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Conformal Bulk Matter and Brane Dynamics

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Summary

- **Randall-Sundrum (RS) scenario:** dynamics of a spherically symmetric 3-brane embedded in a Z_2 symmetric 5D AdS space with bulk matter fields.
- **Exact 5D solutions:** dynamics of inhomogeneous dust and dark radiation on the brane with localized gravity. Geometries correspond to a certain conformal class of bulk matter fields. Static marginally bound limits.
- **Effective 4D dynamics:** conditions for singular or globally regular behavior.

1. Introduction

RS brane world scenario

Observable Universe: 3-brane world embedded in a
Z_2 symmetric 5D AdS space.

- **RS1 Model:** Localization of gravity on hidden positive tension brane and decay towards visible negative tension brane. Exponential hierarchy.

- **RS2 Model:** Exponential localization of gravity in a single positive tension brane. Infinite fifth dimension.

Vaccum solutions

$$\boxed{d\tilde{s}_5^2 = \frac{l^2}{\left(|z - z_0| + z_0\right)^2}\left(dz^2 + ds_4^2\right).}$$

$ds_4^2 \rightarrow$ Ricci flat. AdS radius $l = 1/\sqrt{-\Lambda_B \kappa_5^2/6}$.
$\kappa_5^2 = 8\pi/M_5^3$, $M_5 \rightarrow$ fundamental 5D Planck mass.
Bulk cosmological constant $\Lambda_B = -\kappa_5^2 \lambda^2/6$, $\lambda \rightarrow$ brane tension.

- **Low energy brane gravity:** 4D general relativity.

- **Brane cosmology:** Friedmann-Robertson-Walker.

- **Gravitational collapse:** Static black holes localized on a brane: 2-brane in 4D AdS bulk. Physically realistic exact 5D collapse solutions not yet found. The difficulty: simultaneous localization of gravity and matter $\Rightarrow$ Emparan-Fabbri-Kaloper Conjecture: 5D classical black holes localized on the 3-brane correspond to quantum corrected 4D black holes on the brane.

- **4D effective Gauss-Codazzi approach:** More cosmological and gravitational collapse solutions. Non-static brane world vaccum. Embedding in exact 5D spacetimes frequently unknown.

2. Einstein Equations

Dynamical metric, coordinates (t, r, θ, ϕ, z)

Z_2 symmetry and 4D spherical symmetry

$$\boxed{d\tilde{s}_5^2 = \Omega^2 ds_5^2, \quad ds_5^2 = -e^{2A}dt^2 + e^{2B}dr^2 + R^2 d\Omega_2^2 + dz^2.}$$

$\Omega \rightarrow$ warp factor, $R \rightarrow$ physical radius of the 2-spheres.

$\Omega = \Omega(t, r, z)$, $A = A(t, r, z)$, $B = B(t, r, z)$, $R = R(t, r, z)$ are Z_2 symmetric functions.

Classical 5D Einstein equations

$$\boxed{\tilde{G}^{\nu}_{\mu} = -\kappa_5^2 \left[\Lambda_B \delta^{\nu}_{\mu} - \tilde{T}^{\nu}_{b\mu} - \tilde{T}^{\nu}_{\mu}\right].}$$

$$\tilde{T}^{\nu}_{b\mu} = -\frac{1}{\sqrt{\tilde{g}_{55}}}\left[\lambda\delta(z - z_0) + \lambda'\delta(z - z_1)\right]\left(\delta^{\nu}_{\mu} - \delta^{\nu}_5 \delta^5_{\mu}\right)$$

