Introduction to Loop Calculations

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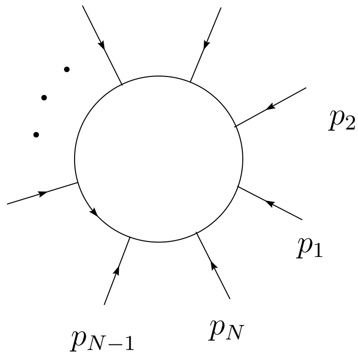
1 One-loop integrals

Consider a generic one-loop diagram with N external legs and N propagators. If k is the loop momentum, the propagators are $q_a = k + r_a$, where $r_a = \sum_{i=1}^{a} p_i$. If we define all momenta as incoming, momentum conservation implies $\sum_{i=1}^{N} p_i = 0$ and hence $r_N = 0$.



If the vertices in the diagram above are non-scalar, this diagram will contain a Lorentz tensor structure in the numerator, leading to tensor integrals of the form

$$I_N^{D,\mu_1\ldots\mu_r}(S) \;=\; \int_{-\infty}^{\infty} \frac{d^Dk}{i\pi^{\frac{D}{2}}} \frac{k^{\mu_1}\ldots k^{\mu_r}}{\prod_{i\in S}(q_i^2 - m_i^2 + i\delta)}\,, \tag{1}$$

but we will first consider the scalar integral only, i.e. the case where the numerator is equal to one. S is the set of propagator labels, which can be used to characterise the integral, in our example $S = \{1, \ldots, N\}$. We use the integration measure $d^Dk/i\pi^{\frac{D}{2}} \equiv d\bar{k}$ to avoid ubiquitous factors of $i\pi^{\frac{D}{2}}$ which will arise upon momentum integration. D is the space-time dimension the loop momentum k lives in. In $D = 4$ dimensions, the loop integrals may be divergent either for $k \to \infty$ (ultraviolet divergences) or for $q_i^2 - m_i^2 \to 0$ (infrared divergences) and therefore need a regulator. A convenient regularisation method is dimensional regularisation.

1.1 Dimensional regularisation

Dimensional regularisation has been introduced in 1972 by ‘t Hooft and Veltman (and by Bollini and Gambiagi) as a method to regularise ultraviolet (UV) divergences in a gauge invariant way, thus completing the proof of renormalizability.
The idea is to work in $D = 4 - 2\epsilon$ space-time dimensions. Divergences for $D \to 4$ will thus appear as poles in $1/\epsilon$.
An important feature of dimensional regularisation is that it regulates infrared (IR) singularities, i.e. soft and/or collinear divergences due to massless particles, as well. Ultraviolet divergences occur if the loop momentum $k \to \infty$, so in general the UV behaviour becomes better for $\epsilon > 0$, while the IR behaviour becomes better for $\epsilon < 0$. Certainly we cannot have $D < 4$ and $D > 4$ at the same time. What is formally done is to first assume the IR divergences are regulated in some other way, e.g. by assuming all external legs are off-shell or by introducing a small mass for all massless particles. Assuming $\epsilon > 0$ we obtain a result which is well-defined

(UV convergent), which we can analytically continue to the whole complex D-plane, in particular to $\mathrm{Re}(D) > 4$. if we now remove the auxiliary IR regulator, the IR divergences will show up as $1/\epsilon$ poles.

The only change to the Feynman rules to be made is to replace the couplings in the Lagrangian $g \to g\mu^\epsilon$, where μ is an arbitrary mass scale. This ensures that each term in the Lagrangian has the correct mass dimension.

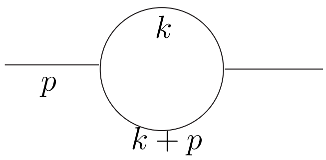
1.2 Feynman parameters

To combine products of denominators of the type $D_i = [(k+r_i)^2 - m_i^2 + i\delta]^{\nu_i}$ into one single denominator, we can use the identity

$$\frac{1}{D_1^{\nu_1} D_2^{\nu_2} \dots D_N^{\nu_N}} = \frac{\Gamma(\sum_{i=1}^N \nu_i)}{\prod_{i=1}^N \Gamma(\nu_i)} \int_0^\infty \prod_{i=1}^N dz_i\, z_i^{\nu_i - 1} \frac{\delta(1 - \sum_{j=1}^N z_j)}{[z_1 D_1 + z_2 D_2 + \dots + z_N D_N]^{\sum_{i=1}^N \nu_i}} \tag{2}$$

The integration parameters z_i are called *Feynman parameters*. For a generic one-loop diagram as shown above we have $\nu_i = 1\ \forall i$.

Simple example: one-loop two-point function



The corresponding integral is given by

$$\begin{aligned}
I_2 &= \int_{-\infty}^{\infty} \frac{d^D k}{(2\pi)^D} \frac{1}{[k^2 - m^2 + i\delta][(k+p)^2 - m^2 + i\delta]} \\
&= \Gamma(2) \int_0^\infty dz_1 dz_2 \int_{-\infty}^{\infty} \frac{d^D k}{(2\pi)^D} \frac{\delta(1 - z_1 - z_2)}{[z_1\,(k^2 - m^2) + z_2\,((k+p)^2 - m^2) + i\delta]^2} \\
&= \Gamma(2) \int_0^1 dz_2 \int_{-\infty}^{\infty} \frac{d^D k}{(2\pi)^D} \frac{1}{[k^2 + 2\,k\cdot Q + A + i\delta]^2} \\
Q^\mu &= z_2\, p^\mu \\
A &= z_2\, p^2 - m^2
\end{aligned} \tag{3}$$

where the δ-constraint has been used to eliminate z_1.

1.3 Momentum integration

Our general integral, after Feynman parametrisation, is of the following form

$$\begin{aligned}
I_N^D &= \Gamma(N) \int_0^\infty \prod_{i=1}^N dz_i\, \delta(1 - \sum_{l=1}^N z_l) \int_{-\infty}^{\infty} d\bar{k} \left[k^2 + 2k\cdot Q + \sum_{i=1}^N z_i\,(r_i^2 - m_i^2) + i\delta \right]^{-N} \\
Q^\mu &= \sum_{i=1}^N z_i\, r_i^\mu \,.
\end{aligned} \tag{4}$$

Now we perform the shift $l = k + Q$ to eliminate the term linear in k in the square bracket to arrive at

$$I_N^D = \Gamma(N) \int_0^\infty \prod_{i=1}^N dz_i\, \delta(1 - \sum_{l=1}^N z_l) \int_{-\infty}^\infty d\bar{l}\, \left[l^2 - R^2 + i\delta\right]^{-N} \quad (5)$$

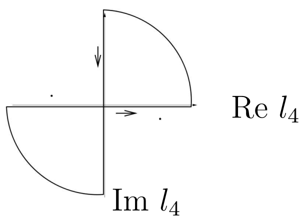
The general form of R^2 is

$$\begin{aligned}
R^2 &= Q^2 - \sum_{i=1}^N z_i\,(r_i^2 - m_i^2)\\
&= \sum_{i,j=1}^N z_i\, z_j\, r_i \cdot r_j - \frac{1}{2}\sum_{i=1}^N z_i\,(r_i^2 - m_i^2)\sum_{j=1}^N z_j - \frac{1}{2}\sum_{j=1}^N z_j\,(r_j^2 - m_j^2)\sum_{i=1}^N z_i\\
&= -\frac{1}{2}\sum_{i,j=1}^N z_i\, z_j\,\left(r_i^2 + r_j^2 - 2\, r_i \cdot r_j - m_i^2 - m_j^2\right)\\
&= -\frac{1}{2}\sum_{i,j=1}^N z_i\, z_j\, \mathcal{S}_{ij}\\
\mathcal{S}_{ij} &= (r_i - r_j)^2 - m_i^2 - m_j^2 \quad (6)
\end{aligned}$$

The matrix $\mathcal{S}_{ij}$, sometimes also called *Cayley matrix* is an important quantity encoding all the kinematic dependence of the integral. It plays the main role in algebraic reduction as well as in the analysis of so-called *Landau singularities*, which are singularities where $\det \mathcal{S}$ or a sub-determinant of $\mathcal{S}$ is vanishing (see below for more details).

Remember that we are in Minkowski space, where $l^2 = l_0^2 - \vec{l}^2$, so temporal and spatial components are not on equal footing. Note that the poles of the denominator are located at $l_0^2 = R^2 + \vec{l}^2 - i\delta \Rightarrow l_0^\pm \simeq \pm\sqrt{R^2 + \vec{l}^2} \mp i\,\delta$. Thus the $i\delta$ term shifts the poles away from the real axis.
For the integration over the loop momentum, we better work in Euclidean space where $l_E^2 = \sum_{i=1}^4 l_i^2$. Hence we make the transformation $l_0 \to i\, l_4$, such that $l^2 \to -l_E^2 = l_4^2 + \vec{l}^2$, which implies that the integration contour in the complex l_0-plane is rotated by 90° such that the contour in the complex l_4-plane looks as shown below. The is called *Wick rotation*. We see that the $i\delta$ prescription is exactly such that the contour does not enclose any poles. Therefore the integral over the closed contour is zero, and we an use the identity

$$\int_{-\infty}^{\infty} dl_0 f(l_0) = -\int_{i\infty}^{-i\infty} dl_0 f(l_0) = i\int_{-\infty}^{\infty} dl_4 f(l_4) \quad (7)$$



Our integral now reads

$$I_N^D = (-1)^N \Gamma(N) \int_0^\infty \prod_{i=1}^N dz_i\, \delta(1-\sum_{l=1}^N z_l) \int_{-\infty}^\infty \frac{d^D l_E}{\pi^{\frac{D}{2}}} \left[l_E^2 + R^2 - i\delta\right]^{-N} \quad (8)$$

Now we can introduce polar coordinates in D dimensions to evaluate the integral: Using

$$\int_{-\infty}^\infty d^D l = \int_0^\infty dr\, r^{D-1} \int d\Omega_{D-1}\ ,\ r = \sqrt{l_E^2} = \left(\sum_{i=1}^4 l_i^2\right)^{\frac{1}{2}} \quad (9)$$

$$\int d\Omega_{D-1} = V(D) = \frac{2\pi^{\frac{D}{2}}}{\Gamma(\frac{D}{2})} \quad (10)$$

where $V(D)$ is the volume of a unit sphere in D dimensions:

$$V(D) = \int_0^{2\pi} d\theta_1 \int_0^\pi d\theta_2 \sin\theta_2 \ldots \int_0^\pi d\theta_{D-1} (\sin\theta_{D-1})^{D-2}$$

Thus we have

$$I_N^D = 2(-1)^N \frac{\Gamma(N)}{\Gamma(\frac{D}{2})} \int_0^\infty \prod_{i=1}^N dz_i\, \delta(1-\sum_{l=1}^N z_l) \int_0^\infty dr\, r^{D-1} \frac{1}{[r^2+R^2-i\delta]^N}$$

Substituting $r^2 = x \Rightarrow$:

$$\int_0^\infty dr\, r^{D-1} \frac{1}{[r^2+R^2-i\delta]^N} = \frac{1}{2}\int_0^\infty dx\, x^{D/2-1} \frac{1}{[x+R^2-i\delta]^N} \quad (11)$$

Now the x-integral can be identified as the Euler Beta-function $B(a,b)$, defined as

$$B(a,b) = \int_0^\infty dz\, \frac{z^{a-1}}{(1+z)^{a+b}} = \int_0^1 dy\, y^{a-1}(1-y)^{b-1} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \quad (12)$$

and after normalising with respect to R^2 we finally arrive at

$$I_N^D = (-1)^N \Gamma(N-\frac{D}{2}) \int_0^\infty \prod_{i=1}^N dz_i\, \delta(1-\sum_{l=1}^N z_l) \left[R^2 - i\delta\right]^{\frac{D}{2}-N} . \quad (13)$$

The integration over the Feynman parameters remains to be done, but we will show below that for one-loop applications, the integrals we need to know explicitly have maximally $N=4$ external legs. Integrals with $N>4$ can be expressed in terms of boxes, triangles, bubbles (and tadpoles in the case of massive propagators). The analytic expressions for these "master integrals" are well-known. The most complicated analytic functions at one loop (appearing in the 4-point integrals) are dilogarithms.

