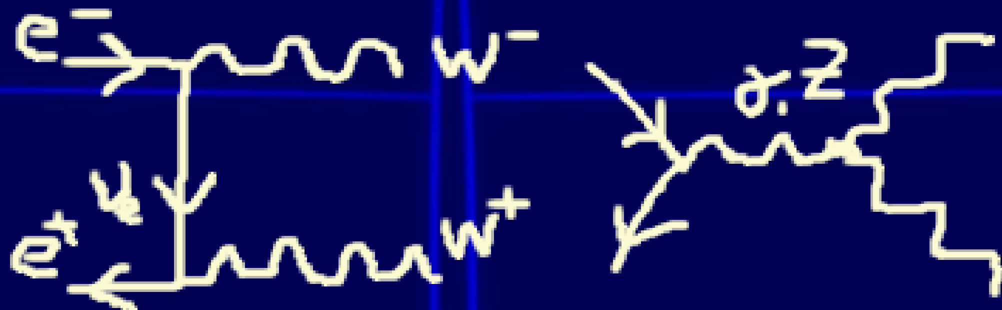
- $u\bar{u} \longrightarrow \nu_e\bar{\nu}_e$

This time, there is only one diagram, because the neutrinos have no electrical charge and because in the Standard Model, they are massless. The catch here will be the treatment of colour – more later.

How to build Feynman diagrams

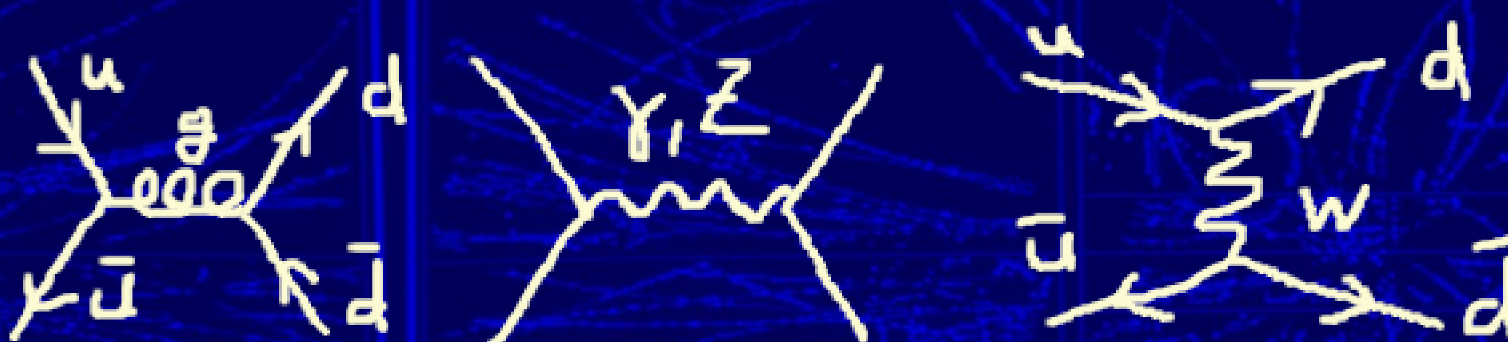
Examples:

- $e^+e^- \rightarrow W^+W^-$



There are three diagrams, one with an exchange of a neutrino in the t-channel and two with a three-gauge boson vertex The Higgs boson has again been omitted here, due to the small electron mass.

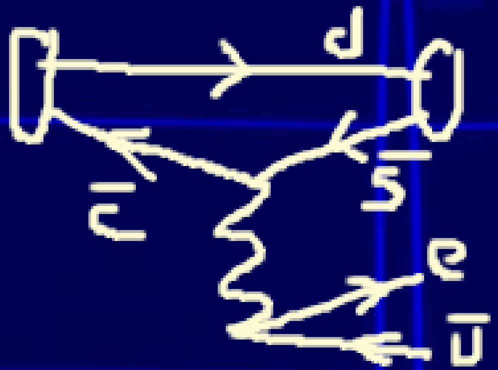
- $u\bar{u} \rightarrow d\bar{d}$



How to build Feynman diagrams

Examples:

- $D^- \rightarrow K^0 e \bar{\nu}_e$



This time, the trick is to know the flavour composition of the mesons and to guess/reconstruct the transition taking place on the quark level. Typically, there is then one spectator involved, in this case the d-quark.

- $b \rightarrow s\gamma$

This is a tricky case, since there is no way that flavour changes when coupling to a photon – therefore such a transition, if allowed, has to be mediated by loop-level diagrams, one of which is shown here. Typically, in such cases, two instances of the CKM matrix are needed. Such processes have a very high phenomenological significance.