Stress-energy tensor $\tilde{T}_\mu^\nu$ conserved in the bulk

$$\boxed{\tilde{\nabla}_\mu \tilde{T}_\nu^\mu = 0.}$$

Assume conformal bulk matter of weight s

$$\boxed{\tilde{T}_\mu^\nu = \Omega^{s+2} T_\mu^\nu.}$$

Then the Einstein and conservation equations transform to

$$\boxed{\begin{gathered} G_\mu^\nu = -\kappa_5^2 \Omega^2 \left[\Lambda_B \delta_\mu^\nu - \Omega^{-1} T_{b\mu}^\nu - \Omega^{s+2} T_\mu^\nu \right] \\ -6\Omega^{-2} \left(\nabla_\mu \Omega \right) g^{\nu\rho} \nabla_\rho \Omega - 3\Omega^{-1} \delta_\mu^\nu g^{\rho\sigma} \nabla_\rho \nabla_\sigma \Omega \\ +3\Omega^{-1} g^{\nu\rho} \nabla_\rho \nabla_\mu \Omega \end{gathered}}$$

and

$$\boxed{\nabla_\mu T_\nu^\mu + \Omega^{-1} \left[(s+7) T_\nu^\mu \partial_\mu \Omega - T_\mu^\mu \partial_\nu \Omega \right] = 0.}$$

Assume further

$$\boxed{\tilde{T}_\mu^\nu = \Omega^{-2} T_\mu^\nu, \quad T_\mu^\nu = T_\mu^\nu(t, r)}$$

Then separate to obtain

$$\boxed{G^{\nu}_{\mu} = \kappa_5^2 T^{\nu}_{\mu},}$$

and

$$6\Omega^{-2}\nabla_\mu\Omega\nabla_\rho\Omega g^{\rho\nu} - 3\Omega^{-1}\nabla_\mu\nabla_\rho\Omega g^{\rho\nu} + 3\Omega^{-1}\nabla_\rho\nabla_\sigma\Omega g^{\rho\sigma}\delta^{\nu}_{\mu} =$$
$$-\kappa_5^2\Omega^2\left[\Lambda_B\delta^{\nu}_{\mu} - \Omega^{-1}T^{\nu}_{b\mu}\right].$$

Similarly for the conservation equation

$$\nabla_\mu T^{\mu}_{\nu} = 0,$$

$$3T^{\mu}_{\nu}\partial_\mu\Omega - T^{\mu}_{\mu}\partial_\nu\Omega = 0.$$

Now if

$$\boxed{A = A(t,r), \quad B = B(t,r), \quad R = R(t,r), \quad \Omega = \Omega(z)}$$

then obtain

$$\boxed{G^{b}_{a} = \kappa_5^2 T^{b}_{a}, \quad \nabla_a T^{a}_{b} = 0, \quad G^{z}_{z} = \kappa_5^2 T^{z}_{z},}$$

$$\boxed{\begin{aligned} 6\Omega^{-2}(\partial_z\Omega)^2 &= -\kappa_5^2\Omega^2\Lambda_B, \\ 3\Omega^{-1}\partial_z^2\Omega &= -\kappa_5^2\Omega^2\left[\Lambda_B - \Omega^{-1}T^{\nu}_{b\mu}\right] \end{aligned}}$$

and

$$\boxed{2T^z_z = T^c_c.}$$

3. Dust Dynamics

Equations of state

$$\boxed{\begin{gathered} T^{\nu}_{\mu} = diag(-\rho, p_r, p_T, p_T, p_z), \\ \rho = \rho_{\text{D}} + \frac{\Lambda}{\kappa_5^2},\ \ p_r = p_T = -\frac{\Lambda}{\kappa_5^2}, \\ p_z = -\frac{1}{2}\left(\rho_{\text{D}} + 4\frac{\Lambda}{\kappa_5^2}\right). \end{gathered}}$$

$\rho_{\text{D}} \rightarrow$ dust energy density.