The generic form of the derivation above makes clear that we do not have to go through the procedure of Wick rotation explicitly each time. All we need is to use the following general formula for D-dimensional momentum integration (in Minkowski space, and after having performed the shift to have a quadratic form in the denominator):

$$\int \frac{d^D l}{i\pi^{\frac{D}{2}}} \frac{(l^2)^r}{[l^2-R^2+i\delta]^N} = (-1)^{N+r} \frac{\Gamma(r+\frac{D}{2})\Gamma(N-r-\frac{D}{2})}{\Gamma(\frac{D}{2})\Gamma(N)} \left[R^2 - i\delta\right]^{r-N+\frac{D}{2}} \quad (14)$$

Example one-loop two-point function

Applying the above procedure to our two-point function, we obtain

$$
\begin{aligned}
I_2 &= \Gamma(2)\int_0^1 dz \int_{-\infty}^{\infty} \frac{d^D l}{(2\pi)^D}\,\frac{1}{[l^2 - J + i\delta]^2} &(15)\\
J &= Q^2 - A = -p^2\, z\,(1-z) + m^2 \;\Rightarrow \\
I_2 &= i\,\frac{\pi^{\frac{D}{2}}}{(2\pi)^D}\,\Gamma(2-\frac{D}{2})\int_0^1 dz\,[-p^2\, z\,(1-z) + m^2\,]^{\frac{D}{2}-2}\;, &(16)
\end{aligned}
$$

where we have dropped the $i\delta$ now. For $m^2 = 0$, the result can be expressed in terms of Γ-functions:

$$
\begin{aligned}
I_{2|m^2=0} &= \frac{i\,\pi^{\frac{D}{2}}}{(2\pi)^D}\,[-p^2]^{\frac{D}{2}-2}\Gamma(2-D/2)\,B(D/2-1, D/2-1)\\
B(a,b) &= \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\;. &(17)
\end{aligned}
$$

The two-point function has an UV pole which is contained in

$$
\Gamma(2-D/2) = \Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon)\;, \tag{18}
$$

where γ_E is "Euler's constant",

$$
\gamma_E = \lim_{n\to\infty}\left(\sum_{j=1}^{n}\frac{1}{j} - \ln n\right) = 0.5772156649...
$$

Tensor integrals

If we have loop momenta in the numerator, as in eq. (1), the procedure is essentially the same, except for combinatorics and additional Feynman parameters in the numerator: The substitution $k = l - Q$ introduces terms of the form $(l-Q)^{\mu_1}\ldots(l-Q)^{\mu_r}$ into the numerator of eq. (5). As the denominator is symmetric under $l \to -l$, only the terms with even numbers of l^μ in the numerator will give a non-vanishing contribution upon l-integration. Further, we know that integrals where the Lorentz structure is only carried by loop momenta can only be proportional to combinations of metric tensors $g^{\mu\nu}$. Therefore we have, as the tensor-generalisation of eq. (14),

$$
\int_{-\infty}^{\infty}\frac{d^D l}{i\pi^{\frac{D}{2}}}\,\frac{l^{\mu_1}\ldots l^{\mu_{2m}}}{\left[l^2 - R^2 + i\delta\right]^N} = (-1)^N\left[(g^{\cdot\cdot})^{\otimes m}\right]^{\{\mu_1\ldots\mu_{2m}\}}\left(-\frac{1}{2}\right)^m\frac{\Gamma(N-\frac{D+2m}{2})}{\Gamma(N)}\left(R^2 - i\delta\right)^{-N+(D+2m)/2}\;, \tag{19}
$$

which can be derived for example by taking derivatives of the unintegrated scalar expression with respect to l^μ. $(g^{\cdot\cdot})^{\otimes m}$ denotes m occurences of the metric tensor and the sum over all possible distributions of the $2m$ Lorentz indices μ_i, to the metric tensors is denoted by $[\cdots]^{\{\mu_1\cdots\mu_{2m}\}}$.

Thus, for a general tensor integral, combining with numerators containing the vectors Q^μ, one finds the following formula [7]:

$$I_N^{D,\mu_1\ldots\mu_r}(S) = \sum_{m=0}^{\lfloor r/2 \rfloor} \left(-\frac{1}{2}\right)^m \sum_{j_1,\ldots,j_{r-2m}=1}^{N-1} \left[(g^{\cdot\cdot})^{\otimes m} r_{j_1}^{\cdot} \ldots r_{j_{r-2m}}^{\cdot}\right]^{\{\mu_1\ldots\mu_r\}} I_N^{D+2m}(j_1,\ldots,j_{r-2m};S) \tag{20}$$

$$I_N^d(j_1,\ldots,j_\alpha;S) = (-1)^N \Gamma(N-\frac{d}{2}) \int \prod_{i=1}^N dz_i\, \delta(1-\sum_{l=1}^N z_l)\; z_{j_1}\ldots z_{j_\alpha} \left(R^2 - i\delta\right)^{d/2-N} \tag{21}$$

$$R^2 = -\frac{1}{2}\, z\cdot\mathcal{S}\cdot z$$

The distribution of the r Lorentz indices μ_i, to the external vectors $r_j^{\mu_i}$ is denoted by $[\cdots]^{\{\mu_1\cdots\mu_r\}}$. These are $\binom{r}{2m}\prod_{k=1}^m(2k-1)$ terms. $(g^{\cdot\cdot})^{\otimes m}$ denotes m occurences of the metric tensor and $\lfloor r/2 \rfloor$ is the nearest integer less or equal to $r/2$. Integrals with $z_{j_1}\ldots z_{j_\alpha}$ in eq. (21) are associated with external vectors $r_{j_1}\ldots r_{j_\alpha}$, stemming from factors of Q^μ in eq. (5).
How the higher dimensional integrals I_N^{D+2m} in eq. (20), associated with metric tensors $(g^{\cdot\cdot})^{\otimes m}$, arise, is left as an exercise.

Schwinger parametrisation

An alternative to Feynman parametrisation is the so-called "Schwinger parametrisation", based on

$$\frac{1}{A^\nu} = \frac{1}{\Gamma(\nu)} \int_0^\infty dx\, x^{\nu-1} \exp(-x\,A), \quad Re(A) > 0 \tag{22}$$

In this case the Gaussian integration formula

$$\int_{-\infty}^{\infty} d^D r_E \exp(-\alpha\, r_E^2) = \left(\frac{\pi}{\alpha}\right)^{\frac{D}{2}} \;,\; \alpha > 0 \tag{23}$$

can be used to integrate over the momenta.

1.4 More about singularities

- The *overall* UV divergence of an integral can be determined by power counting: if we work in D dimensions at L loops, and consider an integral with P propagators and n_l factors of the loop momentum belonging to loop $l \in \{1,\ldots,L\}$ in the numerator, we have $\omega = D\,L - 2P + 2\sum_l \lfloor n_l/2 \rfloor$, where $\lfloor n_l/2 \rfloor$ is the nearest integer less or equal to $n_l/2$. We have logarithmic, linear, quadratic,... overall divergences for $\omega = 0,1,2,\ldots$ and no UV divergence for $\omega < 0$. This means that in 4 dimensions at one loop, we have UV divergences in rank 0 two-point functions, rank 2 (and rank 3) three-point functions and rank 4 four-point functions.

- IR divergences: $1/\epsilon^{2L}$ at worst. Necessary conditions for IR divergences are given by the *Landau equations*:

$$\left\{ \begin{array}{l} \forall i \;\; z_i\,(q_i^2 - m_i^2) = 0 \;, \\ \sum_{i=1}^N z_i\, q_i = 0 \;. \end{array} \right. \tag{24}$$

  If eq. (24) has a solution $z_i > 0$ for every $i \in \{1, \ldots, N\}$, i.e. all particles in the loop are *simultaneously on-shell*, then the integral has a *leading Landau singularity*. If a solution exists where some $z_i = 0$ while the other z_j are positive, the Landau condition corresponds to a lower-order Landau singularity. Soft and/or collinear singularities appearing as poles in $1/\epsilon$ are always stemming from some $z_i = 0$.

  A special type of Landau singularities are *scattering singularities*, where $\det G \to 0$ and $\det \mathcal{S} \sim (\det G)^2$, and $G_{kl} = 2\, p_k \cdot p_l$ is the *Gram matrix*, see section 1.6. These singularities are not spurious (i.e. an artifact of the choice of the basis integrals), but correspond to physical kinematics. For example, in the 6-photon amplitude, a "double parton scattering configuration" occurs, where two incoming photons split into fermion pairs, the latter rescattering into photon pairs (p_3, p_4), (p_5, p_6) with vanishing relative transverse momentum.

- scaleless integrals are zero in dimensional regularisation: $\int_{-\infty}^{\infty} d^{2m-2\epsilon}k\,(k^2)^\alpha = 0$.

- Note that

$$\Gamma(2 - D/2)\, \pi^{\frac{D}{2}}/(2\pi)^D = \frac{\Gamma(\epsilon)}{(4\pi)^{2-\epsilon}} = \frac{1}{(4\pi)^2}\left(\frac{1}{\epsilon} - \gamma_E + \ln(4\pi) + \mathcal{O}(\epsilon)\right) .$$

  The factor in brackets, $\Delta_\epsilon = 1/\epsilon - \gamma_E + \ln(4\pi)$ will always appear in UV divergent loop integrals. Therefore it is convenient to subtract, upon UV renormalisation, not only the $1/\epsilon$ pole, but the whole factor Δ_ϵ. This is called $\overline{\text{MS}}$ subtraction ("modified Minimal Subtraction").

Drawbacks of dimensional regularisation

It is not obvious how to continue the Dirac matrix γ_5, which is in 4 dimensions defined as

$$\gamma_5 = i\,\gamma_0\gamma_1\gamma_2\gamma_3 \tag{25}$$

to D dimensions. One could define $\gamma_5 = \frac{i}{4!}\,\epsilon_{\mu_1\mu_2\mu_3\mu_4}\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_4}\gamma^{\mu_4}$ but doing so, Ward identities relying on $\{\gamma_5, \gamma_\mu\} = 0$ break down due to an extra $(D-4)$-dimensional contribution. A solution to this problem for practical calculations is to leave γ_5 in 4 dimensions and to split the other Dirac matrices into a 4-dimensional and a $(D-4)$-dimensional part, $\gamma_\mu = \hat{\gamma}_\mu + \tilde{\gamma}_\mu$, where $\hat{\gamma}_\mu$ is 4-dimensional and $\tilde{\gamma}_\mu$ is $(D-4)$-dimensional. The Dirac matrices defined in this way obey the algebra

$$\{\gamma^\mu, \gamma_5\} = \left\{ \begin{array}{ll} 0 & \mu \in \{0,1,2,3\} \\ 2\tilde{\gamma}^\mu\gamma_5 & \text{otherwise.} \end{array} \right. \tag{25b}$$

The second line above can also be read as $[\gamma_5, \tilde{\gamma}^\mu] = 0$, which can be interpreted as γ_5 acting trivially in the non-physical dimensions. Note that the 4-dimensional and $(D-4)$-dimensional spaces are orthogonal.
If we use dimension splitting into $2m$ integer dimensions and the remainig 2ϵ-dimensional space, $k^2_{(D)} = k^2_{(2m)} + \tilde{k}^2_{(-2\epsilon)}$, we will encounter additional integrals with powers of $(\tilde{k}^2)^\alpha$ in the numerator. These are related to integrals in higher dimensions by

$$\int \frac{d^D k}{i\pi^{\frac{D}{2}}} (\tilde{k}^2)^\alpha \, f(k^\mu, k^2) \;=\; (-1)^\alpha \frac{\Gamma(\alpha + \frac{D}{2} - 2)}{\Gamma(\frac{D}{2} - 2)} \int \frac{d^{D+2\alpha} k}{i\pi^{\frac{D}{2}+\alpha}} f(k^\mu, k^2) \; . \tag{26}$$

Note that $1/\Gamma(\frac{D}{2} - 2)$ is of order ϵ. Therefore the integrals with $\alpha > 0$ only contribute if the k-integral in $4 - 2\epsilon + 2\alpha$ dimensions is divergent. In this case they contribute a constant part, which forms part of the so-called "rational part" of the full amplitude. Note that a divergence in $4 - 2\epsilon + 2\alpha$ dimensions is always of ultraviolet origin.
Exercise: derive eq. (26).

1.5 Regularisation schemes

Related to the γ_5-problem, it is not uniquely defined how we continue the Dirac-algebra to D dimensions. There are essentially three different schemes:

- **CDR**: "Conventional dimensional regularisation": all momenta and all polarisation vectors are taken in D dimensions.
- **HV**: " 't Hooft-Veltman scheme": momenta and helicities of the unobserved particles are D-dim. while momenta and helicities of the observed particles are 4-dimensional.
- **DR**: "Dimensional reduction": momenta and helicities of the observed particles are 4-dim., as well as all polarisation vectors. Only the momenta of the unobserved particles are D-dimensional.

The conventions are summarized in Table 1.

