Multiple-Field Inflation and the CMB

Bartjan van Tent

ITFA, University of Amsterdam
(from October 1st: DAMTP, University of Cambridge)

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Overview

- **Total Entropy Perturbation**
  Definition & Evolution with Time
- **Amplitudes & Spectral Indices CMB**
  from Multiple-Field Inflation; Observational Tests
- **Conclusions**

Total Entropy Perturbation

During radiation/matter domination equation of motion for gravitational potential Φ (super-Hubble modes):

$$\boxed{\ddot{\Phi} + 4H(1 + \tfrac{3}{4}c_s^2)\dot{\Phi} + \kappa^2(\rho c_s^2 - p)\Phi = -2\kappa^2(\rho c_s^2 - p)S.}$$

$H \equiv \dot{a}/a$, $\rho =$ energy density, $p =$ pressure, $c_s^2 \equiv \dot{p}/\dot{\rho}$, $\kappa^2 \equiv 8\pi G$;

$$S \equiv \frac{1}{4}\frac{\delta p - c_s^2 \delta\rho}{p - c_s^2 \rho}. \qquad \text{total entropy perturbation}$$

(Single-field inflation: $S = 0$.)

Two assumptions:

1. All components i ideal fluids w/ constant $w_i \equiv p_i/\rho_i$;
2. No interactions.

$\Rightarrow\ p_i/\rho_i = \dot{p}_i/\dot{\rho}_i = \delta p_i/\delta\rho_i = w_i$.

Using conservation equation, $\dot{\rho}_i + 3H\rho_i(1 + w_i) = 0 \Rightarrow$

$$S = -\frac{1}{4}\frac{\sum_{k,l}(1 + w_k)(1 + w_l)(w_k - w_l)\rho_k\rho_l S_{kl}}{\sum_{i,j}(w_i - w_j)^2\rho_i\rho_j},$$

with

$$S_{kl} \equiv \frac{\delta\rho_k}{\rho_k(1 + w_k)} - \frac{\delta\rho_l}{\rho_l(1 + w_l)}.$$

[Kodama & Sasaki, Hwang & Noh]

(Two components, γ + CDM: $S = (\delta\rho_C/\rho_C) - \frac{3}{4}(\delta\rho_\gamma/\rho_\gamma)$.)

Time evolution of S

Using $\delta\dot{\rho}_i + 3H\delta\rho_i(1+w_i) - 3\dot{\Phi}\rho_i(1+w_i) = 0 \Rightarrow$

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{3}{4}H\Big[\sum_{i,j}(w_i - w_j)^2\rho_i\rho_j\Big]^{-2}$$
$$\cdot \sum_{i,j,k,l} [(w_k + w_l) - (w_i + w_j)](w_k - w_l)(w_i - w_j)^2$$
$$\cdot (1+w_k)(1+w_l)\rho_i\rho_j\rho_k\rho_l S_{kl}.$$

Only two different values w_i $\Rightarrow$ $S =$ constant.
(E.g. non-interacting baryons + radiation + CDM + HDM.)

More components: in general S and Φ coupled by $\delta\rho_i(t)$.
Special: **three components** (e.g. also cosmological constant)**: can solve for $S(t)$ without knowing $\Phi(t)$**:

$$\frac{\mathrm{d}^2 S}{\mathrm{d}t^2} = f(\rho_i, \cancel{\delta\rho_i})\,\frac{\mathrm{d}S}{\mathrm{d}t}.$$

Example: matter + radiation + cosmological constant:

$$S_{,NN} = 4\,S_{,N}\,\frac{\rho_m^2\rho_r + \rho_m\rho_r^2 - 9\rho_m^2\rho_\Lambda - 16\rho_r^2\rho_\Lambda - 23\rho_m\rho_r\rho_\Lambda}{(\rho_m + \frac{4}{3}\rho_r)(\rho_m\rho_r + 9\rho_m\rho_\Lambda + 16\rho_r\rho_\Lambda)}.$$

(# e-folds N: $\mathrm{d}N = H\mathrm{d}t$; $N_{\text{now}} = 0$, $N_{\text{rec}} = -7.0$, $N_{\text{eq}} = -8.7$.)
$\rho_i(N) = \Omega_i\rho_c \exp(-3(1+w_i)N) \Rightarrow$

$$\frac{S_{\text{rec}} - S_{\text{infl}}}{S_{\text{infl}}} = -12\,\frac{\Omega_\Lambda}{\Omega_m}\,e^{4N_{\text{rec}} - N_{\text{eq}}} = \mathbf{-10^{-7}}.$$

$\Rightarrow$ In this system S is still approx. constant.