$\Lambda \sim$ 4D cosmological constant

Conservation equations

$$\boxed{\begin{gathered}\dot{\rho}_{\mathrm{D}} + \left(\dot{B} + 2\frac{\dot{R}}{R}\right)\rho_{\mathrm{D}} = 0,\\ A'\rho_{\mathrm{D}} = 0.\end{gathered}}$$

Solution of conservation equations

$$\boxed{\begin{gathered}\rho_{\mathrm{D}} = \rho_{\mathrm{D}}(t,r), \quad A(t,r) = 0,\\ G^r_t = 0 \to e^B = \frac{R'}{H},\end{gathered}}$$

$H = H(r) \to$ arbitrary function of r.

$$\downarrow$$

$$\boxed{\rho_{\mathrm{D}} = \frac{2G_N M'}{\kappa_5^2 R^2 R'},}$$

$G_N \to$ Newton's gravitational constant.

$M = M(r) \to$ arbitrary function of r,

dust mass inside shell labelled by r.

Solution of diagonal Einstein equations

$$\boxed{-G^t_t + G^r_r + 2G^\theta_\theta = -2\frac{\ddot{R}'}{R'} - 4\frac{\ddot{R}}{R} = \frac{2G_N M'}{R^2 R'} - 2\Lambda.}$$

$$\downarrow$$

$$\boxed{\dot{R}^2 = \frac{2G_N M}{R} + \frac{\Lambda}{3}R^2 + f}$$

$f = f(r) \to$ arbitrary function of r,

energy inside a shell labelled by r.

On the initial hypersurface $t = 0$, $R(0, r) = r$

$$\boxed{\pm t + \psi = \int \frac{dR}{\sqrt{\frac{2G_N M}{R} + \frac{\Lambda}{3}R^2 + f}}.}$$

Applying the radial equation

$$\boxed{G^r_r = -2\frac{\ddot{R}}{R} + \frac{H^2}{R^2} - \frac{1}{R^2} - \frac{\dot{R}^2}{R^2} = -\Lambda}$$

$$\downarrow$$

$$H = \sqrt{1+f} \to f > -1$$

Then

$$\boxed{G^z_z = -\frac{\ddot{R}'}{R'} - 2\frac{\ddot{R}}{R} + \frac{f - \dot{R}^2}{R^2} + \frac{(f - \dot{R}^2)'}{RR'} = -\frac{G_N M'}{R^2 R'} - 2\Lambda = \kappa_5^2 p_z.}$$

With the RS warp

$$\boxed{\Omega = \Omega_{\text{RS}} = \frac{l}{|z - z_0| + z_0}}$$

obtain dynamical and inhomogeneous 5D dust solutions on the brane with localized gravity

$$\boxed{d\tilde{s}_5^2 = \Omega_{\text{RS}}^2 \left(-dt^2 + \frac{R'^2}{1+f} dr^2 + R^2 d\Omega_2^2 + dz^2 \right).}$$

Marginally bound limit: $f = 0$

$$\downarrow$$

$$\boxed{T = t + \int dR \frac{\sqrt{\frac{\Lambda}{3} R^4 + 2G_N M R}}{\frac{\Lambda}{3} R^3 - R + 2G_N M}.}$$

$$\downarrow$$

AdS/dS-Schwarzschild black string

$$\boxed{\begin{aligned} d\tilde{s}_5^2 &= \Omega_{\text{RS}}^2 \left(dz^2 + ds_4^2 \right), \\ ds_4^2 &= -\left(1 - \frac{2G_N M}{R} - \frac{\Lambda}{3} R^2 \right) dT^2 \\ &+ \left(1 - \frac{2G_N M}{R} - \frac{\Lambda}{3} R^2 \right)^{-1} dR^2 + R^2 d\Omega_2^2. \end{aligned}}$$

3. Dark Radiation Dynamics

Equations of state

$$T_\mu^\nu = diag(-\rho, p_r, p_T, p_T, p_z),$$

$$\rho + p_r = 0, \;\; p_T + \eta\rho + \frac{\Lambda}{\kappa_5^2}(1-\eta) = 0,$$

$$\rho = \rho_{\text{DR}} + \frac{\Lambda}{\kappa_5^2},$$

$$p_z = -(1+\eta)\rho - \frac{\Lambda}{\kappa_5^2}(1-\eta)\,.$$

$\rho_{\text{DR}} \rightarrow$ dark radiation energy density.

