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Evolution of distribution amplitudes

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i. Why is evolution important for exclusive processes?

ii. A useful tool: restricted conformal symmetry

iii. Compendium of next-to-leading order results

iv. Solution of the evolution equation

v. View beyond NLO

vi. Summary

Why is evolution important?

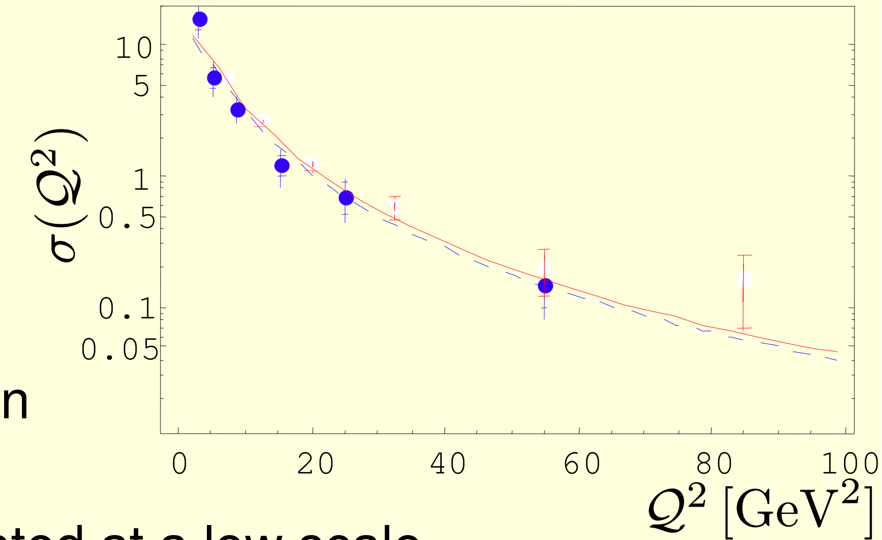
- scaling violation is a pQCD prediction and might be confronted with experimental data, e.g., from collider experiments

H1 and ZEUS DVCS measurement

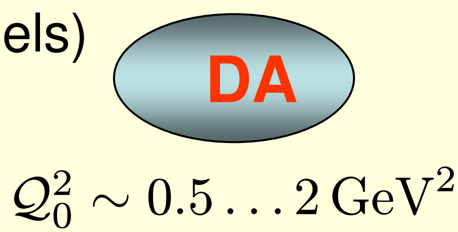
canonical scaling: $\sigma \propto \mathcal{Q}^{-4}$

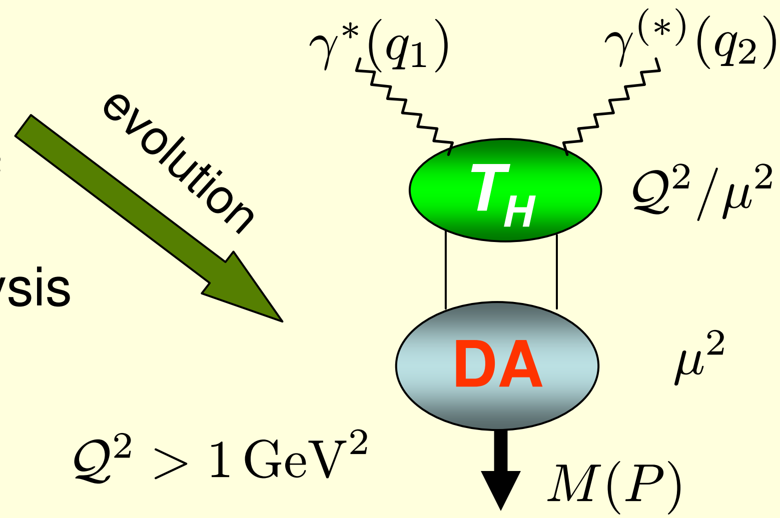
measured scaling: $\propto \mathcal{Q}^{-3.08}$

scaling violation arises from evolution



- non-perturbative input might be evaluated at a low scale (lattice, sum-rules, models)