	CDR	**HV**	**DR**
	$\gamma^\mu = \hat{\gamma}^\mu + \tilde{\gamma}^\mu$	$\gamma^\mu = \hat{\gamma}^\mu + \tilde{\gamma}^\mu$	$\gamma^\mu = \hat{\gamma}^\mu, \quad \tilde{\gamma}^\mu \equiv 0$
$\{\gamma_5, \gamma^\mu\}$	0	eq. (25b)	0
internal momenta	$k = \hat{k} + \tilde{k}$	$k = \hat{k} + \tilde{k}$	$k = \hat{k} + \tilde{k}$
external momenta	$p_i = \hat{p}_i + \tilde{p}_i$	$p_i = \hat{p}_i, \quad \tilde{p}_i = 0$	$p_i = \hat{p}_i, \quad \tilde{p}_i = 0$
int. gluon pol.	$D-2$	$D-2$	2
ext. gluon pol.	$D-2$	2	2

Table 1: Comparison of different regularisation schemes

At one loop, the transition formulae to relate results obtained in one scheme to another scheme are well known [5, 6].

1.6 Reduction of one-loop integrals

In the following two subsections we will show that every one-loop amplitude, with an arbitrary number N of legs, can be written as a linear combination of simple "basis integrals". The "basis integrals" can be chosen such that they do not have more than four external legs.

Reduction of scalar integrals

In this section we will show that scalar one-loop integrals for arbitrary N can be reduced to integrals with $N \leq 4$ only. To this aim we make an ansatz where we cancel some denominators by writing linear combinations of propagators into the numerator, with yet unknown coefficients b_l.

$$I_N^D \;=\; I_{\rm red} + I_{\rm fin} = \int d\bar{k}\, \frac{\sum_{l=1}^N b_l\,(q_l^2 - m_l^2)}{\prod_{l=1}^N (q_l^2 - m_l^2 + i\delta)} + \int d\bar{k}\, \frac{\left[1 - \sum_{l=1}^N b_l\,(q_l^2 - m_l^2)\right]}{\prod_{l=1}^N (q_l^2 - m_l^2 + i\delta)} \tag{27}$$

We will show in the following that one can find coefficients b_l such that $I_{\rm fin}$ contains no IR poles. After Feynman parametrisation and momentum shift as explained above, we obtain (using the short-hand notation $d^N z = \prod_{i=1}^N dz_i$)

$$\begin{aligned} I_{\rm fin} &= \Gamma(N) \int_0^\infty d^N z\, \delta(1 - \sum_{l=1}^N z_l) \int d\bar{l}\, \frac{\left[1 - \sum_{l=1}^N b_l\,(\tilde{q}_l^2 - m_l^2)\right]}{(l^2 - R^2)^N} \\ R^2 &= -\frac{1}{2}\, z\cdot\mathcal{S}\cdot z - i\delta\,, \quad \tilde{q}_j = l + \sum_{i=1}^N (\delta_{ij} - z_i)\, r_i \end{aligned} \tag{28}$$

Now the term in square brackets in Eq. (28) can be written as

$$\left[1 - \sum_{i=1}^N b_i\,(\tilde{q}_i^2 - m_i^2)\right] \;=\; -(l^2 + R^2)\sum_{j=1}^N b_j + \sum_{j=1}^N z_j \Big(1 - (\mathcal{S}\cdot b)_j\Big) \;+\; \text{odd in } l \tag{29}$$

If now the equations

$$(\mathcal{S}\cdot b)_j = 1\,, \quad j = 1, \dots, N \tag{30}$$

are fulfilled, the second term on the right-hand-side of (29) vanishes and one finds

$$I_{\rm fin} \;=\; -\Gamma(N) \left(\sum_{l=1}^N b_l\right) \int_0^\infty d^N z\, \delta(1 - \sum_{l=1}^N z_l) \int d\bar{l}\, \frac{l^2 + R^2}{(l^2 - R^2)^N} \tag{31}$$

Finally the loop momentum integration gives

$$\begin{aligned} I_{\rm fin} &= \left(\sum_{l=1}^N b_l\right) (-1)^{N+1}\, \Gamma(N - 1 - \frac{D}{2})\, (N - D - 1) \int_0^\infty d^N z\, \frac{\delta(1 - \sum_{l=1}^N z_l)}{(R^2)^{N-(D+2)/2}} \\ &= -\left(\sum_{l=1}^N b_l\right) (N - D - 1)\, I_N^{D+2} \end{aligned} \tag{32}$$

Therefore, if eq. (30) can be solved for the b_l, i.e. if $\det \mathcal{S} \neq 0$ we have

$$I_N^D(S) = \sum_{j=1}^{N} b_j \, I_{N-1}^D(S \setminus \{j\}) - (N - D - 1) \, B \, I_N^{D+2}(S) \quad , \quad \det(\mathcal{S}) \neq 0 \tag{33}$$

$$b_j = \sum_{i=1}^{N} \mathcal{S}_{ij}^{-1} \; , \; B = \sum_{j=1}^{N} b_j \tag{34}$$

If $\det \mathcal{S} = 0$, one can construct a reduction formula based on a pseudo-inverse or on the singular-value decomposition of $\mathcal{S}$. In these cases the integrals always can be written as combinations of lower-point integrals only, i.e. the $I_N^{D+2}(S)$ drop out. For more details see e.g. refs. [14, 15, 13]. Further, we have the important relation

$$\sum_{j=1}^{N} b_j \; \det \mathcal{S} = (-1)^{N+1} \det G \tag{35}$$

where $\det G$ is the *Gram determinant*, the determinant of the *Gram matrix* $G_{ij} = 2\, r_i \cdot r_j$. Note that, using $r_N = 0$, G_{ij} is an $(N-1) \times (N-1)$ matrix. If the matrices G_{ij} and $\mathcal{S}_{ij}$ are constructed from 4-dimensional external momenta, they have the following properties:

$$\det G = 0 \text{ for } N \geq 6 \quad \Leftrightarrow \sum_{j=1}^{N} b_j = 0 \text{ for } N \geq 6 \tag{36}$$

$$\det \mathcal{S} = 0 \text{ for } N \geq 7 \tag{37}$$

This is due to the fact that the external momenta become linearly dependent for $N \geq 6$, as one can construct a basis of 4-dimensional Minkowski space from four 4-dimensional momenta. (For $N = 5$, one of the 5 external momenta can be eliminated by momentum conservation, leaving just 4 linearly independent momenta.) Therefore the coefficient in front of $I_N^{D+2}(S)$ in eq. (33) is identically zero for $N \geq 6$, which means that we can express our N-point integrals recursively in terms of $(N-1)$-point integrals until we reach $N = 5$. The case $N = 5$ is special: the coefficient in front of $I_N^{D+2}(S)$ is $(N - D - 1)\, B$, so it is of order ϵ for $N = 5$. As the integrals $I_N^{D+2}(S)$ are always UV and IR finite, we can drop this term for all one-loop applications, such that scalar pentagons can be written as a sum of five boxes, where in each box a different propagator is missing (*"pinched"*).

Note:

- Traditionally, the notation conventions for tadpole, bubble, triangle, box, ...integrals were $I_1^D = A_0, I_2^D = B_0, I_3^D = C_0, I_4^D = D_0, \ldots$ (cf. ref. [11] and the programs `FF` [17] and `LoopTools` [18] for infrared finite integrals).
- A list (and Fortran program) of all IR divergent triangle (there are 6 of them) and box (there are 16) integrals can be found in ref. [19].
- The reduction can also be formulated in terms of *signed minors*, see e.g. [20].

Reduction of tensor integrals

- Historically, tensor integrals occurring in one-loop amplitudes were reduced to scalar integrals using so-called *Passarino-Veltman* reduction [16]. It is based on the fact that at one loop, scalar products of loop momenta with external momenta can always be expressed as combinations of propagators. The problem with Passarino-Veltman reduction is that it introduces powers of inverse Gram determinants $1/(\det G)^r$ for the reduction of a rank r tensor integral. This can lead to numerical instabilities upon phase space integration in kinematic regions where $\det G \to 0$.
- It has been proven [12, 13, 22] that the reduction from rank r pentagons ($N = 5$) to boxes ($N = 4$) can be done without introducing inverse Gram determinants.
- Inverse Gram determinants are unavoidable in the reduction of tensor boxes, triangles when a *scalar* integral basis is chosen. However, from physical arguments we expect singularities which behave like $1/\sqrt{\det G}$ at worst on amplitude level. Higher powers are spurious, i.e. an artifact of the choice of a scalar integral basis, and should cancel when combining the integrals to a gauge invariant quantity. However, this is difficult to achieve for one-loop amplitudes with a large number of external legs.
- solutions to the problem mentioned above are
  - use on-shell methods (see section 1.7)
  - semi-numerical: reduction is stopped before dangerous denominators are produced. The non-scalar integrals (parameter integrals with Feynman parameters in the numerator) are calculated numerically [21]
  - fully numerical: don't do any reduction, calculate the full tensor integral numerically

A *form factor representation* of a tensor integral is a representation where the Lorentz structure has been extracted, each Lorentz tensor multiplying a scalar quantity, the *form factor*. Distinguishing A, B, C depending on the presence of zero, one or two metric tensors, we can write

$$\begin{aligned} & I_N^{D,\,\mu_1\ldots\mu_r}(S) = \\ & \qquad \sum_{j_1\cdots j_r\in S} r_{j_1}^{\mu_1}\ldots r_{j_r}^{\mu_r}\, A^{N,r}_{j_1\ldots,j_r}(S) \\ & \quad + \sum_{j_1\cdots j_{r-2}\in S} \left[g^{\cdot\cdot} r^{\cdot}_{j_1}\cdots r^{\cdot}_{j_{r-2}}\right]^{\{\mu_1\cdots\mu_r\}} B^{N,r}_{j_1\ldots,j_{r-2}}(S) \\ & \quad + \sum_{j_1\cdots j_{r-4}\in S} \left[g^{\cdot\cdot} g^{\cdot\cdot} r^{\cdot}_{j_1}\cdots r^{\cdot}_{j_{r-4}}\right]^{\{\mu_1\cdots\mu_r\}} C^{N,r}_{j_1\ldots,j_{r-4}}(S) \end{aligned} \tag{38}$$

Note that we never need more than two metric tensors in a renormalisable gauge where the rank $r \le N$, because for $N > 5$, we can express the metric tensor in terms of 4 linearly independent external vectors. Three metric tensors would be needed for rank six, i.e. for six-point integrals or higher, but we can immediately reduce those integrals to lower-point ones:

$$I_N^{D,\mu_1\ldots\mu_r}(S) \;=\; -\sum_{j\in S} \mathcal{C}_j^{\mu_1}\, I_{N-1}^{D,\mu_2\ldots\mu_r}(S\setminus\{j\}) \quad (N\ge 6)\;, \tag{39}$$

where $\mathcal{C}_l^\mu = \sum_{k\in S} (\mathcal{S}^{-1})_{kl}\, r_k^\mu$ if $\mathcal{S}$ is invertible, and if not, it can be constructed from the pseudo-inverse [13].

Example for the distribution of indices:

$$\begin{aligned} I_N^{D,\mu_1\mu_2\mu_3}(S) &= \sum_{l_1,l_2,l_3\in S} r_{l_1}^{\mu_1}\, r_{l_2}^{\mu_2}\, r_{l_3}^{\mu_3}\, A_{l_1l_2l_3}^{N,3}(S) \\ &\quad + \sum_{l\in S} \left(g^{\mu_1\mu_2}\, r_l^{\mu_3} + g^{\mu_1\mu_3}\, r_l^{\mu_2} + g^{\mu_2\mu_3}\, r_l^{\mu_1}\right) B_l^{N,3}(S)\ . \end{aligned}$$

Example for *Passarino-Veltman reduction:*
Consider a rank one three-point integral

$$\begin{aligned} I_3^{D,\,\mu}(S) &= \int_{-\infty}^{\infty} d\bar{k}\, \frac{k^\mu}{[k^2+i\delta][(k+p_1)^2+i\delta][(k+p_1+p_2)^2+i\delta]} = A_1\, r_1^\mu + A_2\, r_2^\mu \\ r_1 &= p_1\ ,\ r_2 = p_1+p_2\ . \end{aligned}$$

Contracting with r_1 and r_2 and using the identities

$$k\cdot r_i = \frac{1}{2}\left[(k+r_i)^2 - k^2 - r_i^2\right]\ ,\ i\in\{1,2\}$$

we obtain, after cancellation of numerators

$$\begin{pmatrix} 2\,r_1\cdot r_1 & 2\,r_1\cdot r_2 \\ 2\,r_2\cdot r_1 & 2\,r_2\cdot r_2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} \tag{40}$$

$$\begin{aligned} R_1 &= I_2^D(r_2) - I_2^D(r_2-r_1) - r_1^2 I_3(r_1,r_2) \\ R_2 &= I_2^D(r_1) - I_2^D(r_2-r_1) - r_2^2 I_3(r_1,r_2)\ . \end{aligned}$$

We see that the solution involves the inverse of the Gram matrix $G_{ij} = 2\,r_i\cdot r_j$.

Recap

The procedure to calculate (one-)loop integrals is the following:

1. Feynman (or Schwinger) parametrisation
2. Shift the loop momentum to eliminate the term in the denominator which is linear in the loop momentum
3. Use formula (14) (or (23)) to perform the integration over the loop momentum
4. Integrate over the Feynman parameters.