$\eta \rightarrow$ dark radiation model parameter

Usual dark radiation form: $\eta = -1$. If $\Lambda = 0$ then $T_a^a = 2p_z = 0$

Conservation equations

$$\dot{\rho}_{\text{DR}} + 2(1-\eta)\frac{\dot{R}}{R}\rho_{\text{DR}} = 0,$$

$$\rho_{\text{DR}}{}' + 2(1-\eta)\frac{R'}{R}\rho_{\text{DR}} = 0.$$

Solution of conservation equations

$$\rho_{\text{DR}} = \frac{Q_\eta}{\kappa_5^2}R^{2\eta-2}.$$

$Q_\eta \rightarrow$ dark radiation tidal charge

If $Q_\eta > 0$ then

4D weak, dominant and strong energy conditions

$$\downarrow$$

$$\eta \leq 1,\ |\eta| \leq 1 \text{ and } \eta \leq 0$$

5D weak, dominant and strong energy conditions

$$\downarrow$$

$$\eta \leq 0,\ -2 \leq \eta \leq 0 \text{ and } \eta \leq -1/3$$

If $Q_\eta < 0$ then

all energy conditions are violated.

Solution of diagonal Einstein equations

$$\boxed{-G_t^t + G_r^r + 2G_\theta^\theta = -2\frac{\ddot{R}'}{R'} - 4\frac{\ddot{R}}{R} = -2\eta Q_\eta R^{2\eta-2} - 2\Lambda.}$$

$$\downarrow$$

$$\boxed{\dot{R}^2 = \frac{Q_\eta}{2\eta+1}R^{2\eta} + \frac{\Lambda}{3}R^2 + f}$$

On the initial hypersurface $t = 0$, $R(0,r) = r$

$$\boxed{\pm t + \psi = \int \frac{dR}{\sqrt{\frac{\Lambda}{3}R^2 + f + \frac{Q_\eta}{2\eta+1}R^{2\eta}}}.}$$

Applying the radial equation

$$\boxed{G^r_r = -2\frac{\ddot{R}}{R} + \frac{H^2}{R^2} - \frac{1}{R^2} - \frac{\dot{R}^2}{R^2} = -Q_\eta R^{2\eta-2} - \Lambda}$$

$$\downarrow$$

$$H = \sqrt{1+f} \to f > -1$$

Then

$$\boxed{\begin{aligned} G^z_z &= -\frac{\ddot{R}'}{R'} - 2\frac{\ddot{R}}{R} + \frac{f - \dot{R}^2}{R^2} + \frac{(f-\dot{R}^2)'}{RR'} \\ &= -(1+\eta)\, Q_\eta R^{2\eta-2} - 2\Lambda = \kappa_5^2 p_z. \end{aligned}}$$

With the RS warp

$$\boxed{\Omega = \Omega_{\rm RS} = \frac{l}{|z-z_0| + z_0}}$$

obtain dynamical and inhomogeneous 5D dark radiation solutions on the brane with localized gravity

$$\boxed{d\tilde{s}_5^2 = \Omega^2_{\rm RS}\left(-dt^2 + \frac{R'^2}{1+f}dr^2 + R^2 d\Omega_2^2 + dz^2\right).}$$

Marginally bound limits: $f = 0$

$\downarrow$

$$\boxed{T = t + \int dR \frac{\sqrt{\frac{\Lambda}{3}R^2 + \frac{Q_\eta}{2\eta+1}R^{2\eta}}}{\frac{\Lambda}{3}R^2 - 1 + \frac{Q_\eta}{2\eta+1}R^{2\eta}}.}$$