- evolution is part of a complete pQCD analysis (quantitatively not working to LO)

Restricted conformal symmetry in pQCD

Massless QCD Lagragian is invariant under conformal transformations SO(4,2)

- Poincaré transformations
- dilatation: $x^\mu \to x'^\mu = \lambda x^\mu$
- special conformal transformations: $x^\mu \to x'^\mu = \frac{x^\mu + a^\mu}{1+2a\cdot x + a^2 x^2}$

Restricted to the light-cone ($x^\mu = x_- n^\mu + x_+ \overline{n}^\mu + x_\perp^\mu \,, n^2 = \overline{n}^2 = 0$):

$$x_- \to x_- + c, \quad x_- = \lambda x_-, \quad x_- \to x'_- = \frac{x_-}{1+2a\, x_-}$$

collinear conformal group SL(2,R) ~ SO(2,1) (projection on a line):

$$\Phi(\alpha n) \to \Phi'(\alpha n) = (c\alpha + d)^{-2j}\Phi\left(\frac{a\alpha+b}{c\alpha+d}n\right)\,, \quad ad - bc = 1$$

is characterized by the conformal spin

$$j = (\ell + s)/2,\ \ell\text{-dimension},\ s\text{-spin projection}$$

generators: $\mathbf{L}_+ = -i\mathbf{P}_+,\ \mathbf{L}_- = (i/2)\mathbf{K}_-,\ \mathbf{L}_0 = (i/2)(\mathbf{D} + \mathbf{M}_{-+})$

$$[\mathbf{L}_0, \mathbf{L}_\mp] = \mp\mathbf{L}_\mp\,, \quad [\mathbf{L}_-, \mathbf{L}_+] = -2\mathbf{L}_0$$

Note:

- conformal symmetry is broken by the renormalization procedure (trace anomaly)

$$g^{\mu\nu}\Theta^{\rm QCD}_{\mu\nu} \stackrel{\rm EOM}{=} \frac{d-4}{4}(G^a_{\mu\nu})^2 + \ldots = \frac{-\epsilon g+\beta(g)}{2g}[(G^a_{\mu\nu})^2] + \ldots$$

- and in the non-perturbative sector, since of the observed mass spectrum $P^2=M^2\neq 0$

$$\exp\{i\lambda \mathbf{D}\}\,\mathbf{P}^2\exp\{-i\lambda\mathbf{D}\} = \exp\{2\lambda\}\,\mathbf{P}^2 \quad \Rightarrow \quad \mathbf{P}^2 \stackrel{!}{=} 0$$

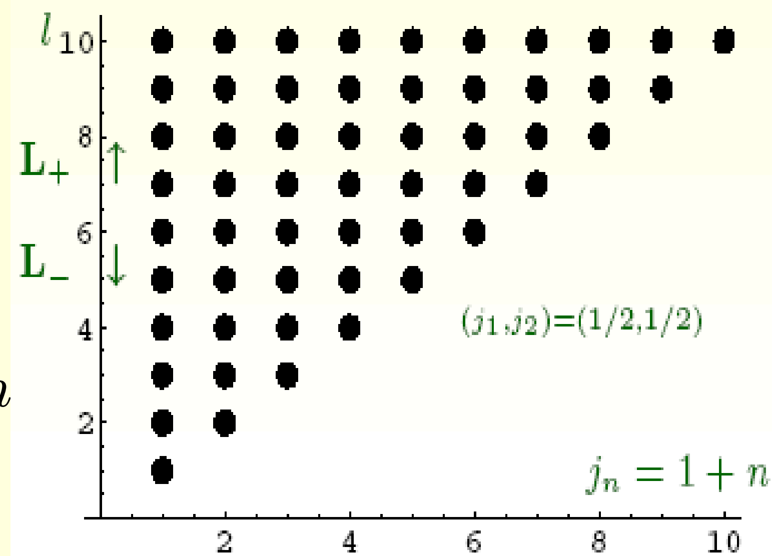
- Restricted conformal symmetry remains a powerful tool,
  e.g., partial wave decomposition of DAs:

$$\varphi(x) = \sum_{n=0}^{\infty}\omega_n(x)P_n^{(2j_1-1,2j_2-1)}(2x-1)\langle 0|\mathbb{O}^{j_1,j_2}_{n,n}|M\rangle$$

$$\mathbb{O}^{j_1,j_2}_n = \partial_+^n\left[\Phi_{j_1}P_n^{(2j_1-1,2j_2-1)}\left(\frac{\vec{\partial}_+ - \overleftarrow{\partial}_+}{\vec{\partial}_+ + \overleftarrow{\partial}_+}\right)\Phi_{j_2}\right]$$

$j_n = j_1 + j_2 + n$ - conformal spin

$$\mathbb{O}^{j_1,j_2}_{n,l} = i^{l-n}(\partial_{+1}+\partial_{+2})^{l-n}\,\mathbb{O}^{j_1,j_2}_{n,n}\,, \quad l \geq n$$



True conformal Ward identities are derived in the standard way within MS scheme [DM 94]:

I. reparameterization invariance of the path integral in the regularized theory

II. renormalization procedure

dilatation Ward identity (renormalization group equation):

$$\mathbf{L}_0^{(y,z)}\langle[\mathbb{Q}_{nl}]\psi(y)\overline{\psi}(z)\rangle = -\sum_{m=0}^{n}\left[\ell_n^{\rm can}\delta_{nm} + \gamma_{nm}\right]\langle[\mathbb{Q}_{ml}]\psi(y)\overline{\psi}(z)\rangle$$
$$+\frac{\beta}{g}\langle[\mathbb{Q}_{ml}\Delta^g]\psi(y)\overline{\psi}(z)\rangle + \cdots$$

anomalous dimensions $\gamma_{nm} = \frac{\alpha_s}{2\pi}\gamma_n^{(0)}\delta_{nm} + \cdots$ modify the canonical ones $\ell_n^{\rm can}$

special conformal Ward identity:

$$\mathbf{L}_-^{(y,z)}\langle[\mathbb{Q}_{nl}]\psi(y)\overline{\psi}(z)\rangle = i\sum_{m=0}^{n}\left[a(n,l)\delta_{nm} + \gamma_{nm}^{c}(l)\right]\langle[\mathbb{Q}_{ml-1}]\psi(y)\overline{\psi}(z)\rangle$$
$$+\frac{\beta}{g}\langle[\mathbb{Q}_{ml}\Delta_-^g]\psi(y)\overline{\psi}(z)\rangle + \cdots$$

with $a(n,l) = (n-l)(n+l+2j-1)$ and special conformal anomaly

$$\gamma_{nm}^{c}(l) = \frac{\alpha_s}{2\pi}\gamma_{nm}^{c(0)}(l) + \cdots, \qquad \gamma_{nm}^{c(0)}(l) = -b_{nm}(l)\gamma_m^{(0)} + w_{nm}$$

conformal covariance is broken to LO in the MS scheme (finite part)

Conformal constraints & anomalous dimensions to NLO

The conformal algebra induces a *constraint* between anomalies [DM 94]:

$$[\mathbf{L}_0, \mathbf{L}_-] = -\mathbf{L}_- \qquad \Rightarrow \qquad \left[\hat{a}(l) + \hat{\gamma}^c(l) + 2\frac{\beta(g)}{g}\hat{b}(l), \hat{\gamma}\right] = 0$$

the off-diagonal entries are related by a recurrence relation

$$2(n-m)(n+m+3)\gamma_{nm}(\alpha_s) = \left[\hat{\gamma}(\alpha_s), \hat{\gamma}^c(\alpha_s) + 2\frac{\beta}{g}(\alpha_s)\hat{b}(l)\right]_{nm}, \; n > m$$

hence, we have to LO and to NLO

$$\gamma_{nm}^{(0)} = 0 \qquad \gamma_{nm}^{(1)} = \frac{\gamma_n^{(0)} - \gamma_m^{(0)}}{2(n-m)(n+m+3)}\left(-b_{nm}\gamma_m^{(0)} + w_{nm} - \beta_0 b_{nm}\right), n > m$$