Four-vectors and phase space

Four-vectors:

- Summary of calculation rules – index free:

$$p^2 = p \cdot p = E^2 - \vec{p}^2 = E^2 - p_x^2 - p_y^2 - p_z^2$$

$$p_1 \cdot p_2 = E_1 E_2 - \vec{p}_1 \cdot \vec{p}_2$$

$$(p_1 + p_2)^2 = p_1^2 + p_2^2 + 2p_1 \cdot p_2 \equiv m_{12}^2 \geq (m_1 + m_2)^2$$

- Energy-momentum relation (a.k.a. "on-shell condition")

$$p^2 = E^2 - \vec{p}^2 = m^2$$

- Four-momentum conservation (for $p_1 + p_2 \to p_3 + p_4$)

$$p_1^\mu + p_2^\mu - p_3^\mu - p_4^\mu = 0 \Longrightarrow \delta^4(p_1^\mu + p_2^\mu - p_3^\mu - p_4^\mu)$$

often the indices are omitted in the delta-function

Four-vectors and phase space

The delta-function:

- Properties:

$$\int_a^b \mathrm{d}x\, f(x)\delta(x-c) = \begin{cases} f(c) & \text{if } c \in [a,b] \\ 0 & \text{else} \end{cases}$$

$$\int \mathrm{d}x\delta(kx-c) = \frac{1}{k}\int \mathrm{d}x\delta(x-c/k)$$

$$\int \mathrm{d}x\, f(x)\delta(x^2-a^2) = \int \mathrm{d}x\, f(x)\frac{\delta(x-a)+\delta(x+a)}{2|a|}$$

- Integration over volumes:

$$\int \mathrm{d}^3x\, f(\vec{x})\delta^3(\vec{x}-\vec{a}) = f(\vec{a})$$

Four-vectors and phase space

The phase space integration, revisited:

$$\mathrm{d}PS^{(2)} = \int \frac{\mathrm{d}^4 p_1}{(2\pi)^4} \frac{\mathrm{d}^4 p_2}{(2\pi)^4} (2\pi)^4 \delta^4(P - p_1 - p_2)$$

$$\times (2\pi)\delta(p_1^2 - m_1^2)\Theta(E_1)(2\pi)\delta(p_2^2 - m_2^2)\Theta(E_2)$$

- First step: Do the 4-d volume integral over p_2 with the 4-d delta-function. This implies that $p_2^\mu = (P - p_1)^\mu$.

$$\mathrm{d}PS^{(2)} = \int \frac{\mathrm{d}^4 p_1}{(2\pi)^2} \delta(p_1^2 - m_1^2)\delta((P - p_1)^2 - m_2^2)$$

$$\times \Theta(E_1)\Theta(E - E_1)$$

Four-vectors and phase space

Disentangle energy and three momentum for the remaining four-momentum. Use the Theta-functions for the energy interval. Use polar coordinates for the three-momentum:

$$\mathrm{d}^3\vec{p}_1 = |\vec{p}_1|^2 \mathrm{d}|\vec{p}_1| \sin\theta \mathrm{d}\theta \mathrm{d}\phi = \rho_1^2 \mathrm{d}\rho_1 \mathrm{d}\cos\theta \mathrm{d}\phi = \rho_1^2 \mathrm{d}\rho_1 \mathrm{d}^2\Omega_1$$

- Write $P^\mu = (M, \vec{0}) \Longrightarrow (P - p_1)^2 = (M - E_1)^2 - \rho_1^2$
- This yields

$$\mathrm{d}PS^{(2)} = \int_0^M \mathrm{d}E_1 \int \frac{\rho_1^2 \mathrm{d}\rho_1 \mathrm{d}^2\Omega_1}{4\pi^2} \delta(E_1^2 - \rho_1^2 - m_1^2)$$
$$\times \delta((M - E_1)^2 - \rho_1^2 - m_2^2)$$

Four-vectors and phase space

- *Do the integral over the three-momentum with help of the first delta-function, leading to:* $\rho_1^2 = E_1^2 - m_1^2$ *. Use this in the second delta function.*

$$\mathrm{d}PS^{(2)} = \frac{\mathrm{d}^2\Omega_1}{4\pi^2}\int\limits_0^M \mathrm{d}E_1 \frac{\rho_1^2}{2\rho_1}\delta(M^2 - 2ME_1 + m_1^2 - m_2^2)$$