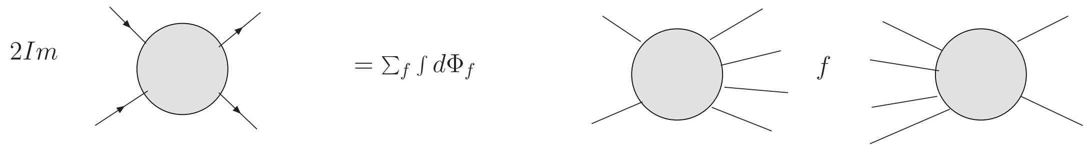
By using algebraic reduction one can show that every one-loop N-point amplitude with 4-dimensional external legs can, for any N, be expressed in terms of basis integrals with four or less external legs. The most complicated analytic functions that appear (at order ϵ^0) are dilogarithms (contained in box integrals).

1.7 Unitarity cuts

The idea is to use the analytic structure of scattering amplitudes to determine their explicit form. Using the unitarity of the S-matrix, where $S = 1 + iT$, we have

$$S^\dagger S = 1 \Rightarrow 2\,\mathrm{Im}(T) = T^\dagger T\,. \tag{41}$$

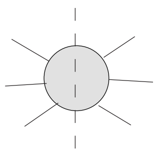
Inserting a complete set of intermediate states, we obtain



The right-hand side can also be considered as all possible cuts of a loop amplitude, where cutting a loop amplitude basically means putting the cut propagators on-shell, exploiting the relation

$$\frac{i}{p^2 + i\delta} \longrightarrow 2\pi\,\delta^{(+)}(p^2)\,. \tag{42}$$

Applied to one-loop amplitudes, we therefore have

$$\mathrm{Im}\,\mathcal{A}_{1-\mathrm{loop}} \sim \sum_{\mathrm{cuts}} \int d\Phi_{\mathrm{cuts}}$$ 

The application of unitarity as an on-shell method of calculating loop amplitudes turns the cutting step around: tree amplitudes are fused together to form one-loop amplitudes.

Using the standard Feynman diagrammatic approach we have shown that any one-loop amplitude with massless internal particles can be decomposed in terms of known scalar integrals with less than five external legs, $I_{N=1,2,3,4}$, with D-dependent coefficients, or, alternatively, as linear combinations of coefficients in $D = 4$ and a rational part $\mathcal{R}$, which comes from terms of the form $(D-4)\,I_{\mathrm{UV\,div}}$ in the Feynman diagrammatic approach.
As we know the analytic form of the basis integrals, the imaginary parts of the different scalar integrals can be uniquely attributed to a given integral. Therefore we have

$$\begin{aligned}\mathcal{A}_{1-\mathrm{loop}} \quad &= \sum_{N=1,2,3,4} C_N(D)\,I_N^D = \sum_{N=1,2,3,4} C_N(4)\,I_N^D + \mathcal{R} \\ &\Rightarrow \quad \mathrm{Im}\,\mathcal{A}_{1-\mathrm{loop}} = \sum_{N=1,2,3,4} C_N(4)\,\mathrm{Im}(I_N^D)\end{aligned} \tag{43}$$

The coefficients of the integrals, C_N, and $\mathcal{R}$, are rational polynomials in terms of Mandelstam variables $s_{ij} = (p_i + p_j)^2$ and masses (or spinor products).

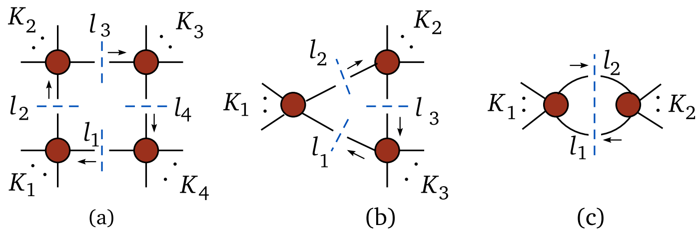


Figure 1: Multiple cuts can be used to fix integral coefficients of amplitudes.

Note that there are many different types of scalar two-, three- and four-point functions present in a given process (which can be classified according to the number and location of off-shell/on-shell external legs and massive/massless propagators). Sums over all these different types are implcit in eq. (43).

Generalized unitarity corresponds to requiring *more* than two internal particles to be on shell. Cutting four lines in an N-point topology amounts to putting the corresponding four propagators on-shell. For $N = 4$, this procedure fixes the associated loop momentum completely and the coefficient of the related box diagram is given as a product of tree diagrams.

Example:
Consider Fig. 1(a) with all external momenta K_i are off-shell ("four-mass-box"). The corresponding box integral is finite and reads

$$I^{4m} = \int d^4l \frac{1}{(l^2 + i\epsilon)((l - K_1)^2 + i\epsilon)((l - K_1 - K_2)^2 + i\epsilon)((l + K_4)^2 + i\epsilon)} \,. \tag{44}$$

Cutting all four propagators, denoted by Δ_{quad}, we obtain:

$$\Delta_{\text{quad}} I^{4m} = \int d^4l \; \delta^{(+)}(l^2)\; \delta^{(+)}((l - K_1)^2)\; \delta^{(+)}((l - K_1 - K_2)^2)\; \delta^{(+)}((l + K_4)^2) \,. \tag{45}$$

As no other box integral in our amplitude shares the same singularity, we can deduce a relation for the coefficient $C_4^{(I^{4m})}$ of this particular box integral in our amplitude, having the representation (43) at hand:

$$\begin{aligned} &\int d^4l \; \delta^{(+)}(l^2)\; \delta^{(+)}((l - K_1)^2)\; \delta^{(+)}((l - K_1 - K_2)^2)\; \delta^{(+)}((l + K_4)^2)\, A_1^{\text{tree}} A_2^{\text{tree}} A_3^{\text{tree}} A_4^{\text{tree}} \\ &= C_4^{(I^{4m})} \Delta_{\text{quad}} I^{4m} \,, \end{aligned} \tag{46}$$

where A_n^{tree} is the tree-level amplitude at the corner with total external momentum K_n.
Note that, since there are four delta functions, and l is a vector in four dimensions, the integral over l is completely frozen and can be solved for l. Therefore we find that

$$C_4^{(I^{4m})} = \frac{1}{n_s} \sum_{s,J} n_J (A_1^{\text{tree}} A_2^{\text{tree}} A_3^{\text{tree}} A_4^{\text{tree}}) \,, \tag{47}$$

where the sum is over the possible spins J of internal particles and the solution set s of the equations constraining l. n_s is the number of these solutions, and n_J is the number of particles of spin J.

The MHV ("maximal helicity violating") tree amplitudes are given by [9]

$$A^{\rm tree}(1^+,\dots,j^-,\dots,k^-,\dots,n^+) = i\,\frac{\langle j\,k\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\dots\langle n\,1\rangle}\,. \tag{48}$$

However, something seems to be wrong when trying to apply this same procedure to box integrals with light-like external legs: some of the tree amplitudes will be all-massless three-point amplitudes. Naively, these amplitudes would vanish. The quadruple cuts would thus vanish as well, making the extraction of the coefficients in this way impossible. The solution to this problem is to use *complex momenta*, such that one can treat opposite-helicity spinors as independent variables. For this purpose one should rather use two-component *Weyl spinors* $\lambda_a(p), \tilde\lambda_{\dot a}(p)$ instead of Dirac spinors, which can be defined as follows

$$u_+(p) = v_-(p) = \frac{1}{\sqrt{2}}\left[\begin{array}{c}\lambda_a(p)\\ \lambda_a(p)\end{array}\right],\quad u_-(p) = v_+(p) = \frac{1}{\sqrt{2}}\left[\begin{array}{c}\tilde\lambda_{\dot a}(p)\\ -\tilde\lambda_{\dot a}(p)\end{array}\right]\,. \tag{49}$$

The three-mass triangles can be isolated through triple cuts. The integrands emerging from triple cuts in general will also contain contributions to those box integrals sharing the same cuts. These contributions, and box-like terms which vanish upon loop integration, must be removed in order to extract the coefficient of the three-mass triangle. Analogous arguments hold for the two-point integrals.

As the rational part does not contribute to the imaginary part of the amplitude, unitarity cuts in 4 dimensions cannot extract this part. Apart from being obtained from Feynman diagrams, it can be obtained by on-shell recursion relations [27] or by D-dimensional unitarity [28].

A similar approach, which is particularly well suited for a numerical solution of the cut conditions, has been formulated by Ossola, Papadopoulos and Pittau [29], and has seen a number of prominent applications meanwhile.
One can write the amplitude on *integrand level* as

$$A_{\rm int} \;=\; \sum_i \frac{\bar C_4^i}{d_{i_1}d_{i_2}d_{i_3}d_{i_4}} + \sum_i \frac{\bar C_3^i}{d_{i_1}d_{i_2}d_{i_3}} + \sum_i \frac{\bar C_2^i}{d_{i_1}d_{i_2}} + \sum_i \frac{\bar C_1^i}{d_{i_1}}\,, \tag{50}$$

where $\bar C_N^i = C_N^i + \tilde C_N^i$, and $\tilde C_N^i$ will vanish upon integration. Again, the box coefficient is particularly simple: multiplying by $d_{i_1}d_{i_2}d_{i_3}d_{i_4}$ and putting the propagators on-shell we are left with C_4^i. For $N<4$, linear systems of equations have to be solved to separate C_N^i from $\tilde C_N^i$ using the on-shell constraints. The rational part can be obtained by several procedures taking into account the D-dimensionality, which will not be described here.

However, we have a general theorem at hand when to expect rational parts in a one-loop amplitude, the "BDDK-theorem" [30], following basically from UV power counting (remember the structure of the $\tilde k^2$-integrals (26), being related to tensor integrals with rank >2): If a one-loop amplitude in a gauge theory has a representation in which all N-point integrals with $N>2$ have at most $N-2$ powers of the loop momentum in the numerator, then there is no rational part, i.e. the amplitude is uniquely determined by its cuts ("cut constructible"). This is fulfilled e.g. for any amplitude in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory.

An important feature of on-shell methods is that the basic building blocks are tree level amplitudes and thus already *incorporate gauge invariance manifestly.* In Feynman diagram computations many graphs have to be combined to result in a gauge invariant expression, leading to large intermediate expressions for multi-leg amplitudes.

2 Beyond one loop

2.1 General form of multi-loop integrals

A scalar D–dimensional L–loop Feynman diagram with N propagators to the power ν_i can be written as

$$G = \int \prod_{l=1}^{L} \frac{d^D k_l}{i\pi^{\frac{D}{2}}} \prod_{j=1}^{N} \frac{1}{P_j^{\nu_j}(\{k\},\{p\},m_j^2)} \tag{51}$$

After Feynman parametrisation:

$$\begin{aligned}
G &= \Gamma(N_\nu) \int \prod_{j=1}^{N} dx_j\, x_j^{\nu_j-1}\, \delta(1-\sum_{i=1}^{N} x_i) \int d\bar{k}_1 \ldots d\bar{k}_L \left[\sum_{j,l=1}^{L} k_j \cdot k_l\, M_{jl} - 2\sum_{j=1}^{L} k_j \cdot Q_j + J\right]^{-N_\nu} \\
&= (-1)^{N_\nu} \frac{\Gamma(N_\nu - LD/2)}{\prod_{j=1}^{N}\Gamma(\nu_j)} \int_0^\infty \prod_{j=1}^{N} dx_j\, x_j^{\nu_j-1} \delta(1-\sum_{i=1}^{N} x_i)\, \frac{\mathcal{U}^{N_\nu-(L+1)D/2}}{\mathcal{F}^{N_\nu-LD/2}} \qquad (52) \\
\mathcal{U} &= \det(M) \quad , \quad N_\nu = \sum_{j=1}^{N} \nu_j \;, \\
\mathcal{F} &= \det(M) \left[\sum_{i,j=1}^{L} Q_i M_{ij} Q_j - J - i\delta\right] \;.
\end{aligned}$$

$P_j^{\nu_j}(\{k\},\{p\},m_j^2)$ are the propagators to the power ν_j, depending on the loop momenta $k_{l\in\{1,\ldots,L\}}$, the external momenta $\{p_1,\ldots p_E\}$ and (not necessarily nonzero) masses m_j. The functions $\mathcal{U}$ and $\mathcal{F}$ can be straightforwardly derived from the momentum representation.

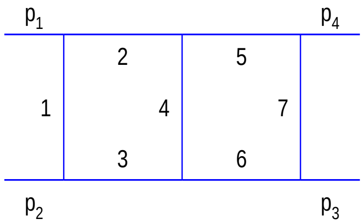
A necessary condition for the presence of infrared divergences is $\mathcal{F}=0$. The function $\mathcal{U}$ cannot lead to infrared divergences of the graph, since giving a mass to all external legs would not change $\mathcal{U}$. Apart from the fact that the graph may have an overall UV divergence contained in the overall Γ-function (see Eq. (52)), UV subdivergences may also be present. A necessary condition for these is that $\mathcal{U}$ is vanishing.