- ✓ this result coincides with the explicit evaluation of the flavor non-singlet kernel
- ✓ the anomalous dimensions are known in an analytic form
- ✓ it explains the unexpected conformal symmetry breaking:
  due to the minimal subtraction scheme
  or by the finite part of the LO operator, which breaks special conformal symmetry
- ✓ for *β=0* conformal covariance can be restored to all orders in perturbation theory (changed scaling dimensions) [DM 97]
- ✓ dilatation operator can be even diagonalized within *β≠0*

Compendium of NLO results

At leading twist-two we have 7 different operators:

$$\bar\psi\gamma_+\lambda^{\rm NS}\psi \quad \left\{ \begin{array}{c} \bar\psi\gamma_+\psi \\ G_{+\mu}g^{\mu\nu}G_{\nu+} \end{array} \right\} \quad \left\{ \begin{array}{c} \bar\psi\gamma_+\gamma_5\psi \\ G_{+\mu}\epsilon^{+-\mu\nu}G_{\nu+} \end{array} \right\} \quad \left\{ \begin{array}{c} \bar\psi\sigma_{+\perp}\psi \\ G_{+\mu}\tau^{\alpha\beta;\mu\nu}G_{\nu+} \end{array} \right\}$$

the three anomalous dimension matrices in the singlet sector were evaluated up to NLO [A.V. Belitsky, DM (98)]

$$\gamma^{(1)}_{nm} = \boldsymbol{\gamma}^{(1)}_n\delta_{nm} + \boldsymbol{\gamma}^{\rm ND(1)}_{nm}|_{n>m}, \quad \hat{\boldsymbol{\gamma}}^{\rm ND(1)} = -\left[\hat{\boldsymbol{\gamma}}^{(0)}, \hat{\boldsymbol{d}}\right]\left(\beta_0\hat{\mathbf{1}} + \hat{\boldsymbol{\gamma}}^{(0)}\right) + \left[\hat{\boldsymbol{\gamma}}^{(0)}, \hat{\boldsymbol{g}}\right]$$

$$\hat{\boldsymbol{\gamma}} = \begin{pmatrix} {}^{QQ}\hat\gamma & {}^{QG}\hat\gamma \\ {}^{GQ}\hat\gamma & {}^{GG}\hat\gamma \end{pmatrix}, \quad \boldsymbol{d}_{nm} = \frac{1}{a(n,m)}\begin{pmatrix} \hat b & 0 \\ 0 & \hat b \end{pmatrix}_{nm}, \quad \boldsymbol{g}_{nm} = \frac{1}{a(n,m)}\begin{pmatrix} {}^{QQ}\hat w & {}^{QG}\hat w \\ {}^{GQ}\hat w & {}^{GG}\hat w \end{pmatrix}_{nm}$$

where the diagonal part $\boldsymbol{\gamma}^{(1)}_n$ coincides with DIS anomalous dimensions

- ➢ from the analytic expressions we were able to construct all ten evolution kernels [A.V. Belitsky, A. Freund, DM (99/00)]
- ✓ consistency checks based on supersymmetry
- ✓ explicit evaluation of the β-proportional terms were performed

- ❖ twist-3 NLO kernels are evaluated for $\Delta^{\uparrow\uparrow\uparrow}$ DA and SUSY scalar operator [A.V. Belitsky, G. Korchemsky, DM (05/06)]

Solution of the evolution equation

$$Q^2 \frac{d}{dQ^2}\varphi(x,Q^2) = \int_0^1 dy V(x,y,\alpha_s(Q^2))\varphi(y,Q^2)$$

