Introduction to particle physics
Lecture 9: Gauge invariance

Frank Krauss

IPPP Durham

U Durham, Epiphany term 2010
Outline

1. Symmetries
2. Classical gauge invariance
3. Phase invariance
4. Generalised phase invariance
Symmetries in classical physics

Invariance and conservation laws

- From classical physics it is known that invariance of a system under certain transformations is related to the conservation of corresponding quantities.

- Examples:

<table>
<thead>
<tr>
<th>Invariance under</th>
<th>Conserved quantity</th>
</tr>
</thead>
<tbody>
<tr>
<td>rotations</td>
<td>⇐⇒ angular momentum</td>
</tr>
<tr>
<td>time translations</td>
<td>⇐⇒ energy</td>
</tr>
<tr>
<td>space translations</td>
<td>⇐⇒ momentum</td>
</tr>
</tbody>
</table>

- Formalised by Emmy Noether (thus: Noether’s theorem)
Invariance and conservation laws

- The same ideas work also in quantum physics: Invariances give rise to conservation laws.
- There, however, internal symmetries also play a role. In fact, they are used to construct interactions in theories.
- Example in particle physics:
  - Invariance under phase transformations of the fields
    \[ \psi(x, t) \rightarrow \psi'(x, t) = \exp(i\theta)\psi(x, t) \iff |\psi|^2 = |\psi'|^2 \]
    yields conserved charges like, e.g., the electrical charge.
  - Note: global changes in phase cannot be observed (because typically squares are taken), but phase differences are observable.
    (In Quantum Mechanics: Aharonov-Bohm effect.)
  - The photon field couples to this charge and is thus related to the invariance under such phase transitions (later more). (Will come to that later.)
Mathematical formulation

- Ideas of symmetry are formalised in group theory.
- **Definition of groups:**
  - Consider sets of elements $S = \{a, b, \ldots\}$ with operation "·": $a \cdot b$.
  - Such sets are called groups, if:
    - $a \cdot b \in S$ (closure)
    - $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ (associativity)
    - $\exists 1 \in S$: $a \cdot 1 = 1 \cdot a = a \forall a \in S$ (neutral element)
    - $\forall a \in S$: $\exists a^{-1} \in S$ such that $a \cdot a^{-1} = a^{-1} \cdot a = 1$ (inverse element).
- **Examples:** $S =$ integer numbers, $\cdot = +$, rotations with arbitrary angles, the set $\{1, 2, 3, \ldots, p - 1\}$ under multiplication modulo $p$, if $p$ is a prime number, \ldots
Discrete vs. continuous symmetries

- Consider two slabs with quadratic and round cross section.
- The quadratic one has a discrete symmetry w.r.t. rotation along its axis, while the round one enjoys a continuous symmetry.

only multiples of 90 degrees

all angles

More physical examples: parity vs. angular momentum
Classical gauge invariance

Fields and potentials in electrodynamics

- Remember Maxwell's equation:

\[
\begin{align*}
\nabla \cdot E &= 4\pi \rho \\
\n\nabla \times E + \frac{\partial B}{\partial t} &= 0 \\
\n\nabla \cdot B &= 0 \\
\n\n\nabla \times B - \frac{\partial E}{\partial t} &= 4\pi j.
\end{align*}
\]

- Implicit: conservation of current, \( \dot{\rho} + \nabla \cdot j = 0 \).

- Can introduce potentials \( \Phi \) and \( A \) such that

\[
\begin{align*}
E &= -\nabla \Phi - \frac{\partial A}{\partial t} \quad \text{and} \quad B = \nabla \times A.
\end{align*}
\]

(Can read them off from homogeneous equations, i.e. equations of the form l.h.s.\(=0 \).)

- **Gauge invariance**: The electromagnetic fields will not change under

\[
\begin{align*}
\Phi \quad \Rightarrow \quad \Phi' &= \Phi + \frac{\partial \Lambda}{\partial t} \quad \text{and} \quad A \quad \Rightarrow \quad A' &= A - \nabla \Lambda
\end{align*}
\]

(This is the gauge transformation of classical electrodynamics with an arbitrary scalar function \( \Lambda \).)
Lorentz force

- Lorentz-force reads:

\[
F = e \left[ E + \frac{dx}{dt} \times B \right] = e \left[ -\nabla \Phi - \frac{\partial A}{\partial t} + \frac{dx}{dt} \times \nabla \times A \right] = e \left[ -\nabla \Phi - \frac{dA}{dt} + \nabla \cdot \left( \frac{dx}{dt} \cdot A \right) \right].
\]

