Lectures on Phenomenology

Problem sheet 3

1) The renormalization group equations for the couplings $g_i, i = 1, 2, 3$, are given by:

$$\frac{\partial g_i}{\partial \ln \mu} = -b_i \frac{g_i^3}{16\pi^2}. \tag{1}$$

In the Standard Model one finds for the coefficients b_i of the different gauge groups:

$$\begin{aligned} b_3 &= 11 - \frac{4}{3} N_{\mathrm{g}}, \\ b_2 &= \frac{22}{3} - \frac{4}{3} N_{\mathrm{g}} - \frac{1}{6} N_{\mathrm{H}}, \\ b_1 &= \quad - \frac{4}{3} N_{\mathrm{g}} - \frac{1}{10} N_{\mathrm{H}}, \end{aligned}$$

where N_{g} is the number of generations and N_{H} the number of Higgs doublets.

In a Grand Unified Theory (GUT) the couplings unify at some GUT scale M_X:

$$g_1(M_X) = g_2(M_X) = g_3(M_X) = g_{\mathrm{GUT}}. \tag{2}$$

a) Why does b_1 have a different form than b_2 and b_3? Show that

$$g_i^{-2}(\mu) = g_i^{-2}(\mu_0) + \frac{b_i}{16\pi^2} \ln \left(\frac{\mu^2}{\mu_0^2} \right) \tag{3}$$

is a solution of eq. (1).

b) Using

$$\begin{aligned} \frac{g_1^2(\mu)}{4\pi} &= \frac{5}{3} \frac{\alpha(\mu)}{c_W^2(\mu)}, \quad \frac{g_2^2(\mu)}{4\pi} = \frac{\alpha(\mu)}{s_W^2(\mu)}, \quad \frac{g_3^2(\mu)}{4\pi} = \alpha_s(\mu), \\ \frac{g_{\mathrm{GUT}}^2}{4\pi} &= \alpha_{\mathrm{GUT}}, \end{aligned}$$

obtain a solution for $s_W^2(\mu)$ in terms of $\alpha(\mu)$, $\alpha_s(\mu)$, b_i.

c) What is the prediction for $s_W^2(M_Z)$ under the above assumptions, using $\alpha(M_Z) \approx 1/128$ and $\alpha_s(M_Z) \approx 0.12$?

d) The additional particles in the MSSM give rise to different coefficients of the renormalization group equations compared to the Standard Model case. Redo the above calculation, using for the MSSM

$$\begin{aligned} b_3 &= 9 - 2N_{\mathrm{g}}, \\ b_2 &= 6 - 2N_{\mathrm{g}} - \frac{1}{2} N_{\mathrm{H}}, \\ b_1 &= \quad - 2N_{\mathrm{g}} - \frac{3}{10} N_{\mathrm{H}}, \end{aligned}$$

and derive the corresponding prediction for $s_W^2(M_Z)$.

2) The dominant and process-independent contributions of the top quark in the Standard Model (SM) and the SUSY particles in the MSSM to the precision observables $M_W, \sin^2\theta_{\text{eff}}$, etc. enter via the leading contribution to the ρ parameter,

$$\Delta\rho = \frac{\Sigma_Z^T(0)}{M_Z^2} - \frac{\Sigma_W^T(0)}{M_W^2}.$$

Here Σ_Z^T, Σ_W^T denote the transverse parts ($\sim g_{\mu\nu}$) of the Z and W self-energies, respectively. This correction leads to a shift in the predictions for M_W and $\sin^2\theta_{\text{eff}}$ according to

$$\Delta M_W \approx \frac{M_W}{2}\frac{c_W^2}{c_W^2 - s_W^2}\Delta\rho, \quad \Delta\sin^2\theta_{\text{eff}} \approx -\frac{c_W^2 s_W^2}{c_W^2 - s_W^2}\Delta\rho.$$

a) We first consider the contribution of the (t/b) doublet in the SM. Draw the relevant diagrams at one-loop order.

   The result for the contribution of the (t/b) doublet in the SM reads

$$\Delta\rho^{\text{SM}} = \frac{3G_\mu}{8\sqrt{2}\pi^2}F_0(m_t^2, m_b^2),$$

   where G_μ is the Fermi constant, and the function F_0 is given by

$$F_0(x,y) = x + y - \frac{2xy}{x-y}\ln\frac{x}{y}.$$

   What is the result in the limits $m_b \to 0$ and $m_b \to m_t$?

b) We now consider the contribution of the superpartners of the t and b quarks. For vanishing mixing in the stop and sbottom sector only $\tilde{t}_L$, $\tilde{b}_L$, i.e. the superpartners of the left-handed components of the t and b quarks, yield a non-vanishing contribution. Draw the relevant one-loop diagrams in this case.

   Which result would you expect in the limit of unbroken SUSY, i.e. $m_{\tilde{t}_L}$, $m_{\tilde{b}_L}$? Accordingly, guess the result for the case of vanishing mixing in the stop and sbottom sector.

   For vanishing mixing in the stop and sbottom sector $m_{\tilde{t}_L}$, $m_{\tilde{b}_L}$ are given by

$$m_{\tilde{f}_L}^2 = m_{\tilde{q}}^2 + M_Z^2\cos 2\beta(I_3^f - Q_f s_W^2) + m_f^2, \quad f = t, b,$$

   where $m_{\tilde{q}}$ is a soft SUSY-breaking parameter in the sfermion sector which is equal for $m_{\tilde{t}_L}$ and $m_{\tilde{b}_L}$.

   What is the result for $m_{\tilde{q}} \gg m_t$?

   The results of the electroweak precision measurements lead to an upper bound on the allowed SUSY contributions to $\Delta\rho$ of currently roughly

$$\Delta\rho_{\text{SUSY}} \lesssim 3\times 10^{-3}.$$

   Which constraints on the parameter $m_{\tilde{q}}$ does this yield in the above scenario?