The solution beyond LO leads in the MS scheme to a mixing of conformal partial waves

$$\varphi(x,Q) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \varphi_n(x)\mathcal{E}_{nm}(Q,Q_0)a_m(Q_0)\,, \quad a_0 = 1$$

$$Q\frac{d}{dQ}\mathcal{E}_{nm}(Q,Q_0) = -\sum_{l=m}^{n}\gamma_{nl}(\alpha_s(Q))\mathcal{E}_{lm}(Q,Q_0), \qquad \mathcal{E}_{nm}(Q_0,Q_0) = \delta_{nm}$$

The solution, given by a path ordered exponential, can be easily evaluated

$$\mathcal{E}_{nm}(Q,Q_0) = \left[\mathcal{P}\exp\left\{-\int_{Q_0}^{Q}\frac{d\mu}{\mu}\hat{\gamma}(\alpha_s(\mu))\right\}\right]_{nm}$$

NOTE: that even for an asymptotic input distribution all harmonics are contributing

$$\varphi^{\mathrm{asy}(0)} = 6x(1-x) \underset{Q\to\infty,\,\alpha_s^*-\mathrm{con.}}{\Rightarrow} \varphi^{\mathrm{asy}(1)} = 6x(1-x)\left(1+\frac{\alpha_s^*}{2\pi}\frac{4}{3}\left[\ln^2\frac{x}{1-x}+2-\frac{\pi^2}{3}\right]\right)$$

How to deal with series?

Conformal partial waves are oscillating, hence one must be very careful or

- first convolution with the hard-scattering part improves convergency

$$\int_0^1 \frac{dx}{x}\varphi(x,Q) = 3\sum_{n=0}^{\infty}\sum_{m=0}^{n}\mathcal{E}_{nm}(Q,Q_0)a_m(Q_0)$$

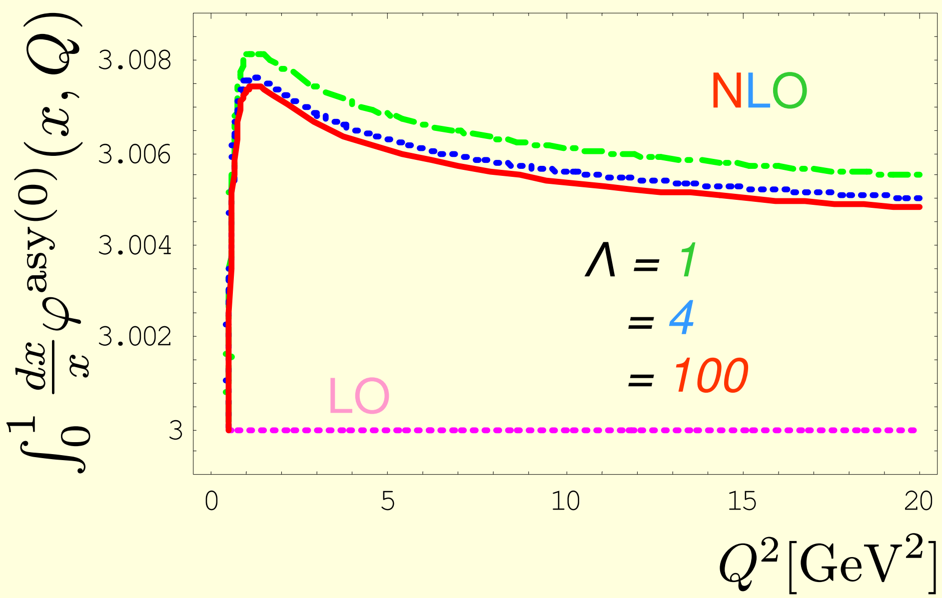
- truncation of the partial wave expansion

$$\int_0^1 \frac{dx}{x}\varphi(x,Q) = 3\sum_{m=0}^{\Lambda_{\rm model}}\sum_{n=m}^{m+2\Lambda}\mathcal{E}_{nm}(Q,Q_0)a_m(Q_0)$$

error is of order $O(1/\Lambda_{\rm cut})$

numerical effects are small, since mixing is suppressed by initial condition,

e.g., $$\int_0^1 \frac{dx}{x}\varphi^{\rm asy(0)} = 3$$



NOTE:

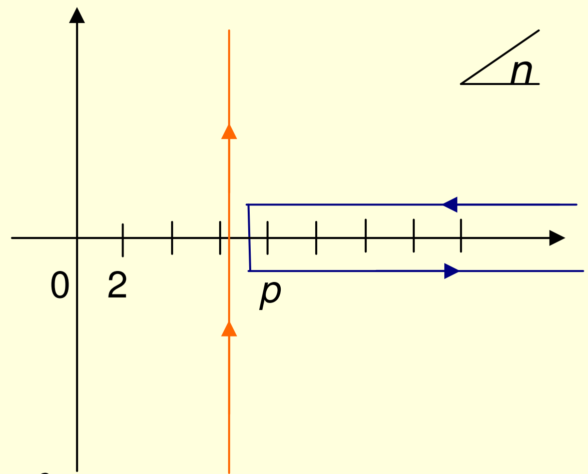
$Q \to \infty$

α_s^*-con. ⇒ $$\int_0^1 \frac{dx}{x}\varphi^{\rm asy(1)} = 3 + 8\frac{\alpha_s^*}{2\pi}$$

Resummation of conformal partial waves

If the DA is very narrow or wide it can not be approximated by the first few partial waves.

How to resum the series?
[K. Kumerički, DM, K. Passek-Kumerički, A. Schäfer (06)]



I. replace the series by a contour integral

$$A = \sum_{\substack{n=0 \\ \text{even}}}^{\infty} a_n \quad \Rightarrow \sum_{\substack{n=0 \\ \text{even}}}^{p} a_n + \frac{1}{4i} \oint_{(p)}^{(\infty)} dn \cot\left(\frac{\pi}{2} n\right) a_n$$

II. deform the contour where *p* is chosen so that $a_n/n \to 0$ for $n \to \infty$

$$A = \sum_{\substack{n=0 \\ \text{even}}}^{p} a_n + \frac{i}{4} \int_{c-i\infty}^{c+i\infty} dn \cot\left(\frac{\pi}{2} n\right) a_n \quad \text{where} \quad p < c < p+2$$

- Mellin-Barnes transformation can be also used for DAs, e.g.,

$$x(1-x) C_{2n}^{3/2}(2x-1) \qquad \Rightarrow \qquad x\, {}_2F_1\left({-n-1 \;\; n+2 \atop 2} \middle| x \right) + \{x \to 1-x\}$$

- analogous technique can be employed for GPDs and DVCS
(see next talk by Kresemir Kumerički)

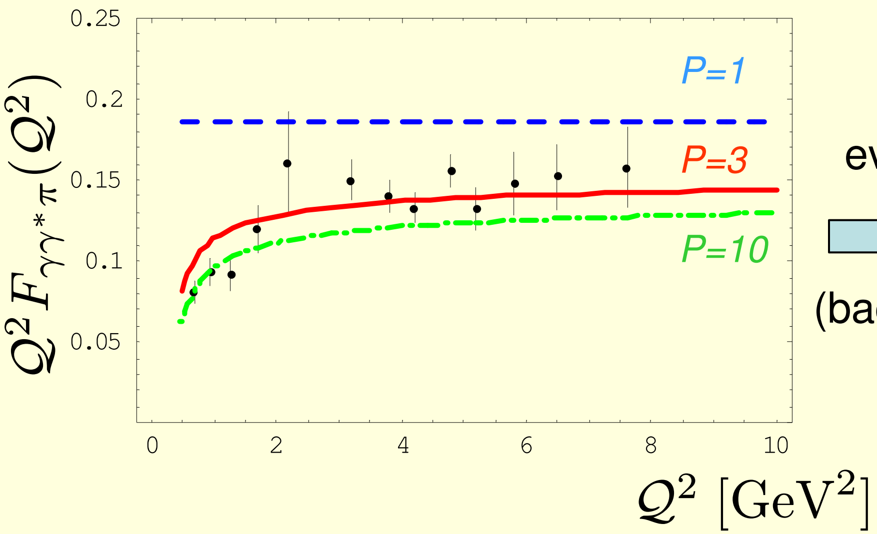
A general representation convenient to use