Amplitudes & Spectral Indices CMB

- Homogeneous solution ($\tilde{S}=0$) $\to$ Φ(**adiabatic**);
- Particular solution (source $\tilde{S}$) $\to$ Φ(**isocurvature**).

(Initial cond.: $\Phi_{\text{iso}} = \Phi'_{\text{iso}} = 0$ at start of radiation domination.)

Assumptions: S is constant, (p)reheating ignored.

$\Rightarrow$ **amplitude** $|\delta_{\mathbf{k}}|^2 \equiv 2k^3\langle\hat{\Phi}^2_{\mathbf{k}}\rangle/(9\pi^2)$
spectral index $\tilde{n} \equiv \partial \ln |\delta_{\mathbf{k}}|^2/\partial \ln k$ (1st order slow roll):

$$
\begin{aligned}
|\delta^{\text{ad}}_{\mathbf{k}}|^2 &= \frac{\boldsymbol{\kappa}^2}{\mathbf{50}\boldsymbol{\pi}^2}\frac{\boldsymbol{H}^2_{\mathcal{H}}}{\tilde{\boldsymbol{\epsilon}}_{\mathcal{H}}}\Big[(1-2\tilde{\epsilon}_{\mathcal{H}})(1+\boldsymbol{U}^T_P\boldsymbol{U}_P) \\
&\qquad +2B\left((2\tilde{\epsilon}_{\mathcal{H}}+\tilde{\eta}^{\parallel}_{\mathcal{H}})+2\tilde{\eta}^{\perp}_{\mathcal{H}}e^T_2U_P+U^T_P\delta_{\mathcal{H}}U_P\right)\Big]; \\
|\delta^{\text{iso}}_{\mathbf{k}}|^2 &= \frac{\mathbf{1}}{\mathbf{36}}\frac{\boldsymbol{\kappa}^2}{\mathbf{50}\boldsymbol{\pi}^2}\frac{\boldsymbol{H}^2_{\mathcal{H}}}{\tilde{\boldsymbol{\epsilon}}_{\mathcal{H}}}\Big[(1-2\tilde{\epsilon}_{\mathcal{H}})\boldsymbol{V}^T\boldsymbol{V}+2BV^T\delta_{\mathcal{H}}V\Big]; \\
|\delta^{\text{tens}}_{\mathbf{k}}|^2 &= \frac{\mathbf{400}}{\mathbf{9}}\frac{\boldsymbol{\kappa}^2}{\mathbf{50}\boldsymbol{\pi}^2}\boldsymbol{H}^2_{\mathcal{H}}(1+2(B-1)\tilde{\epsilon}_{\mathcal{H}}).
\end{aligned}
$$

$B \equiv 2-\gamma_E-\ln 2 \approx 0.7296$, $\delta \equiv \tilde{\epsilon}-\tilde{M}^2/(3H^2)+2\tilde{\epsilon}\, e_1 e^T_1$,
$\tilde{M}^2 =$ eff. mass matrix, $t_{\mathcal{H}} =$ Hubble crossing time ($\mathcal{H}=k$).

$$
\begin{aligned}
\tilde{n}_{\text{ad}} &= -4\tilde{\epsilon}_{\mathcal{H}}-2\tilde{\eta}^{\parallel}_{\mathcal{H}}+\frac{2U^T_P(2\tilde{\epsilon}_{\mathcal{H}}+\tilde{\eta}^{\parallel}_{\mathcal{H}}-\delta_{\mathcal{H}})U_P-4\tilde{\eta}^{\perp}_{\mathcal{H}}e^T_2U_P}{1+U^T_PU_P}; \\
\tilde{n}_{\text{iso}} &= -2\frac{V^T\delta_{\mathcal{H}}V}{V^TV}; \qquad \tilde{n}_{\text{tens}} = -2\tilde{\epsilon}_{\mathcal{H}}.
\end{aligned}
$$

$\tilde{\epsilon} \equiv -\dfrac{\dot{H}}{H^2}$ and $\tilde{\boldsymbol{\eta}} \equiv \dfrac{\mathcal{D}_t\dot{\boldsymbol{\phi}}}{H|\dot{\boldsymbol{\phi}}|}$ multiple-field slow-roll functions.