The functions $\mathcal{U}$ and $\mathcal{F}$ also can be constructed from the topology of the corresponding Feynman graph, as explained in the following subsection.

2.2 Construction of the functions $\mathcal{F}$ and $\mathcal{U}$ from topological rules

Cutting L lines of a given connected L-loop graph such that it becomes a connected tree graph T defines a chord $\mathcal{C}(T)$ as being the set of lines not belonging to this tree. The Feynman parameters associated with each chord define a monomial of degree L. The set of all such trees (or *1-trees*) is denoted by $\mathcal{T}_1$. The 1-trees $T \in \mathcal{T}_1$ define $\mathcal{U}$ as being the sum over all monomials corresponding to a chord $\mathcal{C}(T \in \mathcal{T}_1)$. Cutting one more line of a 1-tree leads to two disconnected trees, or a *2-tree* $\hat{T}$. $\mathcal{T}_2$ is the set of all such 2-trees. The corresponding chords define monomials of degree $L+1$. Each 2-tree of a graph corresponds to a cut defined by cutting the lines which connected the 2 now disconnected trees in the original graph. The momentum flow through the lines of such a cut defines a Lorentz invariant $s_{\hat{T}} = (\sum_{j\in \mathrm{Cut}(\hat{T})} p_j)^2$. The function $\mathcal{F}_0$ is the sum over all such monomials times minus the corresponding invariant:

$$
\begin{aligned}
\mathcal{U}(\vec{x}) &= \sum_{T\in\mathcal{T}_1}\Big[\prod_{j\in\mathcal{C}(T)} x_j\Big] \,,\\
\mathcal{F}_0(\vec{x}) &= \sum_{\hat{T}\in\mathcal{T}_2}\Big[\prod_{j\in\mathcal{C}(\hat{T})} x_j\Big]\,(-s_{\hat{T}}) \,,\\
\mathcal{F}(\vec{x}) &= \mathcal{F}_0(\vec{x}) + \mathcal{U}(\vec{x})\sum_{j=1}^{N} x_j m_j^2 \,. \qquad (53)
\end{aligned}
$$



Example: planar double box with $p_1^2 = p_2^2 = p_3^2 = 0, p_4^2 \neq 0$:
Using $k_1 = k, k_2 = l$ and propagator number one as $1/(k^2+i\delta)$, the denominator, after Feynman parametrisation, can be written as

$$
\begin{aligned}
\mathcal{D} &= x_1\,k^2 + x_2\,(k-p_1)^2 + x_3\,(k+p_2)^2 + x_4\,(k-l)^2 + x_5\,(l-p_1)^2 + x_6\,(l+p_2)^2 + x_7\,(l+p_2+p_3)^2\\
&= (\,k,l\,)\begin{pmatrix} x_{1234} & -x_4\\ -x_4 & x_{4567}\end{pmatrix}\begin{pmatrix} k\\ l\end{pmatrix} - 2\,(Q_1,Q_2)\begin{pmatrix} k\\ l\end{pmatrix} + x_7\,(p_2+p_3)^2 + i\delta\\
Q &= (Q_1,Q_2) = (x_2p_1 - x_3p_2, x_5p_1 - x_6p_2 - x_7(p_2+p_3)) \,,
\end{aligned}
$$

where we have used the short-hand notation $x_{ijk\ldots} = x_i + x_j + x_k + \ldots$.
Therefore

$$
\begin{aligned}
\mathcal{U} &= x_{123}x_{567} + x_4 x_{123567}\\
\mathcal{F} &= (-s_{12})\,(x_2x_3x_{4567} + x_5x_6x_{1234} + x_2x_4x_6 + x_3x_4x_5)\\
&\quad +(-s_{23})\,x_1x_4x_7 + (-p_4^2)\,x_7(x_2x_4 + x_5x_{1234}) \,.
\end{aligned}
$$

A general representation for tensor integrals also exists, it can be found e.g. in [10].

2.3 Reduction to master integrals

Integration by parts

Integration-by-part identities [31] are based on the fact that the integral of a total derivative is zero:

$$
\int d^Dk \frac{\partial}{\partial k^\mu} v^\mu f(k,p_i) = 0 \,, \qquad (54)
$$

where v can either be a loop momentum or an external momentum. Working out the derivative for a certain number of numerators yields systems of relations among scalar integrals which can be solved systematically. The endpoints of the reduction are called *master integrals*.

Simplest example: massive vacuum bubble with general propagator power:

$$F(\nu) = \int d\bar{k} \frac{1}{(k^2 - m^2 + i\delta)^\nu} \tag{55}$$

Here we know that the master integral is

$$F(1) = -\Gamma(1 - D/2)\,(m^2)^{\frac{D}{2}-1} \tag{56}$$

Using the integration-by-part identity

$$\int d\bar{k} \frac{\partial}{\partial k^\mu} \left\{ \frac{k_\mu}{(k^2 - m^2 + i\delta)^\nu} \right\} = 0$$

leads to

$$\begin{aligned} 0 &= \int d\bar{k} \left\{ \frac{1}{(k^2 - m^2 + i\delta)^\nu} \frac{\partial}{\partial k^\mu} (k_\mu) - \nu\, k_\mu \frac{2k^\mu}{(k^2 - m^2 + i\delta)^{\nu+1}} \right\} \\ &= D\, F(\nu) - 2\nu \left(F(\nu) + m^2\, F(\nu + 1) \right) \\ \Rightarrow F(\nu + 1) &= \frac{D - 2\,\nu}{2\,\nu\, m^2}\, F(\nu)\,. \end{aligned} \tag{57}$$

In less trivial cases, to be able to solve the system for a small number of master integrals, an *order relation* among the integrals has to be introduced. For example, a topology T_1 is considered to be smaller than a topology T_2 if T_1 can be obtained from T_2 by pinching some of the propagators. Within the same topology, the integrals can be ordered according to the powers of their propagators.
A completely systematic approach has first been formulated by Laporta [34]. Recent implementations and refinements of the algorithm are provided by the programs `FIRE` [35] and `Reduze` [38]. Other automated reduction programs are `MINCER` [36] (for two-point integrals only) and `AIR` [37].

Note: It is not uniquely defined which integrals are master integrals. For complicated multi-loop examples, it is in general not clear before the reduction which integrals will be master integrals. Further, it can sometimes be more convenient to define an integral with a loop momentum in the numerator rather than its scalar "parent" as a master integral.

2.4 Calculation of master integrals

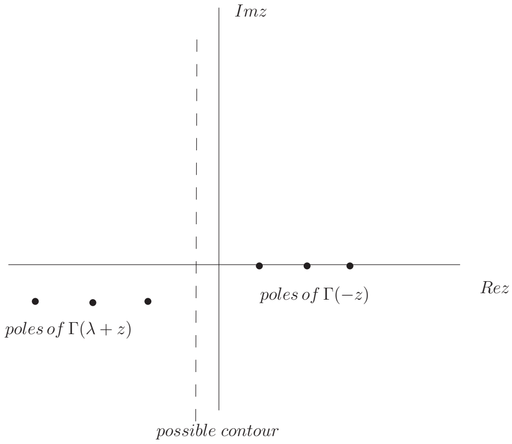
Once we have reduced our expression for a multi-loop amplitude to a linear combination of master integrals, the task is to evaluate these master integrals. A number of techniques have been developed for this task, analytical as well as numerical ones. Simple integrals of course can be evaluated straightforwardly by integration over the Feynman parameters. More complicated ones require additional "tricks". We fill focus only on two of them.

2.4.1 Mellin-Barnes representation

The basic formula underlying the Mellin-Barnes representation of a (multi-)loop integral reads

$$(A_1 + A_2 + ... + A_n)^{-\lambda} = \frac{1}{\Gamma(\lambda)} \frac{1}{(2\pi i)^{n-1}} \int\limits_{c-i\infty}^{c+i\infty} dz_1 ... \int\limits_{c-i\infty}^{c+i\infty} dz_{n-1} \tag{58}$$
$$\times \Gamma(-z_1)...\Gamma(-z_{n-1})\Gamma(z_1 + ... + z_{n-1} + \lambda)\, A_1^{z_1} ... A_{n-1}^{z_{n-1}} A_n^{-z_1-...-z_{n-1}-\lambda}$$

Each contour is chosen such that the poles of $\Gamma(-z_i)$ are to the right and the poles of $\Gamma(\ldots + z)$ are to the left.



The representation in eq. (58) can be used to convert the sum of monomials contained in the functions $\mathcal{U}$ and $\mathcal{F}$ into products, such that all Feynman parameter integrals are of the form of simple integrations over Γ-functions. However, we are still left with the complex contour integrals. The latter are then performed by closing the contour at infinity and summing up all residues which lie inside the contour. In general we will obtain multiple sum over residues and need techniques to manipulate these sums. In simple cases the contour integrals can be performed in closed form with the help of two lemmas by Barnes. Barnes' first lemma states that

$$\frac{1}{2\pi i} \int\limits_{-i\infty}^{i\infty} dz \Gamma(a+z)\Gamma(b+z)\Gamma(c-z)\Gamma(d-z) = \frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)}, \tag{59}$$

if none of the poles of $\Gamma(a+z)\Gamma(b+z)$ coincides with the ones from $\Gamma(c-z)\Gamma(d-z)$. Barnes' second lemma reads

$$\frac{1}{2\pi i} \int\limits_{-i\infty}^{i\infty} dz \frac{\Gamma(a+z)\Gamma(b+z)\Gamma(c+z)\Gamma(d-z)\Gamma(e-z)}{\Gamma(a+b+c+d+e+z)}$$
$$= \frac{\Gamma(a+d)\Gamma(b+d)\Gamma(c+d)\Gamma(a+e)\Gamma(b+e)\Gamma(c+e)}{\Gamma(a+b+d+e)\Gamma(a+c+d+e)\Gamma(b+c+d+e)}. \tag{60}$$

Example: two-point function with one massive propagator
In this example the Mellin-Barnes representation allows us to isolate the mass dependence from the denominator and to perform the Feynman parameter integration as in the massless case:

$$\begin{aligned} F(\nu_1,\nu_2) &= \int d\bar{k}\frac{1}{[k^2-m^2+i\delta]^{\nu_1}\,[(p-k)^2+i\delta]^{\nu_2}} \\ &= \frac{1}{2\pi\, i}\frac{(-1)^{\nu_1+\nu_2}}{\Gamma(\nu_1)}\int\limits_{c-i\infty}^{c+i\infty} dz\,\frac{(m^2)^z}{[-k^2-i\delta]^{\nu_1+z}[-(p-k)^2-i\delta]^{\nu_2}}\,\Gamma(\nu_1+z)\Gamma(-z)\;. \end{aligned} \tag{61}$$

Now we use Feynman parametrisation for the remaining denominator:

$$\frac{1}{[-k^2-i\delta]^{\nu_1+z}[-(p-k)^2-i\delta]^{\nu_2}} = \frac{\Gamma(\nu_1+\nu_2+z)}{\Gamma(\nu_1+z)\Gamma(\nu_2)}\int_0^1 dx\,\frac{x^{\nu_2-1}(1-x)^{\nu_1+z-1}}{[-k^2+2px-xp^2]^{\nu_1+\nu_2+z}}\;.$$

After the substitution $l=k-x\,p$ and integration over l we obtain

$$\begin{aligned} F(\nu_1,\nu_2) &= \frac{1}{2\pi\, i}(-p^2)^{\frac{D}{2}-\nu_1-\nu_2}\frac{(-1)^{\nu_1+\nu_2}}{\Gamma(\nu_1)\Gamma(\nu_2)}\int_0^1 dx\,x^{\frac{D}{2}-\nu_1-z-1}(1-x)^{\frac{D}{2}-\nu_2-1} \\ &\qquad \int\limits_{c-i\infty}^{c+i\infty} dz\,\left(\frac{m^2}{-p^2}\right)^z\Gamma(-z)\,\Gamma(\nu_1+\nu_2+z-D/2) \end{aligned} \tag{62}$$

$$\begin{aligned} &= \frac{1}{2\pi\, i}(-p^2)^{\frac{D}{2}-\nu_1-\nu_2}\frac{(-1)^{\nu_1+\nu_2}\Gamma(D/2-\nu_2)}{\Gamma(\nu_1)\Gamma(\nu_2)} \\ &\qquad \int\limits_{c-i\infty}^{c+i\infty} dz\,\left(\frac{m^2}{-p^2}\right)^z\Gamma(-z)\frac{\Gamma(D/2-\nu_1-z)\Gamma(\nu_1+\nu_2+z-D/2)}{\Gamma(D-\nu_1-\nu_2-z)}\;. \end{aligned} \tag{63}$$

We see that the integration over x in (62) was trivial as we factorised out the mass dependence beforehand. The price to pay is that we are still left with the contour integration in the complex z-plane. Using Cauchy's residue theorem one will end up with a series of residues, which in most cases can be summed in a closed form. Calculations for $\nu_i\in\{1,2\}$ are given in example 4.1 of Ref. [3].

