Supersymmetry, Epiphany term 2007

Problem sheet 1 (please hand in on 29.01.07)

Harmonic Oscillator in SUSY

a) Consider a (bosonic) harmonic oszillator (using for simplicity $\hbar = c = \omega = 1$). The following relations hold,

$$[q,p] = i, \quad a = \frac{1}{\sqrt{2}}(q+ip), \quad a^+ = \frac{1}{\sqrt{2}}(q-ip),$$

from which $[a,a^+] = 1,\ a\,|n\rangle = \sqrt{n}\,|n-1\rangle\,,\ a^+\,|n\rangle = \sqrt{n+1}\,|n+1\rangle$ follows. Here $|n\rangle$ are the eigenstates of the bosonic number operator, $N_B = a^+a$.

Express the Hamilton operator of the harmonic oszillator, $H_B = \frac{1}{2}(p^2+q^2)$, in terms of N_B. Evaluate $[N_B, a]$, $[N_B, a^+]$ and $H_B\,|n\rangle$.

b) We now furthermore consider a two state system (in analogy to usual spin $1/2$ states $\left|\vec{S}^{\,2}, S_z\right\rangle$):

$$\left|\frac{1}{2}, +\frac{1}{2}\right\rangle \equiv |+\rangle\,, \quad \left|\frac{1}{2}, -\frac{1}{2}\right\rangle \equiv |-\rangle\,.$$

What is the algebra of the S_i?
Using the definition $S_\pm = S_x \pm iS_y$ one can define fermionic creation and annihilation operators, $d^+ := S_+,\ d := S_-$. How do d^+, d act on the states $|+\rangle\,, |-\rangle$?

Determine the (anti-commutator) algebra of d^+, d.

In analogy to a spin system in a magnetic field we now define the fermionic Hamilton operator, $H_F = S_z$, and the fermionic number operator, $N_F = d^+d$. Express H_F in terms of N_F. How does N_F act on the states $|+\rangle\,, |-\rangle$?

c) We now couple the fermionic to the bosonic system:

$$H := H_B + H_F.$$

The states have now the form: $|n,+\rangle = |n\rangle \otimes |+\rangle\,,\ |n,-\rangle = |n\rangle \otimes |-\rangle\,.$
What do you obtain for the eigenvalue spectrum of H? Is there a degeneracy of the eigenstates?

Now we construct SUSY generators Q, Q^+ by demanding (with a suitable normalization):

$$\begin{aligned} Q\,|1,+\rangle &= |2,-\rangle \\ Q^+\,|2,-\rangle &= |1,+\rangle\,. \end{aligned}$$

What do you obtain for Q, Q^+? Evaluate $\left[N_{B,F}, Q^{(+)}\right]$.
What does this yield for the SUSY algebra of this quantum mechanical system, which is given by

$$\left\{Q^{(+)}, Q^{(+)}\right\}\,, \quad \left[H, Q^{(+)}\right]\,.$$

What follows for the energy eigenvalues?