$$\int_0^1 \frac{dx}{x}\varphi(x,Q) \;=\; \sum_{n=0}^{p} E_n(Q,Q_0)a_n(Q_0) + \frac{i}{4}\int_{c-i\infty}^{c+i\infty} dn\cot\left(\frac{\pi}{2}n\right)E_n(Q,Q_0)a_n(Q_0)$$

$$E_n(Q,Q_0) = \frac{i}{4}\int_{-1/2-i\infty}^{-1/2+i\infty} dm\cot\left(\frac{\pi}{2}m\right)\mathcal{E}_{n+m,n}(Q,Q_0)$$

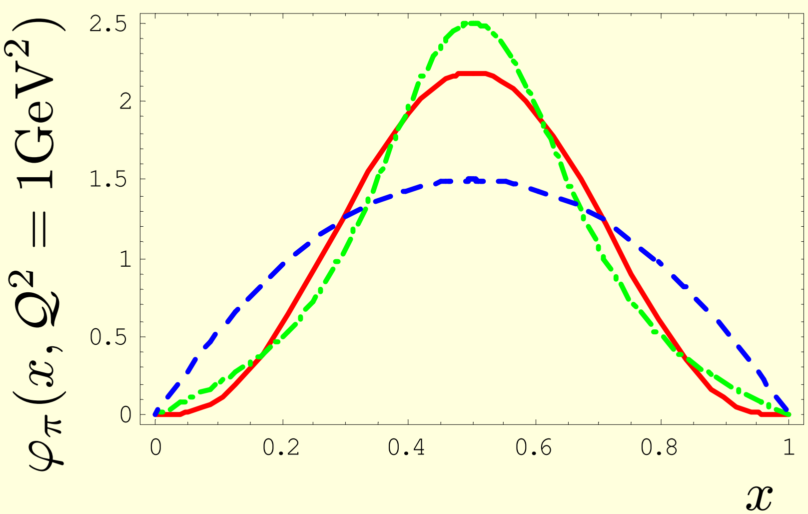
- generalization to flavor singlet case is straightforward
- offers to use the parameterization $\varphi(x,Q_0) = \frac{\Gamma(2P+2)}{\Gamma^2(P+1)}x^P(1-x)^P$

e.g., transition form factor $F_{\gamma\gamma^*\pi}(\mathcal{Q}^2|P,Q_0^2 = 10\mathrm{GeV}^2,\Lambda_{\mathrm{QCD}} = 500\mathrm{MeV})$ (here LO)



evolution

(backwards)