Basis: $\dot{\boldsymbol{\phi}} \rightsquigarrow \mathbf{e}_1$, $\mathcal{D}_t\dot{\boldsymbol{\phi}} \rightsquigarrow \mathbf{e}_2$, etc. [Groot Nibbelink & vT, '00/'01]

$\Rightarrow \tilde{\eta}^{\parallel} \equiv \mathbf{e}_1\cdot\tilde{\boldsymbol{\eta}}$ (eff. single-field), $\tilde{\eta}^{\perp} \equiv \mathbf{e}_2\cdot\tilde{\boldsymbol{\eta}}$ (multiple-field).

Multiple-field effects & Observational relations

Multiple-field contributions determined by vectors U_P (adiabatic) and V (isocurvature) $(\delta\hat{\phi} \propto Q\hat{a}^\dagger + Q^*\hat{a})$:

$$U_P^T = \int_{t_\mathcal{H}}^{t_e} \mathrm{d}t(\ldots)\tilde{\eta}^\perp e_2^T Q, \qquad V^T = (\ldots)\tilde{\eta}_e^\perp e_2^T Q_e.$$

$\tilde{\eta}^\perp$ **crucial** to determine importance multiple-field effects.

If S not constant, but $S(t) = f(t)S_e$ (effects of more precise matching and evolution of S), then:
$|\delta_\mathbf{k}^{\mathrm{iso}}|^2$ and $\tilde{n}_{\mathrm{iso}}$ unchanged, only definition V changes
$\Rightarrow$ observational relations unchanged.

Defining $r \equiv |\delta_\mathbf{k}^{\mathrm{tens}}|^2/|\delta_\mathbf{k}^{\mathrm{ad}}|^2$ we find:

$$\boxed{U_P^T U_P = -1 - \frac{200}{9}\frac{\tilde{n}_{\mathrm{tens}}}{r}\left[1 - (B - \tfrac{1}{2})\tilde{n}_{\mathrm{tens}} + B\,\tilde{n}_{\mathrm{ad}} + (B-1)\frac{1}{\tilde{n}_{\mathrm{tens}}}\frac{\mathrm{d}\tilde{n}_{\mathrm{tens}}}{\mathrm{d}\ln k}\right]}$$

$\Rightarrow$ **observational test for multiple-field effects**.

Consistency relations (*one or two fields only!*):

$$r = -\frac{200}{9}\tilde{n}_{\mathrm{tens}}\left(1 - r_{\mathrm{mix}}^2\right), \qquad r_{\mathrm{mix}}^2 \equiv \frac{|\delta_\mathbf{k}^{\mathrm{mix}}|^4}{|\delta_\mathbf{k}^{\mathrm{ad}}|^2|\delta_\mathbf{k}^{\mathrm{iso}}|^2};$$

[Bartolo et al., '01/'02]

$$\frac{1}{\tilde{n}_{\mathrm{tens}}}\frac{\mathrm{d}\tilde{n}_{\mathrm{tens}}}{\mathrm{d}\ln k} = \frac{\tilde{n}_{\mathrm{tens}} - \tilde{n}_{\mathrm{ad}} + r_{\mathrm{mix}}^2(2\tilde{n}_{\mathrm{mix}} - \tilde{n}_{\mathrm{tens}} - \tilde{n}_{\mathrm{iso}})}{1 - r_{\mathrm{mix}}^2}$$

$\Rightarrow$ distinguish between two or more fields.

(Influence trans-Planckian effects?)

Conclusions

- Gravitational potential Φ only depends on **total entropy perturbation** S.

  S is constant for **two** different types of energy components, and can be computed explicitly (without knowing Φ) for **three**.

  (Assumed: super-Hubble, constant w_i, no interactions.)

- Gave explicit expressions **amplitudes** & **spectral indices** CMB spectrum in terms of inflationary background quantities, valid to first order in slow roll.

  Multiple-field effects: isocurvature perturbation, but also important contribution to adiabatic pert. Behaviour $\tilde{\eta}^{\perp}$ determines importance effects.

- Derived **observational relations** to test significance multiple-field effects and distinguish between one, two, or more fields. They remain valid if S varies with time.