2.4.2 Sector decomposition

Sector decomposition is a method operating in Feynman parameter space which is useful to extract singularities regulated by dimensional regularisation, converting the integral into a Laurent series in ϵ. It is particularly usefule if the singularites are *overlapping* in the sense specified below. The coefficients of the poles in $1/\epsilon$ will be finite integrals over Feynman parameters, which, for most examples beyond one loop, will be too complicated to be integrated analytically, so have to be integrated numerically.

To introduce the basic concept, let us look at the simple example of a two-dimensional parameter integral of the following form:

$$I = \int_0^1 dx\int_0^1 dy\,x^{-1-a\epsilon}\,y^{-b\epsilon}\Big(x+(1-x)\,y\Big)^{-1}\;. \tag{64}$$

The integral contains a singular region where x and y vanish *simultaneously*, i.e. the singularities in x and y are *overlapping*. Our aim is to factorise the singularities for $x \to 0$ and $y \to 0$. Therefore we divide the integration range into two sectors where x and y are ordered (see Fig. 2)

$$I = \int_0^1 dx \int_0^1 dy\, x^{-1-a\epsilon}\, y^{-b\epsilon} \Big(x + (1-x)\,y\Big)^{-1} [\underbrace{\Theta(x-y)}_{(1)} + \underbrace{\Theta(y-x)}_{(2)}] \,.$$

Now we substitute $y = x\,t$ in sector (1) and $x = y\,t$ in sector (2) to remap the integration range to the unit square and obtain

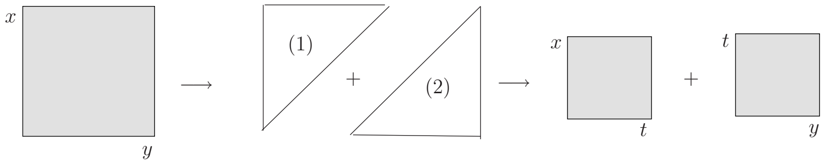


Figure 2: Sector decomposition schematically.

$$\begin{aligned} I &= \int_0^1 dx\, x^{-1-(a+b)\epsilon} \int_0^1 dt\, t^{-b\epsilon} \Big(1 + (1-x)\,t\Big)^{-1} \\ &+ \int_0^1 dy\, y^{-1-(a+b)\epsilon} \int_0^1 dt\, t^{-1-a\epsilon} \Big(1 + (1-y)\,t\Big)^{-1} \,. \end{aligned} \tag{65}$$

We observe that the singularities are now factorised such that they can be read off from the powers of simple monomials in the integration variables, while the polynomial denominator goes to a constant if the integration variables approach zero.
The singularities can then be extracted using

$$x^{-1+\kappa\epsilon} = \frac{1}{\kappa\,\epsilon}\,\delta(x) + \sum_{n=0}^{\infty} \frac{(\kappa\epsilon)^n}{n!} \left[\frac{\ln^n(x)}{x}\right]_+ \,,$$

where

$$\int_0^1 dx\, f(x)\, [g(x)/x]_+ = \int_0^1 dx\, \frac{f(x) - f(0)}{x}\, g(x) \,, \tag{66}$$

and $f(x)$ should be a smooth function. This is known under the name "plus prescription". After the singularities have been extracted, we can expand in ϵ.

The same concept can be applied to N-dimensional parameter integrals over polynomials raised to some power, as for example the functions $\mathcal{F}$ and $\mathcal{U}$ appearing in loop integrals, where the procedure in general has to be iterated to achieve complete factorisation. It also can be applied to phase space integrals, where (multiple) soft/collinear limits are regulated by dimensional regularisation.
In the case of multi-loop integrals, it is convenient to integrate out the δ-function constraining the sum of all Feynman parameters x_i in a special way, such as to preserve the feature that

singularities only occur for $x_i \to 0$ and still having integration limits from 0 to 1: We decompose the parameter integration range for the N-propagator integral into N sectors, where in each sector l, x_l is larger than all other Feynman parameters (note that the remaining $x_{j\neq l}$ are not further ordered), using the identity

$$\int_0^\infty \left(\prod_{j=1}^{N} d\,x_i\right) = \sum_{l=1}^{N} \int_0^\infty \left(\prod_{j=1}^{N} d\,x_i\right) \prod_{\substack{j=1\\ j\neq l}}^{N} \theta(x_l \geq x_j)\;. \tag{67}$$

This is called *primary sector decomposition.* The integral is now split into N domains corresponding to N integrals G_l from which we extract a common factor: $G = (-1)^{N_\nu}\Gamma(N_\nu - LD/2)\sum_{l=1}^{N} G_l$. In the integrals G_l we substitute

$$x_j = \begin{cases} x_l t_j & \text{for} \quad j < l \\ x_l & \text{for} \quad j = l \\ x_l t_{j-1} & \text{for} \quad j > l \end{cases} \tag{68}$$

and then integrate out x_l using the δ-function. As $\mathcal{U}, \mathcal{F}$ are homogeneous of degree $L, L+1$, respectively, and x_l factorises completely, we have $\mathcal{U}(\vec{x}) \to \mathcal{U}_l(\vec{t})\, x_l^L$ and $\mathcal{F}(\vec{x}) \to \mathcal{F}_l(\vec{t})\, x_l^{L+1}$ and thus, using $\int dx_l/x_l\, \delta(1 - x_l(1 + \sum_{k=1}^{N-1} t_k)) = 1$, we obtain

$$G_l \;=\; \int_0^1 \prod_{j=1}^{N-1} \mathrm{d}t_j\, t_j^{\nu_j - 1}\, \frac{\mathcal{U}_l^{N_\nu-(L+1)D/2}(\vec{t})}{\mathcal{F}_l^{N_\nu - LD/2}(\vec{t})} \quad , \quad l = 1, \ldots, N\;. \tag{69}$$

The primary sector decomposition is in general not sufficient to achieve complete factorisation. Therefore the decomposition into sectors where the Feynman parameters go to zero in an ordered way usually has to be iterated.

Iteration

Starting from Eq. (69) we repeat the following steps until a complete separation of overlapping regions is achieved.

II.1: Determine a minimal set of parameters, say $S = \{t_{\alpha_1}, \ldots, t_{\alpha_r}\}$, such that $\mathcal{U}_l$, respectively $\mathcal{F}_l$, vanish if the parameters of S are set to zero. S is in general not unique, and there is no general prescription which defines what set to choose in order to achieve a *minimal* number of iterations. Strategies to choose S such that the algorithm is guaranteed to stop are given in [39, 41, 43, 42]. Using these strategies however in general leads to a larger number of iterations than heuristic strategies to avoid infinite loops, described in more detail below.

II.2: Decompose the corresponding r-cube into r *subsectors* by decomposing unity according to

$$\prod_{j=1}^{r} \theta(1 \geq t_{\alpha_j} \geq 0) = \sum_{k=1}^{r} \prod_{\substack{j=1\\ j\neq k}}^{r} \theta(t_{\alpha_k} \geq t_{\alpha_j} \geq 0)\;. \tag{70}$$

II.3: Remap the variables to the unit hypercube in each new subsector by the substitution

$$t_{\alpha_j} \to \begin{cases} t_{\alpha_k} t_{\alpha_j} & \text{for} \quad j \neq k \\ t_{\alpha_k} & \text{for} \quad j = k\,. \end{cases} \tag{71}$$

This gives a Jacobian factor of $t_{\alpha_k}^{r-1}$. By construction t_{α_k} factorises from at least one of the functions $\mathcal{U}_l$, $\mathcal{F}_l$. The resulting subsector integrals have the general form

$$G_{lk} \;=\; \int_0^1 \left(\prod_{j=1}^{N-1} \mathrm{d}t_j \; t_j^{a_j - b_j \epsilon} \right) \frac{\mathcal{U}_{lk}^{N_\nu - (L+1)D/2}}{\mathcal{F}_{lk}^{N_\nu - LD/2}} \,, \qquad k = 1, \ldots, r\,. \tag{72}$$

For each subsector the above steps have to be repeated as long as a set S can be found such that one of the functions $\mathcal{U}_{l\ldots}$ or $\mathcal{F}_{l\ldots}$ vanishes if the elements of S are set to zero. This way new subsectors are created in each subsector of the previous iteration, resulting in a tree-like structure after a certain number of iterations. The iteration stops if the functions $\mathcal{U}_{lk_1k_2\ldots}$ or $\mathcal{F}_{lk_1k_2\ldots}$ contain a constant term, i.e. if they are of the form

$$\begin{aligned} \mathcal{U}_{lk_1k_2\ldots} &= 1 + u(\vec{t}\,) \\ \mathcal{F}_{lk_1k_2\ldots} &= -s_0 + \sum_\beta (-s_\beta) f_\beta(\vec{t}\,)\,, \end{aligned} \tag{73}$$

where $u(\vec{t}\,)$ and $f_\beta(\vec{t}\,)$ are polynomials in the variables t_j (without a constant term), and s_β are kinematic invariants defined by the cuts of the diagram as explained above, or internal masses. Thus, after a certain number of iterations, each integral G_l is split into a certain number, say α, of subsector integrals, which are of the same form as in Eq. (72).
Evidently the singular behaviour of the integrand now can be read off directly from the exponents a_j, b_j for a given subsector integral. As the singular behaviour is manifestly non-overlapping now, it is straightforward to define subtractions.

Extraction of the poles

The subtraction of the poles can be done implicitly by expanding the singular factors into distributions, or explicitly by direct integration over the singular factors. In any case, the following procedure has to be worked through for each variable $t_{j=1,\ldots,N-1}$ and each subsector integrand:

- Let us consider Eq. (72) for a particular t_j, i.e. let us focus on

$$I_j = \int_0^1 dt_j \, t_j^{(a_j - b_j \epsilon)} \, \mathcal{I}(t_j, \{t_{i \neq j}\}, \epsilon)\,, \tag{74}$$

where $\mathcal{I} = \mathcal{U}_{lk}^{N_\nu-(L+1)D/2} / \mathcal{F}_{lk}^{N_\nu - LD/2}$ in a particular subsector. If $a_j > -1$, the integration does not lead to an ϵ–pole. In this case no subtraction is needed and one can go to the next variable t_{j+1}. If $a_j \leq -1$, one expands $\mathcal{I}(t_j, \{t_{i\neq j}\}, \epsilon)$ into a Taylor series around $t_j = 0$:

$$\begin{aligned}
\mathcal{I}(t_j, \{t_{i\neq j}\}, \epsilon) &= \sum_{p=0}^{|a_j|-1} \mathcal{I}_j^{(p)}(0, \{t_{i\neq j}\}, \epsilon)\frac{t_j^p}{p!} + R(\vec{t}, \epsilon)\ , \text{ where} \\
\mathcal{I}_j^{(p)}(0, \{t_{i\neq j}\}, \epsilon) &= \partial^p \mathcal{I}(t_j, \{t_{i\neq j}\}, \epsilon)/\partial t_j^p\Big|_{t_j=0}\ .
\end{aligned} \tag{75}$$

- Now the pole part can be extracted easily, and one obtains

$$I_j = \sum_{p=0}^{|a_j|-1} \frac{1}{a_j + p + 1 - b_j\epsilon} \frac{\mathcal{I}_j^{(p)}(0, \{t_{i\neq j}\}, \epsilon)}{p!} + \int_0^1 dt_j\, t_j^{a_j - b_j\epsilon} R(\vec{t}, \epsilon)\ . \tag{76}$$

  By construction, the integral containing the remainder term $R(\vec{t}, \epsilon)$ does not produce poles in ϵ upon t_j-integration anymore. For $a_j = -1$, which is the generic case for renormalisable theories (logarithmic divergence), this simply amounts to

$$I_j = -\frac{1}{b_j\epsilon}\mathcal{I}_j(0, \{t_{i\neq j}\}, \epsilon) + \int_0^1 dt_j\, t_j^{-1-b_j\epsilon}\Big(\mathcal{I}(t_j, \{t_{i\neq j}\}, \epsilon) - \mathcal{I}_j(0, \{t_{i\neq j}\}, \epsilon)\Big)\ ,$$

  which is equivalent to applying the "plus prescription" (see eq. (66)), except that the integrations over the singular factors have been carried out explicitly. Since, as long as $j < N - 1$, the expression (76) still contains an overall factor $t_{j+1}^{a_{j+1} - \epsilon\, b_{j+1}}$, it is of the same form as (74) for $j \to j + 1$ and the same steps as above can be applied.