$\downarrow$

$$d\tilde{s}_5^2 = \Omega_{\text{RS}}^2 \left(dz^2 + ds_4^2\right),$$

$$ds_4^2 = -\left(1 - \frac{Q_\eta}{2\eta+1}R^{2\eta} - \frac{\Lambda}{3}R^2\right) dT^2$$

$$+\left(1 - \frac{Q_\eta}{2\eta+1}R^{2\eta} - \frac{\Lambda}{3}R^2\right)^{-1} dR^2 + R^2 d\Omega_2^2.$$

Horizons

$$\boxed{1 - \frac{Q_\eta}{2\eta+1}R^{2\eta} - \frac{\Lambda}{3}R^2 = 0}$$

$\downarrow$

$$\boxed{\Lambda = 0 \to R_h = \left(\frac{2\eta+1}{Q_\eta}\right)^{\frac{1}{2\eta}}.}$$

Exact Dynamical Solutions

Example: $\eta = 1/2, \Lambda > 0$

Solutions organized by $\beta = \frac{3}{\Lambda}\left(\frac{3Q^2_{1/2}}{16\Lambda} - f\right)$

a) $\beta > 0 \rightarrow -1 < f < 3Q^2_{1/2}/(16\Lambda)$

$$\boxed{\left|R + \frac{3Q_{1/2}}{4\Lambda}\right| = \sqrt{\beta}\cosh\left[\pm\sqrt{\frac{\Lambda}{3}}t + \cosh^{-1}\left(\frac{\left|r + \frac{3Q_{1/2}}{4\Lambda}\right|}{\sqrt{\beta}}\right)\right].}$$

b) $\beta < 0 \rightarrow f > 3Q^2_{1/2}/(16\Lambda)$

$$\boxed{R + \frac{3Q_{1/2}}{4\Lambda} = \sqrt{-\beta}\sinh\left[\pm\sqrt{\frac{\Lambda}{3}}t + \sinh^{-1}\left(\frac{r + \frac{3Q_{1/2}}{4\Lambda}}{\sqrt{-\beta}}\right)\right].}$$

c) $\beta = 0 \rightarrow f = 3Q^2_{1/2}/(16\Lambda)$

$$\boxed{\left|R + \frac{3Q_{1/2}}{4\Lambda}\right| = \left|r + \frac{3Q_{1/2}}{4\Lambda}\right|\exp\left(\pm\sqrt{\frac{\Lambda}{3}}t\right).}$$

Singularities and Rebounces

$$\boxed{V = V(R,r) = \frac{Q_\eta}{2\eta+1}R^{2\eta} + \frac{\Lambda}{3}R^2 + f.}$$

$\forall R \geq 0, V(R) > 0 \rightarrow$ **shell focusing singularity at** $R = R_s = 0$

$\exists R = R_* > 0 : V = 0 \rightarrow$ **regular rebounce point at** $R = R_*$

Example: $\eta = 1/2, \Lambda > 0$

$$\boxed{V = \frac{\Lambda}{3}R^2 + \frac{Q_{1/2}}{2}R + f = \frac{\Lambda}{3}\left(Y^2 - \beta\right), \; Y = R + \frac{3Q_{1/2}}{4\Lambda}.}$$