- *fixing the energy to read*

$$E_1 = \frac{M^2 + m_1^2 - m_2^2}{2M}$$

Four-vectors and phase space

- The final expression yields

$$\mathrm{d}PS^{(2)} = \frac{\mathrm{d}^2\Omega_1}{4\pi^2}\frac{\rho_1}{4M} = \frac{\mathrm{d}^2\Omega_1}{4\pi^2}\frac{\sqrt{(M^2-m_1^2-m_2^2)^2-4m_1^2m_2^2}}{8M^2}$$

- The kinematics in the rest frame is entirely fixed by:

$$E_{1,2} = \frac{M^2 \pm (m_1^2 - m_2^2)}{2M}$$

$$\rho_{1,2} \equiv \rho = \frac{\sqrt{(M^2-m_1^2-m_2^2)^2-4m_1^2m_2^2}}{2M}$$

$$\vec{p}_1 = -\vec{p}_2 = \rho\vec{e}_p$$

Widths

- From the phase space, the final expressions for decay widths $P \longrightarrow p_1 + p_2$ can be written as:

$$\mathrm{d}\Gamma = \frac{1}{2M} |\overline{\mathcal{M}}_{P \to 12}|^2 \frac{|\vec{p}_1|}{4M} \frac{\mathrm{d}^2\Omega_1}{4\pi^2}$$

  where the squared matrix element is summed over all outgoing spins and colours and averaged over the incoming spins and colours.

- Typically the kinematics is fixed in the c.m. frame of the decaying particle – in the massless case this reads:

$$P^\mu = (M, 0, 0, 0)$$

$$p^\mu_{1,2} = M/2(1, \pm \sin\theta \cos\phi, \pm \sin\theta \sin\phi, \pm \cos\theta))$$

Cross sections

- From the phase space, the final expressions for scattering cross sections $p_1 + p_2 \to p_3 + p_4$ can be written as:

$$\mathrm{d}\sigma = \frac{1}{4\sqrt{(p_1 p_2)^2 - m_1^2 m_2^2}} |\overline{\mathcal{M}}_{12\to 34}|^2 \frac{|\vec{p}_3|}{4\sqrt{(p_1+p_2)^2}} \frac{\mathrm{d}^2\Omega_3}{4\pi^2}$$

where the squared matrix element is summed over all outgoing spins and colours and averaged over the incoming spins and colours.

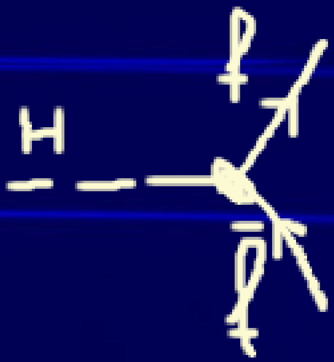
- Typically the kinematics is fixed in the c.m. frame of the incoming particles – in the massless case this reads:

$$p_{1,2}^{\mu} = E(1, 0, 0, \pm 1)$$

$$p_{3,4}^{\mu} = E(1, \pm\sin\theta\cos\phi, \pm\sin\theta\sin\phi, \pm\cos\theta))$$

Examples:
Higgs decay into fermions

- There is only one Feynman diagram:



$$\Longrightarrow |\mathcal{M}_{H\to f\bar{f}}|^2 \sim \frac{m_f^2}{v^2}$$

- The full amplitude squared, summed over all spins (but not over colours, if f is a quark) is given by

$$|\mathcal{M}_{H\to f\bar{f}}|^2 = \frac{4m_f^2}{v^2}(p_f \cdot p_{\bar{f}} - m_f^2)$$

Examples:

Higgs decay into fermions

- The width is given by

$$\int \mathrm{d}\Gamma_{H\to f\bar{f}} = \frac{1}{2M_H} \int \mathrm{d}PS^{(f\bar{f})} |\mathcal{M}_{H\to f\bar{f}}|^2$$

- Using the decay kinematics and four-momentum conservation

$$p_H = p_f + p_{\bar{f}}$$

yields

$$|\mathcal{M}_{H\to f\bar{f}}|^2 = \frac{2m_f^2 m_H^2}{v^2}\left(1 - \frac{4m_f^2}{m_H^2}\right)$$

Examples:

Higgs decay into fermions

- Adding the phase space and doing the angle integral gives:

$$\Gamma_{H\to f\bar{f}} = \frac{m_f^2 m_H N_c}{8\pi v^2}\left(1 - \frac{4m_f^2}{m_H^2}\right)^{3/2}$$