After $N - 1$ steps all poles are extracted, such that the resulting expression can be expanded in ϵ. This defines a Laurent series in ϵ with coefficients $C_{lk,m}$ for each of the $\alpha(l)$ subsector integrals G_{lk}. Since each loop can contribute at most one soft and collinear $1/\epsilon^2$ term, the highest possible infrared pole of an $L-$loop graph is $1/\epsilon^{2L}$. Expanding to order ϵ^r, one has

$$G_{lk} = \sum_{m=-r}^{2L} \frac{C_{lk,m}}{\epsilon^m} + \mathcal{O}(\epsilon^{r+1})\ , \quad G = (-1)^{N_\nu}\Gamma(N_\nu - LD/2)\sum_{l=1}^{N}\sum_{k=1}^{\alpha(l)} G_{lk}\ . \tag{77}$$

Following the steps outlined above one has generated a regular integral representation of the coefficients $C_{lk,m}$, consisting of $(N - 1 - m)$–dimensional finite integrals over parameters t_j. We recall that $\mathcal{F}$ was non-negative in the Euclidean region where all invariants are negative (see eqs. (53,73)), such that the numerical integrations over the finite parameter integrals are straightforward in this region. In principle, it is also possible to do at least part of these parameter integrals analytically, but in most applications such an analytical approach reaches its limits very quickly.

A review on sector decomposition can be found in ref. [10]. Public programs for the calculation of loop integrals are also available [39, 40].

A Appendix

A.1 Useful formulae

$$
\begin{aligned}
\Gamma(x) &= \int_0^\infty e^{-t}\,t^{x-1}dt && \text{(A.1)}\\
x\Gamma(x) &= \Gamma(x+1)\\
\Gamma(x)\Gamma(x+\frac{1}{2}) &= \sqrt{\pi}\,\Gamma(2x)\,2^{1-2x}\\
\Gamma(\frac{1}{2}) &= \sqrt{\pi}\\
\Gamma(1+\epsilon) &= \exp\left(-\gamma_E\epsilon+\sum_{n=2}^{\infty}\frac{(-1)^n}{n}\zeta_n\epsilon^n\right)\;,\;\zeta_n=\sum_{j=1}^{\infty}\frac{1}{j^n}\;.
\end{aligned}
$$

$$
\int_0^\pi d\theta(\sin\theta)^D=\sqrt{\pi}\frac{\Gamma(\frac{D+1}{2})}{\Gamma(\frac{D}{2}+1)} \tag{A.2}
$$

$$
\int_{-\infty}^{\infty}\frac{d^Dl}{i\pi^{\frac{D}{2}}}\frac{(l^2)^r}{[l^2-R^2+i\delta]^N}=(-1)^{N+r}\frac{\Gamma(r+\frac{D}{2})\Gamma(N-r-\frac{D}{2})}{\Gamma(\frac{D}{2})\Gamma(N)}\left[R^2-i\delta\right]^{r-N+\frac{D}{2}} \tag{A.3}
$$

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\frac{d^Dl}{i\pi^{\frac{D}{2}}}\,\frac{l^{\mu_1}\dots l^{\mu_{2m}}}{[l^2-R^2+i\delta]^N}\\
&=(-1)^N\left[(g^{\cdot\cdot})^{\otimes m}\right]^{\{\mu_1\dots\mu_{2m}\}}\left(-\frac{1}{2}\right)^m\frac{\Gamma(N-\frac{D+2m}{2})}{\Gamma(N)}\left(R^2-i\delta\right)^{-N+(D+2m)/2}\;,
\end{aligned} \tag{A.4}
$$

$$
\int\frac{d^{2m-2\epsilon}k}{i\pi^{m-\epsilon}}(\tilde{k}^2)^\alpha f(k^\mu_{(2m)},\tilde{k}^2)\;=\;(-1)^\alpha\frac{\Gamma(\alpha-\epsilon)}{\Gamma(-\epsilon)}\int\frac{d^{2m+2\alpha-2\epsilon}k}{i\pi^{m+\alpha-\epsilon}}f(k^\mu_{2m})\,,\;m\text{ integer}\;. \tag{A.5}
$$

A.2 Multi-loop tensor integrals

A general L-loop, rank R Feynman diagram with N propagators can be written as

$$
G[\mu_1\dots\mu_R]\;=\;\int d\bar{k}_1\dots d\bar{k}_L\frac{k_{l_1}^{\mu_1(l_1)}\dots k_{l_R}^{\mu_R(l_R)}}{\prod_{j=1}^{N}P_j^{\nu_j}(\{k\},\{p\},m_j^2)} \tag{A.6}
$$

where $l_1,\dots,l_R\in\{1,\dots,L\}$ and $P_j^{\nu_j}(\{k\},\{p\},m_j^2)$ are the propagators to the power ν_j. Introducing Feynman parameters, the integral can be expressed in terms of a symmetric $(L\times L)$–matrix M, an L-vector Q (with 4-vectors in each component) and a scalar function J. The contraction of Lorentz indices is indicated by a dot.

$$
\begin{aligned}
G[\mu_1 \dots \mu_R] &= \frac{\Gamma(N_\nu)}{\prod_{j=1}^N \Gamma(\nu_j)} \int \prod_{j=1}^N dx_j \; x_j^{\nu_j - 1}\, \delta(1 - \sum_{i=1}^N x_i) \int d\bar{k}_1 \dots d\bar{k}_L \\
&\quad k_{l_1}^{\mu_1(l_1)} \dots k_{l_R}^{\mu_R(l_R)} \left[\sum_{j,l=1}^L k_j \cdot k_l \, M_{jl} - 2 \sum_{j=1}^L k_j \cdot Q_j + J \right]^{-N\nu} \qquad \text{(A.7)} \\
N_\nu &= \sum_{j=1}^N \nu_j
\end{aligned}
$$

The factors of $k_l^{\mu_i(l)}$ in the numerators can be generated from $G[1]$ by partial differentiation with with respect to $Q_l^{\mu_i(l)}$, where $l \in \{1, \dots, L\}$ denotes the lth component of the l-vector Q, corresponding to the lth loop momentum. Therefore it is convenient to define the double indices $\Gamma_i = (l, \mu_i(l))$, $l \in \{1, \dots, L\}, i \in \{1, \dots, R\}$ denoting the ith Lorentz index, belonging to the lth loop momentum. After having shifted the loop momenta to obtain a quadratic form and differentiated with respect to $Q_l^{\mu_i(l)}$, one obtains after momentum integration the following parameter representation:

$$
\begin{aligned}
G[\mu_1 \dots \mu_R] &= (-1)^{N_\nu} \frac{1}{\prod_{j=1}^N \Gamma(\nu_j)} \int_0^\infty \prod_{j=1}^N dx_j \; x_j^{\nu_j - 1}\, \delta(1 - \sum_{l=1}^N x_l) \\
&\quad \sum_{m=0}^{\lfloor R/2 \rfloor} (-\frac{1}{2})^m \Gamma(N_\nu - m - LD/2) \left[(\tilde{M}^{-1} \otimes g)^{(m)} (\tilde{l}_s)^{(R-2m)} \right]^{\Gamma_1, \dots, \Gamma_R} \\
&\quad \times \frac{\mathcal{U}^{N_\nu - (L+1)D/2 - R}}{\mathcal{F}^{N_\nu - LD/2 - m}} \qquad \text{(A.8)}
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{F}(\vec{x}) &= \det(M) \left[\sum_{j,l=1}^L Q_j \cdot Q_l \, M_{jl}^{-1} - J \right] \qquad \text{(A.9)} \\
\mathcal{U}(\vec{x}) &= \det(M) \\
\tilde{M}^{-1} &= \mathcal{U} M^{-1} \\
\tilde{l}_s &= \tilde{M}^{-1} \cdot Q
\end{aligned}
$$

and $\lfloor R/2 \rfloor$ denotes the nearest integer less or equal to $R/2$. The expression $[(\tilde{M}^{-1} \otimes g)^{(m)} (\tilde{l}_s)^{(R-2m)}]^{\Gamma_1, \dots, \Gamma_R}$ stands for the sum over all different combinations of R (double-)indices distributed to m metric tensors and $(R - 2m)$ vectors $\tilde{l}_s$. The above expression is well known, for more details the reader is referred to the literature [3, 4].

A.3 Exercises

Problem 1: Higher Dimensional Integrals

To see how the higher dimensional integrals I_N^{D+2m}, associated with metric tensors $(g^{\cdot\cdot})^{\otimes m}$, arise in eq. (20), calculate the simplest non-trivial subpart of eq. (19), a rank two tensor, involving two loop momenta in the numerator:

$$L_N^{\mu_1\mu_2} = \Gamma(N)\int_0^\infty \prod_{i=1}^N dz_i\,\delta(1-\sum_{l=1}^N z_l)\int_{-\infty}^\infty \frac{d^Dl}{i\pi^{\frac{D}{2}}}\,l^{\mu_1}l^{\mu_2}\left[l^2-R^2+i\delta\right]^{-N}\ .$$

Solution:
As there is no dimensionful object in the integral which could carry the Lorentz structure, it must be proportional to the metric tensor:

$$\begin{aligned} L_N^{\mu_1\mu_2} &= \Gamma(N)\int_0^\infty \prod_{i=1}^N dz_i\,\delta(1-\sum_{l=1}^N z_l)\int_{-\infty}^\infty d\bar{l}\,l^{\mu_1}l^{\mu_2}\left[l^2-R^2\right]^{-N} \\ &= K\,g^{\mu_1\mu_2} \end{aligned} \tag{A.10}$$

Contracting both sides of eq. (A.10) with $g_{\mu_1\mu_2}$, we obtain

$$\begin{aligned} g_{\mu_1\mu_2}L_N^{\mu_1\mu_2} &= \Gamma(N)\int_0^\infty \prod_{i=1}^N dz_i\,\delta(1-\sum_{l=1}^N z_l)\int_{-\infty}^\infty d\bar{l}\,l^2\ \left[l^2-R^2\right]^{-N} = K\,D \\ &= \Gamma(N)\int_0^\infty \prod_{i=1}^N dz_i\,\delta(1-\sum_{l=1}^N z_l)\int_{-\infty}^\infty d\bar{l}\,\left\{\left[l^2-R^2\right]^{-N+1}+R^2\left[l^2-R^2\right]^{-N}\right\} \end{aligned} \tag{A.11}$$

Now remember the formula for the scalar case:

$$\begin{aligned} I_N^D(S) &= \Gamma(N)\int_0^\infty \prod_{i=1}^N dz_i\,\delta(1-\sum_{l=1}^N z_l)\int_{-\infty}^\infty d\bar{l}\ \left[l^2-R^2\right]^{-N} \\ &= (-1)^N\Gamma(N-\frac{D}{2})\int_0^\infty \prod_{i=1}^N dz_i\,\delta(1-\sum_{l=1}^N z_l)\ \left[R^2\right]^{D/2-N} \end{aligned} \tag{A.12}$$

We see that it can be applied as well to the first term in eq. (A.11) with $N\to N-1$. We obtain:

$$\begin{aligned} g_{\mu_1\mu_2}L_N^{\mu_1\mu_2} &= (-1)^{N-1}\frac{\Gamma(N)}{\Gamma(N-1)}\Gamma(N-1-D/2)\int_0^\infty \prod_{i=1}^N dz_i\,\delta(1-\sum_{l=1}^N z_l)\ \left[R^2\right]^{D/2-N+1} \\ &+ (-1)^N\Gamma(N-\frac{D}{2})\int_0^\infty \prod_{i=1}^N dz_i\,\delta(1-\sum_{l=1}^N z_l)\ \left[R^2\right]^{D/2-N+1} \\ &= (-1)^N\Gamma(N-\frac{D+2}{2})\int \prod_{i=1}^N dz_i\,\delta(1-\sum_{l=1}^N z_l)\ \left[R^2\right]^{(D+2)/2-N}\ \{-(N-1)+N-1-D/2\} \\ &= -\frac{D}{2}\,I_N^{D+2}(S) \end{aligned} \tag{A.13}$$

Hence we find $K=-\frac{1}{2}\,I_N^{D+2}(S)$.

Problem 2: $\tilde{k}$-Integrals

Show that the effect of $(\tilde{k}^2)^\alpha$ in the numerator is to formally shift the integration from D to $D+2\alpha$ dimensions, i.e. derive eq. (26).