To see this, use that \( \frac{dA_x}{dt} = \frac{\partial A_x}{\partial t} + \frac{\partial x}{\partial t} \frac{\partial A_x}{\partial x} + \frac{\partial y}{\partial t} \frac{\partial A_x}{\partial y} + \frac{\partial z}{\partial t} \frac{\partial A_x}{\partial z} \)

and that \( (v \times \nabla \times A)_x = v_y \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) + v_z \left( \frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} \right) + v_x \left( \frac{\partial A_x}{\partial x} - \frac{\partial A_x}{\partial x} \right) \)

and that, since \( \frac{\partial v_x}{\partial x} = \frac{\partial v_y}{\partial x} = \frac{\partial v_z}{\partial x} = 0, v \cdot (\nabla \cdot A) = \nabla (v \cdot A) \)

This can be used to construct a Lagrange function, rederive E.o.M. with Euler-Lagrange method & confirm the force, assess symmetries, construct a Hamilton function to handle the quantum mechanical problem ....
Lagrange and Hamilton function

Therefore: Lagrange function is given by

\[ L = \frac{m}{2} \left( \frac{dx}{dt} \right)^2 - e \left( \Phi + \frac{dx}{dt} \cdot A \right) \]

and the Hamilton function reads

\[ H = \frac{1}{2m} (p - eA)^2 + e\Phi. \]

Can use this to generalise for all situations:

To treat interactions of a particle with a field, replace the momentum with the generalised one,

\[ p \rightarrow \Pi = p - eA. \]

I will argue that this can be enforced by a gauge principle.
Phase invariance

Consider a system in a state $|\psi(t)\rangle$, alternatively described by a wave function $\psi(t, x) = \langle x | \psi(t) \rangle$.

Remember that probabilities are related to $\langle \psi(t) | \psi(t) \rangle$ or $|\psi(t, x)|^2$.

Clearly, one can redefine

$$
|\psi(t)\rangle \longrightarrow |\psi(t)'\rangle = e^{-i\alpha} |\psi(t)\rangle \\
\psi(t, x) \longrightarrow \psi(t, x)' = e^{-i\alpha} \psi(t, x).
$$

without changing measurements related to $\langle \psi(t) | \psi(t) \rangle$ or $|\psi(t, x)|^2$.

This phase invariance is a simple example of a continuous symmetry, described by a continuous parameter, here the phase $\alpha$.

Symmetries are described by groups, represented by matrices. The group related to this symmetry is called $U(1)$, the group of unitary matrices of dimension 1 (complex numbers with absolute value 1).
QED: Global phase invariance

- Lagrange formulation with fields:
  - generalised coordinates \( q(t) \) \(\rightarrow\) fields \( \phi(t, x) \).
  - Lagrange function \( L(q, \dot{q}) \) \(\rightarrow\) Lagrangian density \( \mathcal{L}(\phi(x_\mu), \partial_\mu(\phi)) \) with \( L = \int d^3 x \mathcal{L} \).

- Example: Free complex scalar field(s) \( \phi \) and \( \phi^* \) with mass \( m \):

\[
\mathcal{L} = (\partial_t \phi)(\partial_t \phi^*) - \nabla \phi \cdot \nabla \phi^* - m^2 \phi \phi^* .
\]

- Lagrange function (and with it E.o.M.) invariant under transformation \( G \):

\[
G\mathcal{L}(\phi, \phi^*) = \mathcal{L}(\phi, \phi^*)
\]

\[
G\phi = \phi' = e^{-i\alpha}\phi \quad \text{and} \quad G\phi^* = \phi'^* = e^{i\alpha}\phi^* .
\]

- This yields conserved charges \( \pm 1 \) (group \( G \) again \( U(1) \)).

- Since the transformation acts the same way on all points in space-time, a symmetry transformation like this is called **global**.
Local phase transformations

- To measure phase differences: Must establish a specific \( \theta = 0 \).
- Different conventions related by global phase transformations. Clearly the choice, being unobservable, must not matter for physical observables:
  The theory must be invariant under global phase transformations.
- What happens if the phase depends on space-time: \( \theta \rightarrow \theta(t, x) \)? (This is called a local phase transformation.)
  Simple answer: Then the Lagrangian is not invariant any more.

\[
\mathcal{L}'(\phi) = \mathcal{G}(x)\mathcal{L}(\phi) \rightarrow \mathcal{L}'(\phi) \neq \mathcal{L}(\phi).
\]

\[
\begin{align*}
\mathcal{L}' &= \left[ e^{-i\alpha}(-i\partial_t \alpha \cdot \phi + \partial_t \phi) \right] \left[ e^{+i\alpha}(+i\partial_t \alpha \cdot \phi^* + \partial_t \phi^*) \right] \\
&\quad - \left[ e^{-i\alpha}(-i\nabla \alpha \cdot \phi + \nabla \phi) \right] \cdot \left[ e^{+i\alpha}(+i\nabla \alpha \cdot \phi^* + \nabla \phi^*) \right] - m^2 \phi \phi^* \\
&= \mathcal{L} + (i\partial_t \alpha) \left( \phi^* \partial_t \phi - \phi \partial_t \phi^* \right) - (i\nabla \alpha) \cdot \left( \phi^* \nabla \phi - \phi \nabla \phi^* \right) + \left[ (\partial_t \alpha)^2 - (\nabla \alpha)^2 \right] \phi \phi^*.
\end{align*}
\]
**Restoration of local phase invariance**