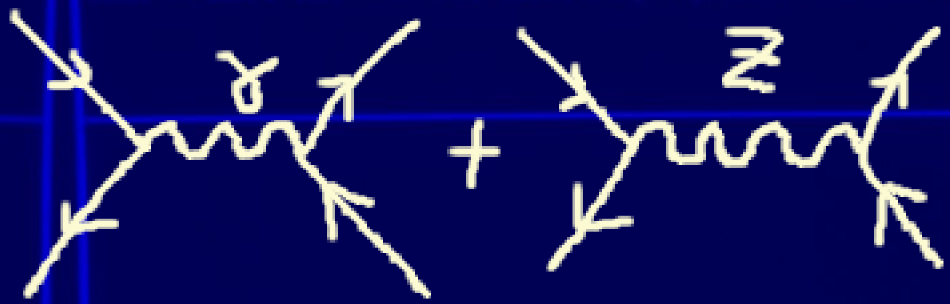
- The extra factor N_c comes into play to take into account the fact that the fermions may be coloured:
  - For quarks, $N_c = 3$, for leptons, $N_c = 1$
- For a Higgs boson with m = 120 GeV

$$\Gamma_{H\to b\bar{b}} \approx 5\,\mathrm{MeV}, \quad \Gamma_{H\to\tau\bar{\tau}} \approx 0.25\,\mathrm{MeV}$$

Examples:

ee-annihilation into fermions

- Only the final state consisting of a massless fermion pair will be considered. There are two Feynman diagrams



- The two internal lines, on the amplitude level, behave like

$$\sim \frac{1}{p^2 - M^2 + i\Gamma M}$$

if M is the mass of the particle and Γ is its width. Obviously, both are 0 for photons.

Examples:

ee-annihilation into fermions

- In the limit of low centre-of-mass energies around or less30 GeV, the momentum p going through the photon or Z line is small compared to the Z boson mass. In this case the second diagram can be neglected.
- The amplitude squared for the purely photonic process then reads

$$|\mathcal{M}_{e^+e^-\to f\bar{f}}|^2 = \frac{32e^4 e_e^2 e_f^2}{s^2}\left[(p_1p_3)(p_2p_4)+(p_1p_4)(p_2p_3)\right]$$

Examples:

ee-annihilation into fermions

- In the c.m. frame of the scattering process, the momenta are given by:

$$p_{1,2} = E(1, 0, 0, \pm 1)$$

$$p_{3,4} = E(1, \pm \sin\theta \cos\phi, \pm \sin\theta \sin\phi, \pm \cos\theta)$$

- Therefore:

$$2p_1 \cdot p_3 = 2p_2 \cdot p_4 = 2E^2(1 - \cos\theta)$$

$$2p_1 \cdot p_4 = 2p_2 \cdot p_3 = 2E^2(1 + \cos\theta)$$

$$2p_1 \cdot p_2 = 2p_3 \cdot p_4 = 4E^2 = s$$

Examples:

ee-annihilation into fermions

- Plugging in the kinematics yields

$$|\mathcal{M}_{e^+e^-\to f\bar{f}}|^2 = \frac{32e^4 e_e^2 e_f^2}{s^2}\left[(p_1p_3)(p_2p_4) + (p_1p_4)(p_2p_3)\right]$$

$$= 2e^4 e_e^2 e_f^2\left[(1-\cos\theta)^2 + (1+\cos\theta)^2\right]$$

- Averaging over the incoming spins, the cross section reads

$$\mathrm{d}\sigma = \frac{1}{2s}\frac{\mathrm{d}\cos\theta\mathrm{d}\phi}{4\pi^2}\frac{E}{8E}\cdot\frac{2e^4 e_f^2 e_e^2}{4}\left[2(1+\cos^2\theta)\right]$$

$$= \frac{\pi\alpha^2 e_f^2 e_e^2}{2s}\left[1+\cos^2\theta\right]\mathrm{d}\cos\theta$$

Examples:

ee-annihilation into fermions

- Integrating over the scattering angle gives the total cross section (colour factor added, as before):

$$\sigma = \frac{4\pi\alpha^2 e_e^2 e_f^2 N_c}{3s}$$

- Adding in numbers gives (for muons with $e_f = 1$)

$$\sigma \approx \frac{2.3 \cdot 10^{-4} \times 3.89 \cdot 10^8 \,\mathrm{pb}\,\mathrm{GeV}^2}{E_{\mathrm{c.m.}}^2} \approx 87\mathrm{nb} \cdot \frac{\mathrm{GeV}^2}{E_{\mathrm{c.m.}}^2}$$