Solution:
We use $k^2_{(D)} = \hat{k}^2_{(4)} + \tilde{k}^2_{(-2\epsilon)}$. $\hat{k}$ and $\tilde{k}$ live in orthogonal spaces. The external vectors live in four dimensions (i.e. we use the 't Hooft-Veltman scheme). Hence all vectors $\tilde{k}^\mu$ which are contracted with *external* (i.e. 4-dim) vectors will be projected to zero. Therefore, the integrals we encounter after dimension splitting are of the form

$$I_N^{D,r,s,\alpha} = \int_{-\infty}^{\infty} \frac{d^Dk}{i\pi^{\frac{D}{2}}} \frac{\hat{k}^{\mu_1}\dots\hat{k}^{\mu_r}\,(\tilde{k}^2)^\alpha\,(\hat{k}^2)^s}{\prod_{i=1}^N (q_i^2 - m_i^2 + i\delta)}\,. \tag{A.14}$$

The factors of $\hat{k}$ in the numerator are entirely in four dimensions and therefore irrelevant to the treatment of the $\tilde{k}$-part. Hence we only need to consider the integral

$$I_N^{D,\alpha} = \int_{-\infty}^{\infty} \frac{d^Dk}{i\pi^{\frac{D}{2}}} \frac{(\tilde{k}^2)^\alpha}{\prod_{i=1}^N (q_i^2 - m_i^2 + i\delta)}\,. \tag{A.15}$$

After Feynman parametrisation as usual and after the substitution $k = l - Q \Rightarrow k^2 = \hat{l}^2 + \tilde{l}^2 - 2Q\cdot\hat{l} + Q^2$, $\tilde{k}^2 = \tilde{l}^2$ (as Q lives in 4 dimensions), we obtain

$$\begin{aligned} I_N^{D,\alpha} &= \Gamma(N)\int_0^\infty \prod_{i=1}^N dz_i\,\delta(1-\sum_{l=1}^N z_l)\int_{-\infty}^{\infty}\frac{d^4\hat{l}}{i\pi^2}\,\frac{d^{(D-4)}\tilde{l}}{\pi^{\frac{D}{2}-2}}\,(\tilde{l}^2)^\alpha \\ &\quad\times\left[\hat{l}^2+\tilde{l}^2-R^2+i\delta\right]^{-N} \end{aligned} \tag{A.16}$$

After Wick rotation we can define polar coordinates with radial components $\rho = |\hat{l}_E|^2$, $t = |\tilde{l}_E|^2$ and obtain (note that $d^{(D-4)}\tilde{l}_E = \frac{1}{2}\,t^{\frac{D-6}{2}}\,dt$ and that we use the convention $\tilde{l}^2 = -\tilde{l}_E^2$)

$$\begin{aligned} I_N^{D,\alpha} &= \frac{1}{4}(-1)^{N+\alpha}\Gamma(N)V(4)V(\frac{D-4}{2})\int_0^\infty\prod_{i=1}^N dz_i\,\delta(1-\sum_{l=1}^N z_l)\int_0^\infty\frac{\rho\,d\rho}{\pi^2}\int_0^\infty\frac{dt}{\pi^{\frac{D}{2}-2}}\,t^{\frac{D-6}{2}+\alpha} \\ &\quad\times\left[\rho+t+R^2-i\delta\right]^{-N} \\ &= (-1)^{N+\alpha}\frac{\Gamma(N)}{\Gamma(\frac{D-4}{2})}\int_0^\infty\prod_{i=1}^N dz_i\,\delta(1-\sum_{l=1}^N z_l)\int_0^\infty\rho\,d\rho\int_0^\infty dt\,t^{\frac{D-6}{2}+\alpha} \\ &\quad\times\left[\rho+t+R^2-i\delta\right]^{-N} \end{aligned} \tag{A.17}$$

The integrals over ρ and t can be mapped to the Euler Beta-function (see lecture). Doing first the ρ-integral (subst. $v = \rho/(t+R^2)$) and then the t-integral leads to

$$
\begin{aligned}
I_N^{D,\alpha} &= (-1)^{N+\alpha}\frac{\Gamma(N-2)}{\Gamma(\frac{D-4}{2})}\int_0^\infty \prod_{i=1}^N dz_i\,\delta(1-\sum_{l=1}^N z_l)\int_0^\infty dt\, t^{\frac{D-6}{2}+\alpha}\,\left[t+R^2-i\delta\right]^{-N+2}\\
&= (-1)^{N+\alpha}\frac{\Gamma(\frac{D}{2}-2+\alpha)\Gamma(N-\frac{D}{2}-\alpha)}{\Gamma(\frac{D-4}{2})}\int_0^\infty \prod_{i=1}^N dz_i\,\delta(1-\sum_{l=1}^N z_l)\,\left[R^2-i\delta\right]^{\frac{D}{2}+\alpha-N}\\
&= (-1)^{\alpha}\frac{\Gamma(\frac{D}{2}-2+\alpha)}{\Gamma(\frac{D-4}{2})}\,I_N^{D+2\alpha}\,, \qquad\qquad (A.18)
\end{aligned}
$$

where $I_N^{D+2\alpha}$ is of the form of an "ordinary" scalar integral in $D+2\alpha$ dimensions, see eq.(A.12).

Problem 3: Generalized Unitarity

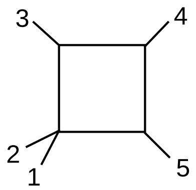


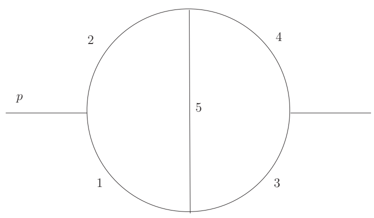
Figure 3: Box integral with propagator 1 pinched, $p_{12}^2 \neq 0$.

Using quadruple cuts, compute the coefficient of a box integral occurring in the pure Yang-Mills theory amplitude $A_5^{1-\text{loop}}(1^-,2^-,3^+,4^+,5^+)$, shown in figure 1. The integral is given by ($p_{ij} = p_i + p_j$, $i\delta$ terms are implicit)

$$
I_4^D(S\setminus\{1\}) = \int d\bar{l}\,\frac{1}{l^2(l+p_{12})^2(l+p_{123})^2(l-p_5)^2}\,. \qquad (A.19)
$$

Solution:
See Z. Bern, L. J. Dixon and D. A. Kosower, *"On-Shell Methods in Perturbative QCD"*, Annals Phys. **322** (2007) 1587 [arXiv:0704.2798 [hep-ph]], section 4.4.

Problem 4: "Kirchhoff rules" for multi-loop graphs

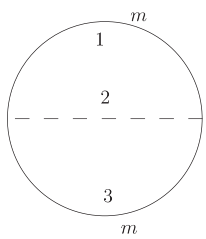


Determine the functions $\mathcal{F}$ and $\mathcal{U}$ for the graph shown in figure 2 using the topological cutting rules.

Solution:

$$\begin{aligned}
\mathcal{U} &= (x_1+x_2)(x_3+x_4)+x_5\,(x_1+x_2+x_3+x_4)\\
\mathcal{F} &= (-p^2)\,\{x_1\,x_2\,(x_3+x_4+x_5)+x_3\,x_4\,(x_1+x_2+x_5)+x_5\,(x_1\,x_4+x_2\,x_3)\}
\end{aligned}$$

Problem 5: Sector decomposition



Using sector decomposition, factorize the singularities of the two-loop vacuum bubble graph with two massive propagators (see figure)

$$G = \int d\bar{k}\,d\bar{q}\frac{1}{[k^2-m^2+i\delta]\,[(q-k)^2+i\delta]\,[q^2-m^2+i\delta]}\ .$$

Solution:

$$\begin{aligned}
G &= \int d\bar{k}\,d\bar{q}\frac{1}{[k^2-m^2+i\delta]\,[(q-k)^2+i\delta]\,[q^2-m^2+i\delta]} \qquad \text{(A.20)}\\
&= -\Gamma(3-D)\,(m^2)^{1-2\epsilon}\int_0^\infty \prod_{i=1}^{3} dx_i\,\delta(1-\sum_{l=1}^{3}x_l)\,(x_1+x_3)^{1-2\epsilon}\,(x_1x_2+x_1x_3+x_2x_3)^{-2+\epsilon}
\end{aligned}$$

Now we split the integration domain into three parts and eliminate the δ–distribution in such a way that the remaining integrations are from 0 to 1 (primary sector decomposition). In our example, this means

$$\begin{aligned}
\int_0^\infty dx_1\,dx_2\,dx_3 = \int_0^\infty dx_1\,dx_2\,dx_3\,[\quad & \theta(x_1-x_2)\theta(x_1-x_3)\\
+\; & \theta(x_2-x_1)\theta(x_2-x_3)\\
+\; & \theta(x_3-x_1)\theta(x_3-x_2)\quad]\ .
\end{aligned}$$

Our integral is now split into 3 domains corresponding to 3 integrals G_l from which we extract a common factor: $G=-\Gamma(3-D)\sum_{l=1}^{3}G_l$. In the integrals G_l we substitute

$$x_j = \begin{cases} x_l t_j & \text{for} \quad j\neq l\\ x_l & \text{for} \quad j=l \end{cases} \qquad \text{(A.21)}$$

and then integrate out x_l using the δ–distribution. As $\mathcal{U}, \mathcal{F}$ are homogeneous of degree $L, L+1$, respectively, and x_l *always* factorises completely, and we have $\mathcal{U}(\vec{x}) \to \mathcal{U}_l(\vec{t})\, x_l^L$ and $\mathcal{F}(\vec{x}) \to \mathcal{F}_l(\vec{t})\, x_l^{L+1}$. Thus, using $\int dx_l/x_l\, \delta(1 - x_l(1 + \sum_{k=1}^{N-1} t_k)) = 1$, we obtain

$$\begin{aligned}
G_1 &= = \int_0^1 dt_2 dt_3\, (1+t_3)^{1-2\epsilon}(t_2 + t_3 + t_2\, t_3)^{-2+\epsilon} \\
G_2 &= = \int_0^1 dt_1 dt_3\, (t_1+t_3)^{1-2\epsilon}(t_1 + t_3 + t_1\, t_3)^{-2+\epsilon} \\
G_3 &= G_1 \quad \text{with } t_1 \leftrightarrow t_3
\end{aligned}$$

Now we iterate the procedure until the polynomials in the Feynman parameters (which are the functions $\mathcal{F}$ and $\mathcal{U}$ in terms of the new variables) are of the form “constant plus polynomial in the t_i”. For example, in G_1, we decompose in the variables t_2, t_3:

$$G_1 = = \int_0^1 dt_2 dt_3\, (1+t_3)^{1-2\epsilon}(t_2 + t_3 + t_2\, t_3)^{-2+\epsilon} \left[\underbrace{\theta(t_2 - t_3)}_{(a)} + \underbrace{\theta(t_3 - t_2)}_{(b)}\right]$$

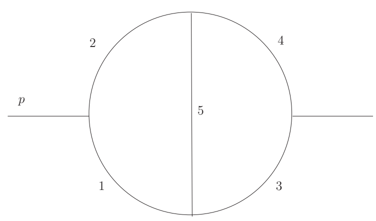
$$\begin{aligned}
\text{Subst.} \qquad & t_3 = t_2\, t_3 \quad \text{in (a)}\,, \\
& t_2 = t_3\, t_2 \quad \text{in (b)}
\end{aligned}$$

$$G_1^{(a)} = = \int_0^1 dt_2 dt_3\, t_2^{-1+\epsilon}(1+t_2 t_3)^{1-2\epsilon}(1 + t_3 + t_2\, t_3)^{-2+\epsilon} \tag{A.22}$$

$$G_1^{(b)} = = \int_0^1 dt_2 dt_3\, t_3^{-1+\epsilon}(1+t_3)^{1-2\epsilon}(1 + t_2 + t_2\, t_3)^{-2+\epsilon}\,. \tag{A.23}$$

We see that the singularities have been factored out, residing now simply in factors like $t_2^{-1+\epsilon}$, while the remaining polynomials are finite in the limit $t_i \to 0$.
We apply the same procedure to G_2 and G_3. The final result for each pole coefficient will be a sum of finite parameter integrals stemming from the endpoints of the decomposition tree.

Problem 6: Integration by Parts



The integral for the graph shown above with massless propagators and general propagator powers is given by ($i\delta$ dropped)

$$F(\nu_1, \ldots, \nu_5) = \int d\bar{k} \int d\bar{l} \frac{1}{[k^2]^{\nu_1}[(k-p)^2]^{\nu_2}[l^2]^{\nu_3}[(l-p)^2]^{\nu_4}[(l-k)^2]^{\nu_5}}\,. \tag{A.24}$$

Use the integration-by-parts identity

$$\int d\bar{k} \int d\bar{l} \frac{1}{[l^2]^{\nu_3}[(l-p)^2]^{\nu_4}} \frac{\partial}{\partial k_\mu} \left(\frac{k_\mu - l_\mu}{[k^2]^{\nu_1}[(k-p)^2]^{\nu_2}[(l-k)^2]^{\nu_5}} \right) = 0$$

to express $F(1,\ldots,1)$ in terms of integrals where one of the ν_i is zero.

Disclaimer: The reference list is far from complete.

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