a) $\beta < 0, f > 3Q_{1/2}^2/(16\Lambda)$

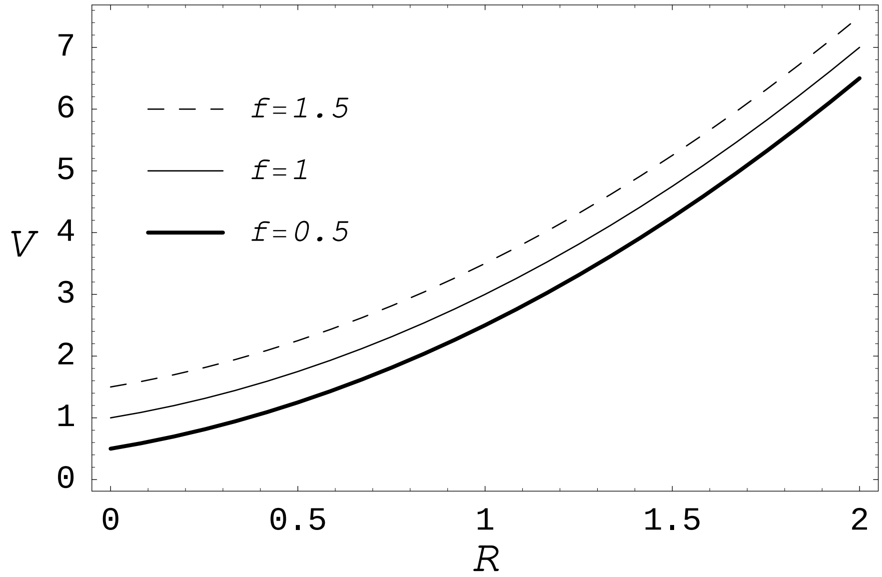

Figure 1: Plots of V for $Q_{1/2} > 0$. At the singular origin $V(0, r) = f$.

b) $\beta > 0, -1 < f < 0$

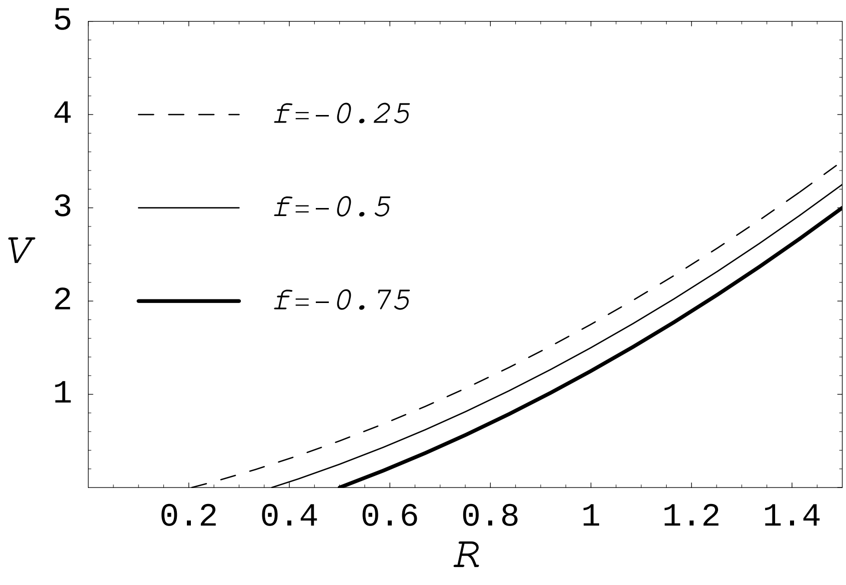

Figure 2: Plots of V for $Q_{1/2} > 0$. Rebounce points at $R_* = -\frac{3Q_{1/2}}{4\Lambda} + \sqrt{\beta}$. Globally regular solutions in 4D.