- But: Can make the Lagrangian invariant by introducing another field, $A$.

  $$G(x)\mathcal{L}(\phi, \phi^*, A) \rightarrow \mathcal{L}(\phi', \phi'^*, A').$$

- Properties of this field:
  - Must be massless to allow for infinite range - it must connect different phase conventions all over space.
  - It’s a four vector, $A^\mu$, identified with the photon field.
  - The photon field must also transform under $G(x)$ such that the combination with changes due to the electron field are compensated.

- Couple it with the replacement (from Lorentz force)
  $$p^\mu = (E, p) \rightarrow \Pi^\mu = (p^\mu - eA^\mu) = (E - e\Phi, p - eA)$$

- Summary of this construction:
  - Global phase invariance yields conserved charges.
  - Local phase invariance gives rise to the photon field, i.e. interactions.

- Final remark: A trivial mass term for the photon would look like
  $$\mathcal{L}_m \propto m^2 A^2$$ and it is not invariant under local phase transformations.
Relation to gauge invariance in classical theory

- Write the photon field as $A^\mu = (\Phi, \vec{A})$, where
  \[
  \vec{E} = -\vec{\nabla}\Phi - \partial_t \vec{A} \quad \text{and} \quad \vec{B} = \vec{\nabla} \times \vec{A}
  \]
gives the relation to the electric and magnetic field.

- The potentials are invariant under the **gauge transformation**
  \[
  \Phi \rightarrow \Phi' = \Phi + \partial_t \Lambda \quad \text{and} \quad \vec{A} \rightarrow \vec{A}' = \vec{A} - \vec{\nabla} \Lambda,
  \]
or, in four-vector notation ($\partial_\mu = (\partial_t, -\vec{\nabla})$)
  \[
  A_\mu \rightarrow A'_\mu = A_\mu + \partial_\mu \Lambda.
  \]

- Invariance of the Lagrangian $\mathcal{L} = \vec{E}^2 - \vec{B}^2$ follows trivially.

- Finally: The $\Lambda(x)$ here is more or less identical with the $\theta(x)$ of the local gauge transformation before.
Generalised gauge invariance

More complicated symmetries

- Obviously, this can be extended by making the state vector/wave function a vector with components labelled by $j \in [1, N]$, such that
  \[ |\psi(t, x)|^2 = \sum_j |\psi(t, x)_j|^2 \]

- Generalised phase transformation assumes $N \times N$-matrix character,
  \[ |\psi(t)\rangle_j \rightarrow |\psi(t)\rangle'_k = \left[ e^{-i\alpha} \right]_{kj} |\psi(t)\rangle_j \]
  \[ \psi(t, x)_j \rightarrow \psi(t, x)'_k = \left[ e^{-i\alpha} \right]_{kj} \psi(t, x)_j. \]

- Can use base matrices (generators) $T^a_{kj}$ such that
  \[ \left[ e^{-i\alpha} \right]_{kj} = e^{-i \sum_\alpha \alpha^a T^a_{kj}}, \quad \alpha^a \in \mathbb{R}. \]

Information about the allowed transformations contained in the form and properties of the $N \times N$ base matrices $T^a_{kj}$.

- Prominent examples $SO(N), SU(N)$:
A “classical” example

- Seemingly, gauge invariance an elegant way to produce interactions.
- Added benefit: protects high-energy behaviour of QED.  

Extent this to other interactions, e.g. of nucleons with pions:
- Pions transform nucleons into nucleons, put $p$ and $n$ into iso-doublet.

Then: Need gauge transformations acting on the nucleon field $N = (p, n)$, mixing the states.
- $G(x)$ must have $2 \times 2$ matrix form $\implies$ use Pauli matrices as basis:
  - There are 3 Pauli matrices - each corresponds to a field: 3 $\rho$’s!

Due to Gell-Mann-Nishijima formula: $\rho$’s carry isospin $\implies$ self-interactions!

- Can show that Lagrangian is invariant under global $SU(2)$:
  - $G^{SU(2)} \mathcal{L}(N) \to \mathcal{L}(N')$.

But: $\rho$’s not elementary and $\pi$’s are the “true” isospin force carriers.
Summary

- Introduced the concept of symmetries and their role in creating interactions.