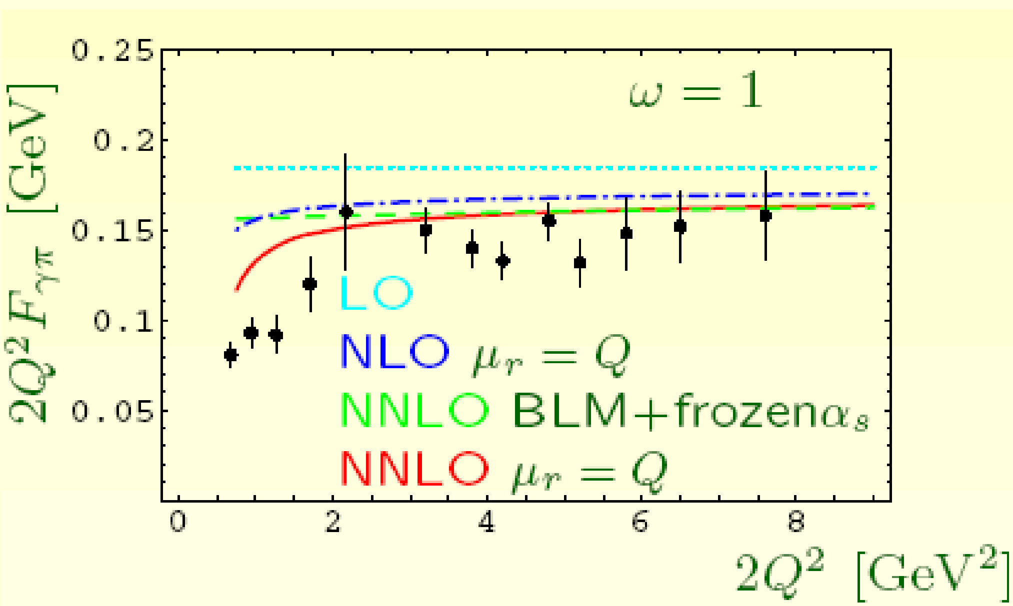
View beyond NLO

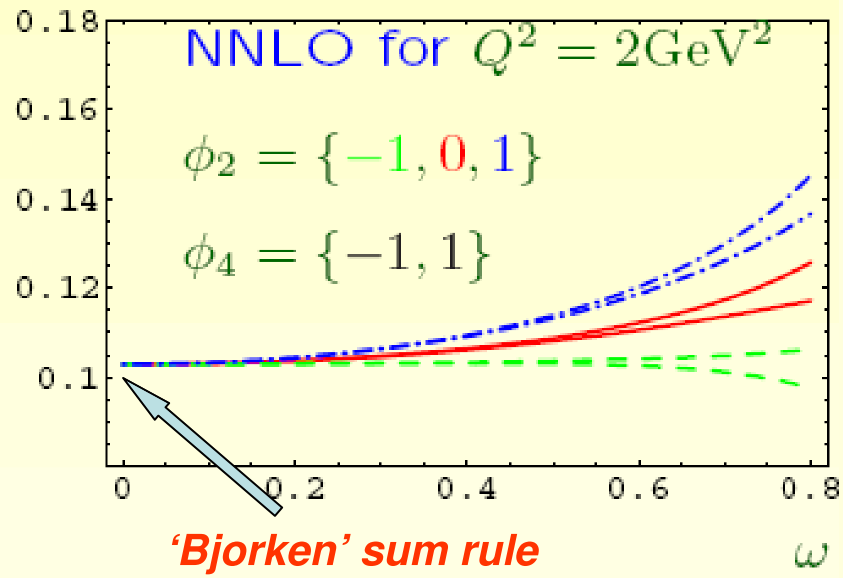
restoration of conformal symmetry ➡ conformal operator product expansion is true

normalization of Wilson-coefficients are borrowed from DIS, known to NNLO
(see next talk by Kresimir Kumerički)

predictive power can be used to describe 2-photon processes at light-cone distances (generalized Bjorken limit), e.g., photon-to-pion transition form factor
[B. Melić, DM, K. Passek-Kumerički, (03)]

$$\gamma^*(q_1)\gamma^{(*)}(q_2) \to \pi(P) \qquad Q^2 = -\frac{1}{2}\left(q_1^2 + q_2^2\right) \quad \omega = \frac{q_1^2 - q_2^2}{q_1^2 + q_2^2}$$





Summary

- By means of conformal Ward-identities and constraints ***all*** twist-two NLO anomalous dimensions and evolution kernels were evaluated in MS scheme.

- For twist-three operators with maximal helicity (or R-charge, gluon operator should come soon) twist-3 evolution kernels have been evaluated and the spectrum of anomalous dimensions have been studied [*AdS/CFT duality*].

- Twist-two evolution equations for DAs are straightforwardly to solve, however, the numerical treatment requires some effort and caution.

- Although mixing effects due to the evolution are numerically small (since of the initial condition), they are for observables on the 10% level.

- For meson DAs that are not closed to the asymptotic form a Mellin-Barnes representation offers a convenient numerical treatment

⇒ new DA ansätze