c) $\beta > 0, 0 < f < 3Q_{1/2}^2/(16\Lambda)$

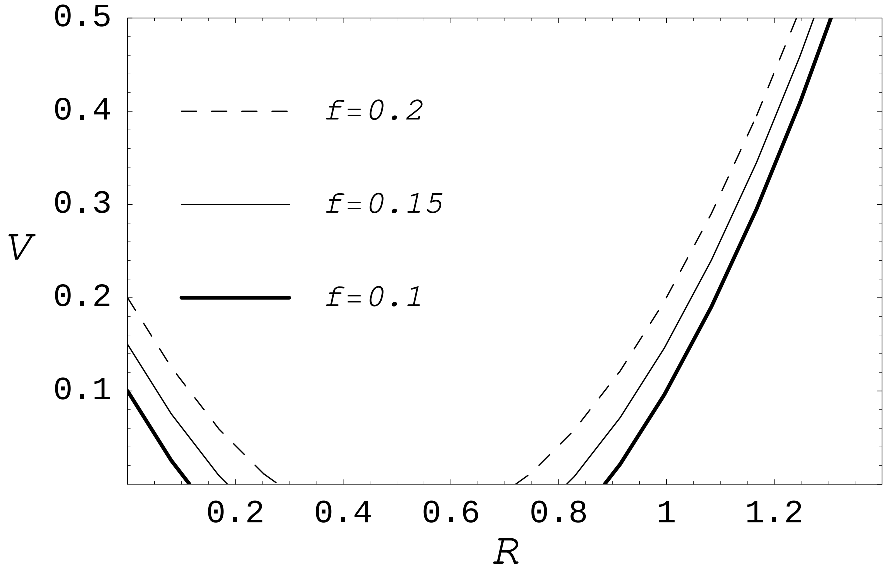

Figure 3: Plots of V for $Q_{1/2} < 0$. Rebounce points at $R_{*\pm} = -\frac{3Q_{1/2}}{4\Lambda} \pm \sqrt{\beta}$. At the singular origin $V(0, r) = f$.

d) $\beta = 0, f = 3Q_{1/2}^2/(16\Lambda)$

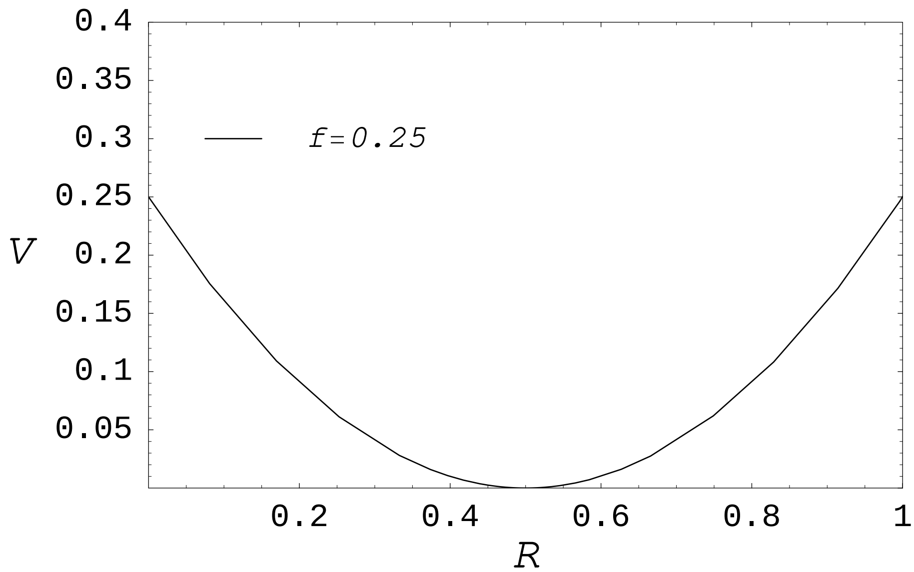

Figure 4: Plot of V for $Q_{1/2} < 0$. Rebounce points at $R_{*} = -\frac{3Q_{1/2}}{4\Lambda}$. At the singular origin $V(0, r) = f$.

4. Conclusions

- Exact 5D solutions for which gravity is localized near the brane and the dynamics of the bulk fields on the brane is that of inhomogeneous dust and dark radiation.
- Conformal fields in the AdS bulk with stress-energy tensor of weight -4.
- Dynamical phase space with evolution of dust and dark radiation shells on the brane. Conditions for singular or globally regular behavior.
- Bulk fields are not localized near the brane. The density and pressures increase with z and become singular at the AdS horizon. This is not a problem in the RS1 model as space-time is cut before the AdS horizon is reached. Within the RS2 model a solution to this problem requires the simultaneous localization of gravity and matter near the brane, an open problem for future research.

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